Exact Wavefunctions for non-Abelian Chern-Simons Particles†

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Abstract

Exact wavefunctions for $N$ non-Abelian Chern-Simons (NACS) particles are obtained by the ladder operator approach. The same method has previously been applied to construct exact wavefunctions for multi-anyon systems. The two distinct base states of the NACS particles that we use are multi-valued and are defined in terms of path ordered line integrals. Only strings of operators that preserve the monodromy properties of these base states are allowed to act on them to generate new states.

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1. Introduction

It is now well-known that anyons—particles with arbitrary statistics can exist in 2+1 dimensions.\cite{1} Owing to the topological gauge potential, even non-interacting two-anyon states are not (symmetrized) tensor products of single anyon states.\cite{2} Thus the quantum mechanics of systems of many anyons presents a challenge to field theorists.\cite{3} In general, the center of mass motion of the system is not sensitive to the statistics and can be factored out. For two anyons, the relative coordinates present us with a one-body problem which can be solved in many cases. For $N$ anyons, there are $N-1$ relative coordinates, whereas there are $N(N-1)/2$ pairs of particles. These two numbers match only when $N = 2$. For $N > 2$, the various pair-separation coordinates (in terms of which the statistical gauge potential is easy to write down) are not independent of each other. For this reason, not a single three-anyon system has been completely solved. The many-anyon system has been studied in the mean field approach.\cite{4} Other methods that have been employed include the semi-classical approximation\cite{5}, numerical studies\cite{6}, perturbative analysis from the bosonic or fermionic ends\cite{7}, and the most interesting of all, the ladder operator approach: In the last couple of years, a substantial subset of the exact multi-anyon wavefunctions in a magnetic field has been found with this systematic analysis.\cite{8} This soon generalized to free anyons\cite{9} and anyons of multi-species.\cite{10} While a lot of states are still missing, all the states in the lowest Landau level are obtained by this method.

In this paper, we apply the ladder operator approach to non-Abelian Chern-Simons particles which may be regarded as a generalisation of anyons.\cite{11} In the gauge that the Hamiltonian is a free Hamiltonian, the wavefunctions (which have more than one component) are multivalued with non-trivial monodromy properties given by a monodromy matrix.\cite{12} By introducing statistical gauge potentials, one has the liberty to work with single-valued wavefunctions. However, since we find multivalued wavefunctions convenient to work with, we will stick to them in the rest of this paper.

In section 2, we review the ladder operator formalism as applied to anyons. This helps to highlight the differences between the cases of anyons and non-Abelian C-S particles, the object under study in section 3. In particular, of the operators used for anyons, only a subclass of operators, which preserve the monodromy properties of the wavefunctions, are allowed to act on the C-S particles. Nonetheless, our wavefunctions do cover the lowest Landau level. As an application of our formalism, we compute the
second virial coefficient of NACS particles. The same set of ladder operators apply
to free NACS particles with minor modifications. We also consider systems of multi-
species NACS particles. Finally, the relevance of our work to systems of vortices of
finite gauge groups is also discussed.

2. Anyons

The Hamiltonian for $N$ anyons with charge $e$ and mass $m$ moving on a plane with
a constant magnetic field $B$ (perpendicular to the plane) is given by

$$H = \sum_{\alpha=1}^{N} \frac{1}{2m}(\nabla_\alpha - i a_\alpha - i e A)^2,$$  \hspace{1cm} (1)

where the external gauge field $A^i = -\frac{1}{2} B e^{ij} x^j$ in the symmetric gauge and the sta-
tistical gauge potential

$$a_\alpha^i(x_1, \ldots, x_N) = \nu \sum_{\beta \neq \alpha} \epsilon^{ij} \frac{x^i_\alpha - x^i_\beta}{|x_\alpha - x_\beta|^2}. \hspace{1cm} (2)$$

By a singular gauge transformation, we can remove $a_\alpha$ from the Hamiltonian at
the expense of using multi-valued wavefunction

$$\psi_{\text{new}}(x_1, \ldots, x_N) = \exp \left( i \nu \sum_{\alpha < \beta} \theta_{\alpha \beta} \right) \psi_{\text{old}}(x_1, \ldots, x_N). \hspace{1cm} (3)$$

Using the complex notation $z = x^1 + i x^2$, $\bar{z} = x^1 - i x^2$, $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, the gauge
transformed Hamiltonian becomes\[4\]

$$H = \sum_{\alpha=1}^{N} \left( -\frac{2}{m} \partial_\alpha \bar{\partial}_{\alpha} + \frac{e^2 B^2}{8m} |\bar{z}_\alpha|^2 \right) - \frac{e B}{2m} J, \hspace{1cm} (4)$$

where $J$ is the angular momentum operator in the singular gauge

$$J = \sum_{\alpha=1}^{N} (z_\alpha \partial_\alpha - \bar{z}_\alpha \bar{\partial}_{\alpha}). \hspace{1cm} (5)$$

