Macroscopic Loop Amplitudes 

in Two-Dimensional Dilaton Gravity 

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Abstract 

Macroscopic loop amplitudes are obtained for the dilation gravity in two-dimensions. The dependence on the macroscopic loop length $l$ is completely determined by using the Wheeler-DeWitt equation in the mini-superspace approximation. The dependence on the cosmological constant $\Lambda$ is also determined by using the scaling argument in addition.
1 Introduction

Initiated by the work on the black hole evaporation [1], many people have recently devoted efforts to study two-dimensional gravity interacting with a dilaton field and matter fields [2]–[10]. If one replaces the gravitational coupling constant $G_N$ by a spacetime dependent field, we obtain a new gravitational theory. By employing the exponential parametrization for the spacetime dependent field, we have an action for the dilaton field $\phi$ and the metric $\bar{g}_{\mu\nu}$

$$S_{\text{Einstein}} = \frac{1}{16\pi G_N} \int d^D x \sqrt{-\bar{g}} \left[ \bar{R} - \Lambda \right]$$

$$\rightarrow S_{\text{dilaton}} = \frac{1}{2\pi} \int d^D x \sqrt{-\bar{g}} \left[ e^{-2\phi} \bar{R} - 2\Lambda \right], \quad (1.1)$$

where we have chosen not to multiply the cosmological constant $\Lambda$ by a function of the dilaton field $\phi$ and have adjusted the normalization of $\Lambda$ for convenience. A function of scalar fields multiplying the Einstein action can be absorbed into the trace part of the metric $\bar{g}_{\mu\nu}$ by a local Weyl transformation if the spacetime dimension $D$ is different from two. In two spacetime dimensions, the dilaton field is of special importance for one more reason: the Einstein action without the dilaton field is a topological invariant which is dynamically empty. In order to clarify the significance of the dilaton field in two dimensions more clearly, we make a local Weyl transformation $g_{\mu\nu} = e^{2\phi} \bar{g}_{\mu\nu}$

$$S_{\text{dilaton}} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \ e^{-2\phi} \left[ R + 4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2\Lambda \right]. \quad (1.2)$$

This form of the dilaton gravity system is suggested by string theory and has been found to possess the black hole solution [1].

Since the back reaction of matter quantum effects can be described by a quadratic effective action, the dilaton gravity in two-dimensions is especially suited for studying the back reaction of the Hawking radiation and its consequences on the black hole evaporation. The results of many investigations suggest that the semi-classical approximations are inadequate to study the black hole evaporation [2]–[8]. Therefore it is important to consider the dilaton gravity in a fully quantum-mechanical way. In quantizing two-dimensional gravity, we can use methods of conformal field theory developed for string theory [12]–[18]. We have already obtained the correlation functions of local operators as a result of the conformal field theory treatment.
of dilaton gravity [10]. From the viewpoint of two-dimensional gravity, these correlation functions correspond to manifolds with several punctures. If we extend the punctures to macroscopic loops, we obtain wave functions of the universe, and/or topology changing amplitudes. In the case of the ordinary two-dimensional gravity without the dilaton field, matrix models provide precise methods of calculation [19], [20]. In the continuum approach, the Wheeler-DeWitt equation gives macroscopic loop amplitudes. It has been shown that the solutions of the Wheeler-DeWitt equation in the mini-superspace approximation agree completely with the results of the matrix model [21]. We do not have a discretized approach like the matrix model in the case of the dilaton gravity. However, we can formulate the Wheeler-DeWitt equation in the continuum approach for the dilaton gravity [22].

The purpose of this paper is to study the macroscopic loop amplitudes in the case of the dilaton gravity. We will solve the Wheeler-DeWitt equation in the mini-superspace approximation in the continuum approach. We will also apply the scaling argument following the conformal field theory approach [12] in order to examine the small length limit of the macroscopic loop amplitude. We find that the results of the Wheeler-DeWitt equation in the mini-superspace approximation are in agreement with the scaling argument and that the macroscopic loop can be represented in the small length limit by a local operator. Moreover, the scaling argument serves to fix the power of the cosmological constant in the normalization of the amplitudes which is undetermined by the Wheeler-DeWitt equation. We also use the correlation functions of local operators to determine the dependence of the macroscopic loop amplitudes on the momenta of other local tachyon operators.

