Topological Excitations in Compact Maxwell-Chern-Simons Theory

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We construct a lattice model of compact (2+1)-dimensional Maxwell-Chern-Simons theory, starting from its formulation in terms of gauge invariant quantities proposed by Deser and Jackiw. We thereby identify the topological excitations and their interactions. These consist of monopole-antimonopole pairs bounded by strings carrying both magnetic flux and electric charge. The electric charge renders the Dirac strings observable and endows them with a finite energy per unit length, which results in a linearly confining string tension. Additionally, the strings interact via an imaginary, topological term measuring the (self-)linking number of closed strings.

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Topological excitations play a fundamental role in statistical mechanics and field theory. They often drive phase transitions, as is the case in the two-dimensional $XY$ model [1] and they can lead to drastic modifications of the perturbative infrared behavior of a theory [2]. The origin of these nonperturbative phenomena lies in the compactness of a symmetry underlying a given model. The topology of the gauge group is thus of paramount importance in Abelian gauge theories, where the noncompact group $R$ and the compact group $U(1)$ lead to the same perturbation series. In three Euclidean dimensions, for example, the compactness of $U(1)$ leads to the existence of instanton solutions of the Maxwell equations, which coincide with the familiar Dirac magnetic monopoles [3] of three-dimensional Minkowski space. As was shown in [2], in the weak coupling limit these lead to the confinement of electric charges, by effectively transforming the Coulomb potential from logarithmic to linear.

Three-dimensional space-times are characterized by the possibility of adding a gauge invariant, nonconventional Chern-Simons term to the gauge field action [4]. The resulting theory, with Lagrangian density

$$\mathcal{L}_{\text{MCS}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{2} \epsilon^{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha$$

(1)

(in Minkowski space-time) describes massive photons with mass $m = ke^2$ (note that in $2+1$ dimensions $e^2$ has the dimension of mass). The Coulomb interaction between charges minimally coupled to (1) is correspondingly exponentially screened. Therefore, the weak coupling nonperturbative behavior of the compact Maxwell-Chern-Simons theory depends crucially on the interplay between magnetic monopoles and the Chern-Simons term, since they clearly work in opposite directions: The former promotes confinement of electric charges while the latter screens the Coulomb interaction. The outcome of this competition can have profound physical consequences, given that Maxwell-Chern-Simons theories appear as effective theories for the fractional quantum Hall effect and for two-dimensional spin models possibly relevant to high-$T_c$ superconductivity [5].

There are two simple ways to analyze compact Abelian gauge theories. As was pointed out in [2], one automatically obtains the compact $U(1)$ group by spontaneous breakdown of a compact non-Abelian group. Alternatively, one can formulate the $U(1)$ model on a lattice, with the gauge fields being phases of link variables. Monopoles in compact Maxwell-Chern-Simons theory were studied in [6,7] by using the first of the above described approaches. It was found that finite action single-monopole solutions do not exist. In [6] it was, however, shown that there exists a complex solution corresponding to a monopole-antimonopole pair: This has real, finite action proportional to the distance between the monopole and the antimonopole. In [7] the absence of isolated monopoles was established by a perturbative treatment of the Chern-Simons term. All this indicates that monopoles are linearly confined by a string of magnetic flux due to the Chern-Simons term.

It is therefore important to confirm and clarify this mechanism by analyzing compact Maxwell-Chern-Simons theory on the lattice. Moreover, from this approach one should also obtain the interaction between the strings. While noncompact versions of lattice Maxwell-Chern-Simons theory have been previously studied [8], no compact formulation of the same lattice models is yet available. The difficulty lies in the explicit appearance of the gauge potential $A_\mu$ in the Chern-Simons term. This makes it difficult to formulate a periodic and gauge invariant lattice action with the correct continuum limit. Here we avoid this problem by considering an equivalent formulation of Maxwell-Chern-Simons theory in terms of gauge invariant quantities.