Its eigenvalues are shifted from those in the symmetric gauge by a constant $\frac{1}{2} \nu N(N - 1)$. (See (10.a).) It is convenient to extract a factor $\exp(-\frac{1}{2} e B \sum_{\alpha=1}^{N} |z_\alpha|^2)$ from the
wavefunction. Then the eigenvalue problem becomes

\[ \hat{H} \tilde{\psi} = (E - \frac{1}{2}N\omega)\tilde{\psi}, \]  
\[ (6.a) \]

\[ J\tilde{\psi} = j\tilde{\psi}, \]  
\[ (6.b) \]

(with \( \omega \equiv \frac{eB}{m} \)) where the new Hamiltonian \( \hat{H} \) and wavefunctions \( \tilde{\psi} \) are defined by

\[ \hat{H} = \sum_{\alpha=1}^{N} \left(-\frac{2}{m}\bar{\bar{\beta}}_{\alpha} \partial_{\alpha} + \frac{eB}{m}\bar{z}_{\alpha} \bar{\beta}_{\alpha}\right), \]  
\[ (7.a) \]

\[ \tilde{\psi} = \exp\left(\frac{eB}{4} \sum_{\alpha=1}^{N} |z_{\alpha}|^2\right)\psi. \]  
\[ (7.b) \]

Note that the ground state energy is shifted by \( \frac{1}{2}N\omega \). We impose two physical requirements for the wavefunctions. Firstly, they must vanish at points of coincidences if \( \nu \neq 0 \) due to the centrifugal potentials (hard-core requirement). Secondly, they form Abelian representations of the braid group.

Now we introduce the operators

\[ a_{\alpha}^\dagger = z_{\alpha} - \frac{2}{eB}\bar{\beta}_{\alpha}, a_{\alpha} = \bar{\beta}_{\alpha}, \]  
\[ (8.a) \]

\[ b_{\alpha}^\dagger = z_{\alpha} - \frac{2}{eB}\bar{\beta}_{\alpha}, b_{\alpha} = \partial_{\alpha}, \]  
\[ (8.b) \]

which satisfy \([a_{\alpha}, a_{\beta}^\dagger] = [b_{\alpha}, b_{\beta}^\dagger] = \delta_{\alpha\beta}\), all other commutators being zero. With respect to these operators, the Hamiltonian \( \hat{H} \) in (7.a) and the angular momentum \( J \) in (7.b) can be rewritten as

\[ \hat{H} = \omega \sum_{\alpha=1}^{N} a_{\alpha}^\dagger a_{\alpha}, \]  
\[ (9.a) \]

\[ J = \sum_{\alpha=1}^{N} (b_{\alpha}^\dagger b_{\alpha} - a_{\alpha}^\dagger a_{\alpha}). \]  
\[ (9.b) \]

It is trivial to construct two distinct base states (for \( 0 \leq \nu < 2 \)) with energy and
angular momentum eigenvalues:

\[
\hat{\psi}_I^{(0)} = \prod_{\alpha < \beta} (z_\alpha - z_\beta)^\nu, \\
E_I^0 = \frac{1}{2} N \omega, \\
J_I^0 = \frac{1}{2} \nu N(N - 1),
\]

(10.a)

\[
\hat{\psi}_{II}^{(0)} = \prod_{\alpha < \beta} (\bar{z}_\alpha - \bar{z}_\beta)^{2-\nu}, \\
E_{II}^0 = \frac{1}{2} N \omega + \frac{2 - \nu}{2} N(N - 1) \omega, \\
J_{II}^0 = -\frac{2 - \nu}{2} N(N - 1).
\]

(10.b)

The general strategy of the ladder operator approach is to construct multi-anyon wavefunctions by acting with step operators on the base states in (10.a) and (10.b). We must, however, respect the statistics and hard-core requirements for the resulting wavefunctions. In order to respect the statistics, we use only symmetric combinations of the step operators. Consider the symmetric operators

\[
C_{ln} = \sum_{\alpha=1}^{N} a_{\alpha}^l b_{\alpha}^n,
\]

(11)

where \(l, n\) are non-negative integers such that \(l + n \leq N\). (These operators form a basis in the ring of symmetric polynomials in \(2N\) variables.) They are step operators in energy and angular momentum which respect the statistics properties of the base states

\[
[\hat{H}, C_{ln}] = \omega l C_{ln}, \\
[J, C_{ln}] = (n - l) C_{ln}.
\]