There has been an interesting work on the Wheeler-DeWitt equation to obtain information on the quantum behavior of the black hole [22]. On the other hand, we determine the dependence of the macroscopic loop amplitudes on the length \( l \) and the cosmological constant \( \Lambda \). Therefore our results have little overlap with theirs. While we are writing up this work, we have received three preprints on the quantum theory of dilaton gravity. Two of them formulate the Wheeler-DeWitt equation and obtain its solutions without using approximations [23]. Although we use the mini-superspace approximation, we have obtained explicitly the dependence on the length \( l \) and the cosmological constant \( \Lambda \) as well as on momenta of other local operators. The other preprint [24] performed a careful canonical quantization of the dilaton gravity. Although they considered only the case of a particular number of matter fields, \( N = 24 \), they emphasized the rigor of the analysis. Their treatment seems to
be useful especially if one wants to go beyond the mini-superspace approximation.

In Sect.2, we will describe the model of the dilaton gravity. In Sect.3, we will present and solve the Wheeler-DeWitt equation in the mini-superspace approximation. In Sect.4, we will present the scaling argument. A discussion is given in Sect.5.

## 2 Dilaton gravity as a conformal field theory

We first consider a classical theory of dilaton gravity with $N$ massless matter fields $f^j$. We shall use the Lorentzian signature spacetime instead of the Euclidean signature in our previous paper [10]. In addition to the action for the dilaton gravity (1.2), we obtain the classical action

\[
S_{\text{classical}} = S_{\text{dilaton}} + S_{\text{matter}},
\]

\[
S_{\text{matter}} = -\frac{1}{8\pi} \int d^2x \sqrt{-\hat{g}} \sum_{j=1}^{N} g^{\mu\nu} \partial_{\mu} f^j \partial_{\nu} f^j. \tag{2.1}
\]

In order to define a quantum theory for this system, we shall take the conformal field theory approach [8]. The path integral measure for the matter field is defined in terms of the metric $g_{\mu\nu}$. We use the conformal gauge $g_{\mu\nu} = e^{2\rho} \hat{g}_{\mu\nu}$ with the Liouville field $\rho$ and the reference metric $\hat{g}_{\mu\nu}$. By changing the path integral measure to the translation invariant measure using the reference metric $\hat{g}_{\mu\nu}$, we obtain the conformal anomaly from the matter fields represented by the Liouville action. As for the Liouville field, an ansatz using the conformal field theory has been successful [12],[13]. In the case of the dilaton gravity, however, it is not obvious which metric one should use to define the path integral measure for various fields. By generalizing the proposals in Refs. [7] and [8], we consider to use the metric $e^{\alpha \phi} g_{\mu\nu}$ with various $\alpha$ for the quantization of various fields. If we denote the amount of the anomaly for the $j$-th field as $\gamma_j$, we find the following kinetic term for the Liouville and the dilaton field with the parameters [10],[11]

\[
a = \sum \gamma_j \alpha_j \quad \text{and} \quad b = \sum \gamma_j \alpha_j^2
\]

\[
S_{\text{kin}} = \frac{1}{2\pi} \int d^2x \sqrt{-\hat{g}} \left[ e^{-2\phi} \left( 4\hat{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - 4\hat{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \rho + \hat{R} \right) 
+ \kappa (\hat{g}^{\mu\nu} \partial_{\mu} \rho \partial_{\nu} \rho + \hat{R} \rho) - a \left( 2\hat{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \rho + \hat{R} \phi \right) 
- bh^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{4} \sum_{j=1}^{N} h^{\mu\nu} \partial_{\mu} f^j \partial_{\nu} f^j \right]. \tag{2.2}
\]
We leave the total amount of the anomaly $\kappa$ as a parameter which is to be determined by the conformal invariance. With the translation invariant path integral measure using the reference metric $\hat{g}_{\mu\nu}$, the quantum effective action consists of the kinetic term and the cosmological term $S_{\text{cosm}}$ which will be determined later

$$S_{\text{quantum}} = S_{\text{kin}} + S_{\text{cosm}}.$$  \hspace{1cm} (2.3)

We can perform a non-linear field transformation in order to reduce this kinetic term (2.2) to a free field action [10]. For $\kappa \neq 0$ this change of variables is

$$\omega = e^{-\phi}, \quad \chi = -\frac{\rho}{2} - \frac{\omega^2 + a \ln \omega}{2\kappa} = \frac{\hat{\chi}}{4\sqrt{|\kappa|}},$$

$$d\Omega = \frac{1}{\kappa}d\omega \sqrt{\omega^2 - \kappa + a + \frac{a^2 + b\kappa}{4\omega^2}} = d\hat{\Omega} \frac{\sqrt{\omega^2 - \kappa}}{4\sqrt{|\kappa|}}.$$ \hspace{1cm} (2.4)