In [9] it was shown that the self-dual model

$$\mathcal{L} = \frac{1}{2e^2} f_{\mu\nu} f^{\mu\nu} - \frac{1}{2me^2} \epsilon^{\mu\nu\alpha} f_\mu \partial_\nu f_\alpha$$

(2)

considered first in Ref. [10], is equivalent (by a Legendre
transformation) to the Maxwell-Chern-Simons theory (1). Indeed, the field equations obtained by varying (2) with respect to \( f_\mu \) are identical to the Maxwell-Chern-Simons field equations if \( f_\mu \) is interpreted as the dual of the field strength \( F_{\mu \nu} \) \( f_\mu = \epsilon^{\mu \nu \rho \sigma} F_{\nu \rho} / 2 \). This is possible since the equations of motion imply the "Bianchi identity" \( \partial_\mu f_\mu = 0 \). Although this is valid only on shell, this is sufficient to guarantee the equivalence of the free theories (1) and (2), as shown in [9]. We are thus led to consider the continuum model defined by the Euclidean partition function

\[
Z = \int \mathcal{D}f_\mu e^{-S_E(f_\mu)} ,
\]

\[
S_E(f_\mu) = \int d^4x \left[ \frac{1}{2e^2} \left( f_\mu f_\mu + \frac{i}{m} \epsilon_{\mu \nu a} \partial_\nu f_a \right) \right] .
\]

With this formulation at hand it is easy to construct a compact lattice model of Maxwell-Chern-Simons theory. To this end we consider a cubic lattice with lattice spacing \( a \) and lattice sites denoted by the vectors \( z \). The forward and backward lattice derivatives are defined as

\[
d_{\mu}g(z) \equiv \frac{g(z + \hat{\mu}a) - g(z)}{a} ,
\]

\[
d_{\mu}g(z) \equiv \frac{g(z) - g(z - \hat{\mu}a)}{a} ,
\]

\[
S_E(f_\mu) = \int d^4x \frac{1}{2e^2} f_\mu^2 + \frac{i}{4e^2} \left[ f_\mu + \frac{1}{2m} \epsilon_{\mu \nu a} \partial_\nu f_a \right] - \frac{i}{4e^2} \left[ f_\mu - \frac{1}{2m} \epsilon_{\mu \nu a} \partial_\nu f_a \right] .
\]

We can now obtain a Villain-type model [12] by summing over three sets of integer link variables, whose purpose is to take into account the periodicity of the link variables appearing in the three squares. Since we are integrating over a unique fundamental variable \( f_\mu \), however, it turns out that two sets of integers are sufficient to enforce periodicity. We therefore posit the following lattice regularized version of the Euclidean model (3):

\[
Z_L = \sum_{l_\mu,k_\mu} \int_{|l_\mu|} e^{i a z} \mathcal{D}f_\mu \exp \sum_{z,\mu} \left[ -\frac{a^3}{2e^2} \left( f_\mu + \frac{2\pi}{a} l_\mu \right)^2 - \frac{i a^3}{4e^2} \left( f_\mu + \frac{1}{2m} \epsilon_{\mu \nu a} d_\nu f_a + \frac{2\pi}{a^2} (2l_\mu - k_\mu) \right)^2 \right] ,
\]

where the sum runs over all lattice sites \( z \) and all directions \( \mu \), \( l_\mu \), and \( k_\mu \) are integer link variables, and we have introduced the notation \( \mathcal{D}f_\mu = \prod_z df_\mu(z) \). This partition function is clearly invariant under the transformations (5), since these can be absorbed by a redefinition

\[
l_\mu \equiv l_\mu + n_\mu , \quad k_\mu \equiv k_\mu + n_\mu - \frac{1}{2ma} \epsilon_{\mu \nu a} d_\nu n_\mu .
\]

The formal continuum limit of the above lattice model is obtained by letting \( a \to 0 \) and \( \sum_z \to (1/a^3) \int d^4z \). As we now show, in this limit we recover the continuum model (3) times a partition function describing the topological excitations due to the compactness of the underlying gauge symmetry.

Indeed, we now rewrite (8) in a fashion which exposes explicitly these topological excitations and their interactions. To this end we decompose \( k_\mu \) as

\[
k_\mu = l_\mu - \frac{1}{2ma} \epsilon_{\mu \nu a} d_\nu n_\mu + j_\mu ,
\]

where \( j_\mu \) are integers. Correspondingly, the sum over all configurations \( \{k_\mu\} \) in (8) can be replaced by a sum over all configurations \( \{j_\mu\} \):
where $g_{\mu} = f_{\mu} + (2\pi/a^2)l_{\mu}$. By changing variables from $f_{\mu}$ to $g_{\mu}$ in the integration and performing the sum over all configurations $l_{\mu}$, we obtain

$$Z_L = \sum_{l_{\mu}} \int^{+\infty}_{-\infty} Df_{\mu} \exp \left\{ -\frac{a^3}{2e^2} g_{\mu}^2 - \frac{ia^3}{4e^2} \left[ g_{\mu} + \frac{1}{2m} \epsilon_{\mu\nu\alpha} d_{\nu}g_{\alpha} - \frac{2\pi}{a^2} f_{\mu} \right]^2 + \frac{ia^3}{4e^2} \left[ g_{\mu} - \frac{1}{2m} \epsilon_{\mu\nu\alpha} d_{\nu}g_{\alpha} + \frac{2\pi}{a^2} f_{\mu} \right]^2 \right\} ,$$