(12.a)

(12.b)

They may, however, produce singular states (states with non-vanishing wavefunctions at points of coincidences) which have to be excluded by hand. We must identify which particular \(C_{ln}\) produce regular states. These operators can be safely applied to the
base states. Consider $C_{0n}$ first. With (8.b), we have

$$C_{0n} \hat{\psi}_I^{(0)} = \left( \sum_{\alpha=1}^{N} z_{\alpha}^{n} \right) \hat{\psi}_I^{(0)}.$$  \hfill (13)

Thus they can be safely applied. For $C_{1m}$, we have

$$C_{1m} \hat{\psi}_I^{(0)} = \sum_{\alpha=1}^{N} (z_{\alpha} - \partial_{\alpha}) (z_{\alpha} - \tilde{\partial}_{\alpha})^{m} \hat{\psi}_I^{(0)}$$

$$= \sum_{\alpha=1}^{N} (z_{\alpha} - \partial_{\alpha}) z_{\alpha}^{m} \hat{\psi}_I^{(0)}$$

$$=- \sum_{\alpha=1}^{N} z_{\alpha}^{m} \partial_{\alpha} \hat{\psi}_I^{(0)} + \sum_{\alpha=1}^{N} (z_{\alpha} z_{\alpha}^{m} - m z_{\alpha}^{m-1}) \hat{\psi}_I^{(0)},$$  \hfill (14)

where we set $eB = 2$ for simplicity. The seemingly singular first term is in fact regular because

$$\sum_{\alpha=1}^{N} z_{\alpha}^{m} \partial_{\alpha} \hat{\psi}_I^{(0)} = \sum_{\alpha=1}^{N} z_{\alpha}^{m} \sum_{\beta \neq \alpha} \frac{\nu}{z_{\alpha} - z_{\beta}} \hat{\psi}_I^{(0)} = \nu \sum_{\alpha < \beta} \frac{z_{\alpha}^{m} - z_{\beta}^{m}}{z_{\alpha} - z_{\beta}} \hat{\psi}_I^{(0)}. $$  \hfill (15)

By a similar proof, one can apply a sequence of operators of the form $C_{0n}$ followed by a sequence of operators of the form $C_{1m}$ to $\hat{\psi}_I^{(0)}$ without generating any singularities. Moreover, states of the form

$$\hat{\psi}_I^{(1)} = \prod_{\alpha < \beta} (z_{\alpha} \beta)^{l+2l}, l = 1, 2, \ldots$$  \hfill (16)

are obtained from the action of $C_{0n}$ on $\hat{\psi}_I^{(0)}$. We can apply a string of operators $C_{n_1 m_1} C_{n_2 m_2} \ldots C_{n_i m_i}$ with $\sum_{j=1}^{i} n_j \leq 2l$ to $\hat{\psi}_I^{(1)}$ without generating singularities, because such a string contains at most derivatives of order $\sum_{j=1}^{i} n_j$ with respect to $z_{\alpha}$.

Thus we see that under suitable conditions, the step operators $C_{1m}$ can be safely applied to the base state $\hat{\psi}_I^{(0)}$ to generate regular new wavefunctions. A similar analysis holds for the other base state $\hat{\psi}_I^{(0)}$. Furthermore, closed-form eigenfunctions generated by the action of combinations of operators $C_{11}, C_{10}$ and $C_{0m}$ only have been found, and they can be expressed in terms of the Laguerre functions. In particular, they do not involve the operators $C_{1m}$ with $m > 1$, which however are allowed to act on $\hat{\psi}_I^{(0)}$ to produce regular wavefunctions.
One should also note that the step operator approach only generates a subset of the whole spectrum of wavefunctions.\[^6\] If we naively set $\nu$ to be zero or one, we obtain only a subset of the bosonic and fermionic wavefunctions. Unlike the states generated by the step operators, the energies of the missing states show non-linear dependence on the statistical parameter $\nu$ in recent numerical studies.\[^6\] However, this is unimportant for what follows.