The Liouville field $\rho$ is contained only in the field $\hat{\chi}$. Therefore field $\hat{\chi}$ exhibits the same transformation property as the Liouville field under conformal transformations. We shall call $\hat{\chi}$ a modified Liouville field. After this transformation the kinetic part of the action becomes

$$S_{\text{kin}} = \frac{1}{8\pi} \int d^2 x \sqrt{-\hat{g}} \left( \epsilon \hat{g}^{\mu\nu} \partial_\mu \hat{\chi} \partial_\nu \hat{\chi} - \epsilon 2\sqrt{|\kappa|} \hat{R} \hat{\chi} - \epsilon \hat{g}^{\mu\nu} \partial_\mu \hat{\Omega} \partial_\nu \hat{\Omega} - \sum_{j=1}^{N} \hat{g}^{\mu\nu} \partial_\mu f^j \partial_\nu f^j \right),$$ \hspace{1cm} (2.5)

where $\epsilon = \frac{\kappa}{|\kappa|}$. It must be noted that the modified Liouville field $\hat{\chi}$ has negative metric in the case of $\kappa > 0$, whereas $\hat{\Omega}$ representing the dilatonic degree of freedom has negative metric in the case of $\kappa < 0$.

The case of $\kappa = 0$ needs a separate treatment. We can find another change of variables in order to transform the kinetic term to a free field action

$$\chi^\pm = Q \left(-\rho - \frac{b}{2a} \phi - \frac{2a + b}{4a} \ln \left|e^{-2\phi} + \frac{a}{2}\right|\right) \pm \frac{2}{Q} \left(e^{-2\phi} - aQ\right).$$ \hspace{1cm} (2.6)

After this change of variables, we obtain the free field action for the case of $\kappa = 0$

$$S_{\text{kin}} = \frac{1}{8\pi} \int d^2 x \sqrt{-\hat{g}} \left(-\hat{g}^{\mu\nu} \partial_\mu \chi^+ \partial_\nu \chi^+ + Q \hat{R} \chi^+ + \hat{g}^{\mu\nu} \partial_\mu \chi^- \partial_\nu \chi^- - Q \hat{R} \chi^- - \sum_{j=1}^{N} \hat{g}^{\mu\nu} \partial_\mu f^j \partial_\nu f^j \right).$$ \hspace{1cm} (2.7)
As an $O(1, 1)$ transformation in $\chi^+, \chi^-$ field space can change the parameter $Q$, the value of $Q$ itself is not essential.

We define the path integral measure for the fields by the translation invariant measure with the reference metric $\hat{g}_{\mu \nu}$. The physical result must be independent of the choice of the reference metric $\hat{g}_{\mu \nu}$. This requirement determines the parameter $\kappa$ to be [8], [10]

$$\kappa = \frac{N - 24}{12}. \quad (2.8)$$

The physical states of the theory have been obtained from the BRST cohomology [10]. One of them is a tachyon operator with the momentum $p$. In the case of $N \neq 24$ ($\kappa \neq 0$), the tachyon operator is given by

$$O_p = \int d^2 x \sqrt{-\hat{g}} e^{\beta \chi + \beta_0 \hat{\Omega} + i \sum_{j=1}^N p_j f_j},$$

$$\frac{1}{2} \beta_\chi \left( \epsilon \beta_\chi - 2 \sqrt{|\kappa|} \right) - \frac{1}{2} \epsilon \beta_\Omega^2 + \frac{1}{2} \sum_j p_j^2 = 1. \quad (2.9)$$

In the case of $N = 24$ ($\kappa = 0$), the tachyon operator is given by

$$O_p = \int d^2 x \sqrt{-\hat{g}} e^{\beta_+ \chi^+ + \beta_- \chi^- + i \sum_{j=1}^N p_j f_j},$$

$$-\frac{1}{2} \beta_+ (\beta_+ + Q) + \frac{1}{2} \beta_- (\beta_- - Q) + \frac{1}{2} \sum_j p_j^2 = 1. \quad (2.10)$$

In the case of $N \neq 0$, there are also oscillator excitation states as in the usual string theory. Moreover, there are also other special states corresponding to the nontrivial BRST cohomology similar to the discrete states in the case of the Liouville gravity [10].