(10)

In the last step we perform the Gaussian integration over $f_{\mu}$. To this end we note that, by a summation by parts, the operator appearing in the quadratic term in $f_{\mu}$ can be rewritten as

$$\delta_{\mu\alpha} + \frac{i}{m} \epsilon_{\nu\mu\alpha} D_{\nu} = \delta_{\mu\alpha} + \frac{i}{m} \epsilon_{\nu\mu\alpha} D_{\nu} ,$$

(12)

$$D_{\mu} = \frac{1}{2i} (d_{\mu} + \tilde{d}_{\mu}) .$$

Its inverse is given by

$$G_{\mu}(z,z') = (m^2 \delta_{\mu\nu} - D_{\mu} D_{\nu} - im \epsilon_{\mu\nu\alpha} D_{\alpha}) G(z,z') ,$$

$$(-D_{\mu} D_{\mu} + m^2) G(z,z') = 2\pi \delta_{zz'} .$$

(13)

Note that the operator $D_{\mu} D_{\mu}$ appearing in this formula represents a lattice regularized version of the Laplacian. Using (13) we obtain our final result:

$$Z_L = Z_0 \sum_{l_{\mu}} \exp \left\{ -\frac{4\pi^2}{2ae^2} j_{\mu}(z) G_{\mu}(z,z') j_{\mu}(z') \right\} ,$$

(14)

where

$$Z_0 = \int Df_{\mu} \exp \left\{ -\frac{a^3}{2e^2} f_{\mu}^2 - \frac{ia^3}{2me^2} \epsilon_{\mu\nu\alpha} d_{\nu}f_{\alpha} \right\}$$

(15)

is the Gaussian partition function for a free massive photon on the lattice. As anticipated, this reduces to the continuum partition function (3) in the formal continuum limit. The remaining factor in (14) describes the topological excitations, which we now discuss.

First, let us note that these topological excitations are strings: These can be closed (rings), in which case $a \tilde{d}_{\mu} j_{\mu} = 0$, or open, in which case we identify the integers

$$a \tilde{d}_{\mu} j_{\mu} = q$$

(16)

with the magnetic monopoles. In order to justify this interpretation let us consider the continuum limit of the topological excitations. The continuum field equations, derived from the continuum action (3), are given by

$$f_{\mu} + \frac{i}{m} \epsilon_{\nu\mu\alpha} \partial_{\nu} f_{\alpha} = 0 .$$

(17)

When $f_{\mu}$ is interpreted as the dual field strength of a compact gauge theory, however, it can contain string singularities [2] $j_{\mu}$: $f_{\mu} = f_{\mu}^{reg} - 2\pi j_{\mu}$, where $j_{\mu}$ are of the type

$$j_{\mu} = n [\theta(x_0 - L/2) - \theta(x_0 + L/2)] \delta(x_1) \delta(x_2) ,$$

with $n$ an integer. These string singularities can be brought to the right-hand side of (17), where they act as sources for the regular part of $f_{\mu}$,

$$f_{\mu}^{reg} + \frac{i}{m} \epsilon_{\nu\mu\alpha} \partial_{\nu} f_{\alpha}^{reg} = 2\pi j_{\mu} .$$

(18)

Inserting (18) and its inverse

$$f_{\mu}^{reg} = \frac{m^2 \delta_{\mu\nu} - \partial_{\mu} \partial_{\nu} - im \epsilon_{\mu\nu\alpha} \partial_{\alpha}}{2\pi} j_{\nu} ,$$

(19)

in the action $S_E(f_{\mu})$ in (3) we obtain

$$S_{top} = \int d^3x \frac{2\pi^2}{2e^2} j_{\mu} \frac{m^2 \delta_{\mu\nu} - \partial_{\mu} \partial_{\nu} - im \epsilon_{\mu\nu\alpha} \partial_{\alpha}}{2\pi} j_{\nu} ,$$

(20)

which is exactly the continuum limit of the topological action in (14).