3. Non-Abelian Chern-Simons Particles

Recently, there has been much interest in the non-Abelian generalization of anyons. Non-Abelian Chern-Simons (NACS) particles carry non-Abelian charges and interact with each other through the non-Abelian Chern-Simons term. It has been argued that they may have applications in the fractional quantum Hall effect.\[^2\] Consider a system of $N$ particles each of which carries a statistical charge corresponding to a representation $R_{l\alpha}$, $\alpha = 1, \ldots, N$ of a non-Abelian gauge group, which for definite we take to be $G = SU(2)$. In the holomorphic gauge, the dynamics of $N$ free $SU(2)$ NACS particles is governed by the Hamiltonian\[^1,2\]

$$
\hat{H} = -\sum_{\alpha=1}^{N} \frac{1}{m_{\alpha}} \left( \nabla_{z_{\alpha}} \nabla_{\bar{z}_{\alpha}} + \nabla_{\bar{z}_{\alpha}} \nabla_{z_{\alpha}} \right),
$$

$$
\nabla_{z_{\alpha}} = \frac{\partial}{\partial z_{\alpha}} + \frac{2}{k} \sum_{\beta \neq \alpha} \frac{T^{a}_{\alpha} T^{a}_{\beta}}{z_{\alpha} - z_{\beta}},
$$

$$
\nabla_{\bar{z}_{\alpha}} = \frac{\partial}{\partial \bar{z}_{\alpha}},
$$

where $k$, a positive integer, is a parameter of the theory and $T^{a}_{\alpha}$ are the $SU(2)$-generators in the representation $R_{l\alpha}$.

The wavefunctions take values in the tensor product of these representations.

$$
\Psi \in R_{l_1} \otimes \ldots \otimes R_{l_N}.
$$

We expand the single-valued wavefunction $\psi$ in terms of the conformal blocks $F_{i} \in R_{l_1} \otimes \ldots \otimes R_{l_N}$ (which satisfy $\nabla_{\alpha} F_{i} = 0$):

$$
\Psi = \sum_{i} \psi_{i} F_{i}.
$$

The Hamiltonian acting on the new wavefunction is just the free Hamiltonian. However, the complexity of the problem is hidden in the multivaluedness of the wavefunctions $\psi_{i}$. (In fact, it is more “natural” to work with the multi-valued wavefunctions
\( \psi_i \) than the original single-valued wavefunctions \( \psi \), partly because in the holomorphic gauge the Hamiltonian is not hermitian with respect to the usual inner product. Instead, the inner product is defined in the singular gauge and transformed back to the holomorphic gauge by a non-unitary transformation function which has to be taken into account in the definition of the inner product.\(^{[22]}\)

From now on, we stick to the singular gauge. Consider \( N \) NACS particles in the same irreducible representation \( R_l \) of \( SU(2) \) moving in a uniform external magnetic field \( B \). We introduce operators \( a_\alpha, a_\alpha^\dagger, b_\alpha, b_\alpha^\dagger \) as in eqn.(8) of section 2 and find that the Hamiltonian is again given by eqn.(9). The only difference lies in the constraints of the monodromy properties of the wavefunctions. In the case of anyons, the wavefunctions have only one component and monodromy leads to acquisition of phases, whereas NACS particles have multi-component wavefunctions whose monodromy properties are given by matrices.

We define

\[
\Omega_{\alpha \beta} \equiv \frac{2}{k} \sum_a T^a_{\alpha \beta}, \tag{20}
\]

Note that \( \sum_{\alpha < \beta} \Omega_{\alpha \beta}, J \) and \( \hat{H} \) commute with each other and are thus good quantum numbers. \( (J - \sum_{\alpha < \beta} \Omega_{\alpha \beta} \) is the angular momentum operator in the holomorphic gauge.) We will discuss the diagonalization of \( \sum_{\alpha < \beta} \Omega_{\alpha \beta} \) later. For the time being, let us assume this has been done and let \( \psi_I \in R_{l_1} \otimes R_{l_2} \otimes \ldots \otimes R_{l_N} \) be a (position-independent) eigenvector of \( \sum_{\alpha < \beta} \Omega_{\alpha \beta} \) with eigenvalue \( \Omega \). In analogy with the anyon case, we propose applying the same ladder operator approach with the following base states which are expressed as path-ordered line integrals:\(^{[23]}\)

\[
\hat{\psi}_I^{(0)}(z_1, \ldots, z_N) = P \exp \left( \int_\Gamma \sum_{\alpha < \beta} (\Omega_{\alpha \beta} - 2m_{\alpha \beta} I) d\log(z_\alpha - z_\beta) \right) \psi_I, \tag{21}
\]

\[
\hat{\psi}_{II}^{(0)}(z_1, \ldots, z_N) = P \exp \left( \int_\Gamma \sum_{\alpha < \beta} (2m_{\alpha \beta} I - \Omega_{\alpha \beta}) d\log(z_\alpha - z_\beta) \right) \psi_I, \tag{22}
\]

where \( \Gamma \) is a path in the \( N \)-dimensional complex space with one end point fixed and the other being \( \zeta = (z_1, \ldots, z_N) \). The \( m_{\alpha \beta} \) (\( n_{\alpha \beta} \)) depend on \( \psi_I \) and are the maximal (minimal) integers which make the wavefunctions non-singular at the points of coincidences. This is analogous to the requirement \( 0 \leq \nu < 2 \) in the anyon case. Modulo the terms involving the identity matrix, the first integrand is just
the flat Knizhnik-Zamolodchikov connection\[^{10}\] whereas the second is related to its antiholomorphic analogue. One can easily check that these base states have the desirable monodromy properties. From (5),(6) and (7.a), we have

\[
\hat{H}^{(0)}_{\psi_I} = 0, \\
E^{(0)}_{\psi_I} = \frac{1}{2} N \Omega, \\
J^{(0)}_{\psi_I} = \sum_{\alpha=1}^{N} z_\alpha \sum_{\beta \neq \alpha} \frac{(\Omega_{\alpha\beta} - 2m_{\alpha\beta})}{z_\alpha - z_\beta} \hat{\psi}^{(0)}_{\psi_I} \\
= \sum_{\alpha<\beta} (\Omega_{\alpha\beta} - 2m_{\alpha\beta}) \hat{\psi}^{(0)}_{\psi_I} \\
= (\Omega - 2 \sum_{\alpha<\beta} m_{\alpha\beta}) \hat{\psi}^{(0)}_{\psi_I}, \\
\sum_{\alpha<\beta} \Omega_{\alpha\beta} \hat{\psi}^{(0)}_{\psi_I} = \Omega \hat{\psi}^{(0)}_{\psi_I},
\]

and

\[
\hat{H}^{(0)}_{\psi_{II}} = \sum_{\alpha=1}^{N} \frac{eB}{m} z_\alpha \sum_{\beta \neq \alpha} \frac{(2n_{\alpha\beta}I - \Omega_{\alpha\beta})}{z_\alpha - z_\beta} \hat{\psi}^{(0)}_{\psi_{II}} \\
= \sum_{\alpha<\beta} \frac{eB}{m} (2n_{\alpha\beta}I - \Omega_{\alpha\beta}) \hat{\psi}^{(0)}_{\psi_{II}} \\
= \frac{eB}{m} [2 \sum_{\alpha<\beta} n_{\alpha\beta} - \Omega] \hat{\psi}^{(0)}_{\psi_{II}}, \\
J^{(0)}_{\psi_{II}} = -\sum_{\alpha=1}^{N} z_\alpha \sum_{\beta \neq \alpha} \frac{(2n_{\alpha\beta}I - \Omega_{\alpha\beta})}{z_\alpha - z_\beta} \hat{\psi}^{(0)}_{\psi_{II}} \\
= \sum_{\alpha<\beta} (-2n_{\alpha\beta} + \Omega_{\alpha\beta}) \hat{\psi}^{(0)}_{\psi_{II}} \\
= [-2 \sum_{\alpha<\beta} n_{\alpha\beta} + \Omega] \hat{\psi}^{(0)}_{\psi_{II}}, (24)
\]

\[
\sum_{\alpha<\beta} \Omega_{\alpha\beta} \hat{\psi}^{(0)}_{\psi_{II}} = \Omega \hat{\psi}^{(0)}_{\psi_{II}},
\]

where we have used the relation\[^{15}\]

\[
\sum_{\alpha<\beta} \Omega_{\alpha\beta} \Omega_{\gamma \delta} = 0, (25)
\]

and the fact that \(\psi_I\) is an eigenstate of the operator \(\sum_{\alpha<\beta} \Omega_{\alpha\beta}\). Mathematically,
these commutator relations are just consequences of the integrability condition (infinitesimal pure braid relations) satisfied by the connection. Physically, they follow from the fact that $\Omega$ is related to the angular momentum $J$ which is invariant upon monodromy.

Now that we have found the analogous base states, we will apply the ladder operators to them to generate new states. As before, the new states have to respect the statistics. (The NACS particles in the same irreducible representation are regarded as indistinguishable.) Thus, we may only use symmetric combinations of step operators. Also, we have to check that the wavefunctions produced are regular at the point of coincidence. There is, however, one crucial difference between the cases of anyons and NACS particles. Even with symmetric step operators, there is no guarantee that the monodromy properties of the wavefunctions are preserved. Any combination of step operators which do not preserve the monodromy properties of the wavefunctions are to be rejected.