Equation (19) represents the Maxwell-Chern-Simons monopole solution. The fact that $\partial_{\mu} f_{\mu}^{reg} = 2\pi \delta_{\mu}(x_{mon})$ justifies our identification (16) in the lattice model. As expected, the magnetic field is exponentially screened with a characteristic length $(1/m)$ determined by the topological Chern-Simons mass. In the limit $m \to 0$, (19) reduces to the familiar monopole solution of compact, $(2+1)$-dimensional QED [2]. Correspondingly (20) reduces to the action for a Coulomb gas of monopoles [2]. Our monopole solution (19) is different from previously considered Abelian solutions [6,13] which describe long range magnetic fields. In analogy to Pisarski's non-Abelian solution [6], our Abelian solution is complex; however, the corresponding action contains a positive definite real part.

Let us now return to our lattice model and consider the action for a monopole-antimonopole pair united by a string of length $L$. This contains a positive piece given by

$$S_{MM} = \frac{4\pi^2 m^2}{2e^2} G(0) L ,$$

(21)

with $G$ defined in Eq. (13). This shows that a single-monopole solution ($L \to \infty$) has infinite action; therefore isolated monopoles are completely suppressed in the partition function. Monopoles can only appear as monopole-antimonopole pairs linearly confined by a string; since $2e^2$ plays the role of temperature and $G(0) = a^2$...
of $g(ma)$ we can identify $m^2 4 \pi^2 g(ma)$ as the corresponding string tension. The string singularities (18), due to the compactness of the model, correspond exactly to the induced current first postulated in [13]. The necessity of such an induced current in the presence of a monopole was also recognized by Pisarski [6], who first advanced an explanation of its physical meaning. As is now well known, in topologically massive quantum electrodynamics, external currents generate magnetic flux due to the presence of $f_\mu$ in the equations of motion. What happens in the presence of a monopole is the converse: Magnetic flux generates an induced current which flows through the Dirac string. Because of its electric charge, this becomes observable and acquires a finite energy per unit length. Because of this energy, configurations with infinitely long strings are suppressed. This mechanism is clearly exposed in Eqs. (14) and (20). The interaction energy for the strings contains three terms: the first is the electric-electric interaction between the charged strings, the second is the magnetic-magnetic interaction between the monopoles at the end of the strings, and the third is the electric-magnetic interaction between the electric charge of one string and the magnetic field induced by the second via the Biot-Savart law. All this is reminiscent of self-dual $Z_N$ models in even dimensions [14], the difference being that here it is the same objects that carry both the electric and the magnetic charge. Note also that the parity-violating, imaginary electric-magnetic interaction contains a topological term measuring the (self-)linking number of closed strings [15].

A word of care is due at this point. Indeed, the fact that $\partial_\mu f_\mu \neq 0$ does not imply any violation of charge conservation. In fact $f_\mu^{\mu \phi}$ appearing on the left-hand side of (18) cannot be interpreted as the dual field strength of a gauge theory exactly because it is stripped of its singularities. Correspondingly, $f_\mu$ is not a current minimally coupled to a gauge theory and need not therefore be conserved. As is shown in [6,13], the total current minimally coupled to $A_\mu$ in the gauge formation is indeed conserved.

Let us now write the field equations of topologically massive electrodynamics in the form

$$\frac{1}{e^2} \epsilon_{\mu \nu \sigma} \partial_\alpha f_{\beta} - \frac{i m}{e^2} f_\mu = J_\mu^\phi.$$  \hspace{1cm} (22)

The current appearing in this formulation is the particle-number current density. Comparing this with (18) we recognize first that $J_\mu^{\phi}$ is imaginary in Euclidean space. In Minkowski space it would be real and $\partial_\mu J_\mu^{\phi} = 0$ would correspond to the creation of particles at the event of the monopole [13]. Comparing with the divergence of (18) we recognize that the number of these particles is quantized only if the Chern-Simons coupling constant is quantized as $2 \pi \equiv 2 \pi n/e^2 \equiv \text{integer}$, in full agreement with [6,13].

We conclude by remarking that the topological excitation we have found in compact Maxwell-Chern-Simons theory are the same as the corresponding ones in the compact Abelian Higgs model in (2+1) dimensions [16]. There, the same mechanism takes place: Coupling the gauge fields to a scalar effectively attaches flux tubes to the monopoles, which become linearly confined. Indeed, the real part of the interaction between strings is identical in the two models; the only difference lies in the parity-violating imaginary part of the interaction, present in the Maxwell-Chern-Simons model. We postpone the analysis of the phase structure of our model to a forthcoming publication.

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