First of all, let us consider $C_{0n}$. As before, we get

$$C_{0n} \hat{\psi}_I^{(0)} = \left( \sum_{\alpha=1}^{N} \frac{z_{\alpha}^n}{\bar{z}_{\alpha}^n} \right) \hat{\psi}_I^{(0)}. \quad (26)$$

This shows that $C_{0n}$ can be safely applied to $\hat{\psi}_I^{(0)}$ without changing its monodromy properties or producing singularities. Next we consider $C_{1m}$.

$$C_{1m} \hat{\psi}_I^{(0)} = \sum_{\alpha=1}^{N} (z_{\alpha}^m - \bar{z}_{\alpha}^m) \hat{\psi}_I^{(0)}$$

$$= - \sum_{\alpha=1}^{N} z_{\alpha}^m \partial_{\alpha} \hat{\psi}_I^{(0)} + \sum_{\alpha=1}^{N} (z_{\alpha}^m z_{\alpha}^{m-1}) \hat{\psi}_I^{(0)}$$

$$= - \sum_{\alpha<\beta} (\Omega_{\alpha\beta} - 2m_{\alpha\beta} I) \frac{z_{\alpha}^m - z_{\beta}^m}{z_{\alpha}^m - z_{\beta}^m} \hat{\psi}_I^{(0)} + \sum_{\alpha=1}^{N} (z_{\alpha}^m z_{\alpha}^{m-1}) \hat{\psi}_I^{(0)}. \quad (27)$$

For $m = 0$,

$$C_{10} \hat{\psi}_I^{(0)} = \sum_{\alpha=1}^{N} \bar{z}_{\alpha} \hat{\psi}_I^{(0)}. \quad (28)$$
which clearly preserves the monodromy property of $\hat{\psi}_I^{(0)}$. When $m = 1$, we have

$$C_{11}\hat{\psi}_I^{(0)} = -\sum_{\alpha < \beta} (\Omega_{\alpha\beta} - 2m_{\alpha\beta} I)\hat{\psi}_I^{(0)} + \sum_{\alpha = 1}^N (\bar{z}_\alpha z_\alpha - 1)\hat{\psi}_I^{(0)}$$

$$= [-\Omega + 2\sum_{\alpha < \beta} m_{\alpha\beta} + \sum_{\alpha = 1}^N (\bar{z}_\alpha z_\alpha - 1)]\hat{\psi}_I^{(0)}.$$  

(29)

This shows that $C_{11}$ can be safely applied to the base state. We can say more: strings made up of combinations of the operators $C_{0n}$ ($n=1,2,\ldots$), $C_{10}$ and $C_{11}$ act on the base state to generate physical states. On the other hand, the operators $C_{1m}$ with $m > 1$ and $C_{nm}$ with $n > 1$ generally change the monodromy property of base state. There is no obvious way of constructing an admissible combination of operators involving them which would preserve the monodromy property of the base state. We therefore reject them as being unphysical and restrict the admissible set of operators to those generated by $C_{0n}, C_{10}$ and $C_{11}$.

The crucial reason why the argument for $C_{1m}$ (with $m > 1$) and $C_{nm}$ (with $n > 1$) as physical operators for anyons do not carry over to the case of non-Abelian C-S particles is that the various monodromy matrices do not commute. In other words,

$$[\Omega_{\alpha\beta}, \Omega_{\gamma\delta}] \neq 0.$$  

(30)

As in the anyon case, there are again missing states in the spectrum. However, our wavefunctions do cover the entire lowest Landau level as they involve the operators $C_{0n}$ only.

We now consider the construction of closed-form eigenfunctions. In the case of anyons, for $P(z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N) \prod_{\alpha < \beta} (z_\alpha - z_\beta)^{\lambda}$ to be an eigenfunction of $\hat{H}$, it follows that the function $P$ has to satisfy a modified differential equation.

$$\sum_{\alpha = 1}^N \left(-\frac{2}{m} \partial_\alpha + \frac{eB}{m} \bar{z}_\alpha \right) \partial_\alpha P - \frac{2}{m} \sum_{\alpha < \beta} \left(\bar{z}_\alpha - \bar{z}_\beta \right) \partial_\beta P = \left(E - \omega \frac{N}{2}\right) P$$  

(31)

An ansatz has been made to construct closed-form eigenfunctions. All the solutions constructed can be expressed in terms of the Laguerre Polynomials. They are generated by strings of operators $C_{0n}, C_{10}$ and $C_{11}$ only. For NACS particles, we get a
similar equation for $P$, but with $\nu$ replaced by $\Omega_{\alpha\beta} - 2m_{\alpha\beta}$. Nevertheless, since these operators are chosen to preserve the monodromy properties of the states. There is every reason to believe that the construction of closed-form solutions will go through with $\frac{1}{2}N(N-1)\nu$ replaced by $\Omega - 2\sum_{\alpha<\beta}m_{\alpha\beta}$.

Finally, we come to the diagonalization of $\sum_{\alpha<\beta}\Omega_{\alpha\beta}$. Consider the identity

$$\sum_{\alpha}(T_1^\alpha + T_2^\alpha + \ldots + T_N^\alpha)(T_1^\alpha + T_2^\alpha + \ldots + T_N^\alpha) = N \sum_{\alpha=1}^N T_\alpha^2 T_\alpha^2 + 2 \sum_{\alpha<\beta} T_\alpha^2 T_\beta^2. \quad (32)$$

For $SU(2)$, the left-hand side gives the Casimir operator $J(J+1)$ of the “spin” of the composite made up of the $N$ particles, and the first term on the right-hand side gives the sum of the Casimir operators $\sum_{\alpha=1}^N J_\alpha(J_\alpha + 1)$ of the “spins” for the individual particles. (Here we abuse the word “spin” for the internal $SU(2)$ symmetry group. The physical spin (which is a scalar in 2+1 d) of a NACS particle in the $j$ representation is given by $\frac{2}{k}J(J+1)$. Thus, for $SU(2)$, we have

$$\Omega \equiv \sum_{\alpha<\beta}\Omega_{\alpha\beta} = \frac{1}{k}[J(J+1) - \sum_{\alpha=1}^N J_\alpha(J_\alpha + 1)]. \quad (33)$$

We just decompose the composite state into irreducible representations and $\sum_{\alpha<\beta}\Omega_{\alpha\beta}$ would be diagonal in that basis. Actually, we can do better than that. It is easy to check that the operator $T_1^\alpha + T_2^\alpha + \ldots + T_N^\alpha$ commutes with $\hat{H}$, $J$ and $\sum_{\alpha<\beta}\Omega_{\alpha\beta}$. Thus they can be simultaneously diagonalised.

4. Second virial coefficient and the large $k$ limit

In this section, we compute the second virial coefficients for some simple systems of NACS particles. To do so, we need to know all the two-particle states only.

First of all, consider two identical NACS particles in the $j = \frac{1}{2}$ representation of $SU(2)$. From the addition rule for angular momenta, we find that the resulting states consist of a triplet with $\Omega = \frac{1}{2}$ and a singlet with $\Omega = -\frac{1}{2}$. For $N = 2$, $\Omega$ plays the role of the anyon phase, $\nu$. Let us recall the formula derived by Arovas et al.\[21\] for the second virial coefficient of anyons,

$$B(\nu = 2j + \delta, T) = \lambda \frac{\nu}{T}(-\frac{1}{4} + |\delta| - \frac{1}{2}b^2), \quad (34)$$

where $|\delta| < 2$. Note that it has a cusp at Bose values $\nu = 2j$. By taking the average over the four two-body states, the second virial coefficient of the NACS particles is

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given by

\[
B(j = \frac{1}{2}, T) = \lambda_T^2 \left[ -\frac{1}{4} + \frac{3}{4k} - \frac{3}{8k^2} \right].
\]

For two particles with \( j = 1 \), the resulting states have “spins” 2, 1, and 0 (with \( \Omega = \frac{2}{k}, -\frac{2}{k}, \) and \(-\frac{4}{k}\) and degeneracies 5, 3 and 1 respectively). We remark that all these states are bosonic if \( k = 1 \). When \( k = 2 \), the singlet is a bosonic state whereas others are fermionic. For \( k > 4 \), all the states are anyonic with \( |\nu| < 1 \). For \( k > 1 \), we have

\[
B(j = 1, T) = \lambda_T^2 \left[ -\frac{1}{4} + \frac{20}{9k} - \frac{8}{3k^2} \right].
\]

Now we come to the large \( k \) limit. For two particles belonging to a representation \( j \) with \( \lim_{k \to \infty} \frac{j}{k} = a < 1 \), we approximate the sum over all the resulting “spins” \( r \leq 2j \) by an integral. For example, the \( |\delta| \) term is given by

\[
\frac{1}{(2j + 1)^2} \sum_{r=0}^{2j} (2r + 1) \frac{1}{k} \left[ r(r + 1) - 2j(j + 1) \right] \\
\approx \frac{1}{k^{\frac{3}{2}}} \int_{r=0}^{\sqrt{2j}} -r(r^2 - 2j^2) = \frac{j^2}{k},
\]

where in the second line we approximate the sum by an integral and divide it into two parts (which happen to be equal) according to the sign of \( r^2 - 2j^2 \). The \( \delta^2 \) term can be evaluated in a similar manner. Hence, we get

\[
B = \lambda_T^2 \left[ -\frac{1}{4} + \frac{j^2}{k} - \frac{j^4}{3k^2} \right] = \lambda_T^2 \left[ -\frac{1}{4} + a - \frac{a^2}{3} \right].
\]

If \( a \to 0 \), the last term may be discarded and the second virial coefficient of the NACS particle in the \( j \) representation (with physical spin \( \frac{2}{k}j(j + 1) \)) is the same as that of an anyon with half of the physical spin as its statistical parameter.
5. Concluding Remarks

(1) The \( N \)-free-NACS-particle problem can be solved by a similar method. The free Hamiltonian in the “anyon” gauge is given by

\[
H = \sum_{\alpha=1}^{N} -\frac{2}{m}\bar{\partial}_\alpha \partial_\alpha. \tag{39}
\]

The subtlety is that our base states become unnormalizable.\(^{[9]}\) Let us define \( r = (\sum_{\alpha=1}^{N} |z_\alpha|^2)^{\frac{1}{2}} \) and consider

\[
HM(r)\psi_I^{(0)} = \psi_I^{(0)} \times -\frac{1}{2m}[\partial^2 / \partial r^2 + (1/r)(2N - 1 + 2\Omega - 4 \sum_{\alpha < \beta} m_{\alpha \beta})]M(r). \tag{40}
\]

We have eigenfunctions of the form

\[
M_\mu(r) = r^{-\mu}J_\mu(kr), \tag{41}
\]

(where \( \mu = 2(N - 1 + \Omega - 2 \sum_{\alpha < \beta} m_{\alpha \beta}) \)) with eigenvalues \( \hbar k^2 / 2m \) for \( H \).

(2) Let us now consider the construction of \( C_{lm} \) for multi-species non-Abelian C-S particles (particles in various irreducible representations). In this case, when we construct \( \tilde{C}_{0n} \), we do so for each irreducible representation \( R \) and symmetrize over particles in this irreducible representation only.\(^{[6]}\) Let us call the resulting operator \( \tilde{C}_{0n}^R \). If we construct \( \tilde{C}_{10}^R \) and \( \tilde{C}_{11}^R \) in a similar manner, we find that these operators have to be rejected: They do not preserve the monodromy properties of the base states because \( [\Omega_{\alpha \beta}, \Omega_{\gamma \delta}] \neq 0 \), and we are no longer summing over all the particles. Therefore, for \( \tilde{C}_{10} \) and \( \tilde{C}_{11} \) we do sum over all the particles in the various irreducible representations.

(3) Note that \( \tilde{C}_{01} \) and \( \tilde{C}_{10} \) represent center of mass excitations. The operator \( \tilde{C}_{10} \) was also analyzed by Johnson and Canright,\(^{[9]}\) while \( \tilde{C}_{11} \) is directly related to the Lie group generator of \( SU(1,1) \).\(^{[3]}\)

(4) We remark that the operators of \( \tilde{C}_{lm} \) and \( \tilde{C}_{nm} \ (m, n > 2) \) do preserve the monodromy property of the base state, if \( \psi_I \) in eqns. (20) and (21) is chosen to be a simultaneous eigenstate of all \( \Omega_{\alpha \beta} \). The statistics is “Abelianized” in this case. This situation occurs, for example, for some models of non-Abelian vortices of finite gauge groups such as the quaternion group.
(5) The same ladder operator approach may well apply to non-abelian vortices of finite gauge groups. Unfortunately, we generally do not know how to construct connections which would produce the desirable monodromy in this case.

(6) In a recent paper, Dasnieres de Veigy and Ourvy derived the equation of state of an anyon gas in a strong magnetic field at low temperatures. The idea is that at sufficiently low temperatures, excitations to higher Landau levels can be neglected. Thus one may consider only the lowest Landau states of the anyons, which are covered by the step operators. In fact, apart from the statistical phase factor, the multi-anyon states in the lowest Landau level are tensor-product states of the individual anyon states. By regularizing the grand partition function with a harmonic potential, the equation of state can be obtained. The same decoupling principle should apply to NACS particles. For a fixed base state, modulo the statistical term involving $\Omega_{\alpha\beta}$, the multi-particle wavefunctions in the lowest Landau level are again tensor products of individual particle states. Therefore, in principle, one should be able to derive the equation of state of NACS particles in a strong magnetic field at low temperatures.

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REFERENCES