Next we consider the cosmological term in (2.3). The cosmological term can be uniquely determined by requiring that it should belong to the BRST cohomology classes and that it must coincide with the classical one in the limit of weak gravitational coupling constant (i.e. $e^{\phi} \to 0$)

$$S_{\text{cosm}} = \begin{cases} -\frac{\Lambda}{16\pi} \int d^2 x \sqrt{-g} e^{\frac{\sqrt{|\kappa|}}{2}} (-\hat{\chi} + \hat{\Omega}) & \text{for } \kappa \neq 0, \\ -\frac{\Lambda}{16\pi} \int d^2 x \sqrt{-g} e^{-\frac{\sqrt{|\kappa|}}{2}} (\chi^+ + \chi^-) & \text{for } \kappa = 0. \end{cases} \quad (2.11)$$
3 The Wheeler-DeWitt equation

The method of conformal field theory and the analytic continuation have been used successfully to compute the correlation functions of local operators in the case of the Liouville gravity theory \[16\]–\[18\]. However, the precise evaluation of macroscopic loop amplitudes can be done only by means of matrix models. Similarly, we cannot evaluate the path integral of the macroscopic loop amplitudes directly in the continuum approach for dilaton gravity. In the case of the Liouville gravity, the Wheeler-DeWitt equation in the mini-superspace approximation turned out to provide solutions which agree with the matrix model results correctly \[21\]. In this section we shall use the Wheeler-DeWitt equation in the mini-superspace approximation to obtain the macroscopic loop amplitudes.

We shall first consider the case of \(N \neq 24 (\kappa \neq 0)\). To formulate the Wheeler-DeWitt equations, we first construct the energy-momentum tensor \(T_{\mu\nu}\) from the action (2.5)

\[ T_{\mu\nu} \equiv -4\pi \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S}{\delta \hat{g}^{\mu\nu}} \bigg|_{\hat{g}_{\mu\nu} = \eta_{\mu\nu}} \]

\[ = -\frac{\epsilon}{2} \partial_{\mu} \hat{\chi} \partial_{\nu} \hat{\chi} + \frac{\epsilon}{4} \eta_{\mu\nu} \partial_{\lambda} \hat{\chi} \partial^{\lambda} \hat{\chi} + \epsilon \sqrt{\left|\kappa\right|} (\eta_{\mu\nu} \partial_{\lambda} \hat{\chi} \partial^{\lambda} \hat{\chi} - \partial_{\mu} \partial_{\nu} \hat{\chi}) \]

\[ + \frac{\epsilon}{2} \partial_{\mu} \hat{\Omega} \partial_{\nu} \hat{\Omega} - \frac{\epsilon}{4} \eta_{\mu\nu} \partial_{\lambda} \hat{\Omega} \partial^{\lambda} \hat{\Omega} - 2\Lambda e^{\frac{1}{\sqrt{\left|\kappa\right|}}(-\hat{\chi} + \hat{\Omega})} \eta_{\mu\nu} \]

\[ + \sum_{j=1}^{N} \left[ \frac{1}{2} \partial_{\mu} f^j \partial_{\nu} f^j - \frac{1}{4} \eta_{\mu\nu} \partial_{\lambda} f^j \partial^{\lambda} f^j \right], \tag{3.1} \]

where we choose the flat metric \(\eta_{\mu\nu} = \text{diag}(-1,1)\) as the reference metric \(\hat{g}_{\mu\nu}\). In light-cone coordinates \(x^\pm = x^0 \pm x^1\), the trace part becomes

\[ T_{++} = T_{+-} = \epsilon \sqrt{\left|\kappa\right|} \partial_{+} \partial_{-} \hat{\chi} + \Lambda e^{\frac{1}{\sqrt{\left|\kappa\right|}}(-\hat{\chi} + \hat{\Omega})}. \tag{3.2} \]

In accordance with the conformal invariance, the equation of motion for the modified Liouville field \(\hat{\chi}\)

\[ \epsilon \partial_{+} \partial_{-} \hat{\chi} + \frac{\Lambda}{\sqrt{\left|\kappa\right|}} e^{\frac{1}{\sqrt{\left|\kappa\right|}}(-\hat{\chi} + \hat{\Omega})} = 0 \tag{3.3} \]

guarantees that the trace part of the energy-momentum tensor vanishes \(T_{++} = 0\). It is noteworthy that the equation of motion (3.3) given by the quantum effective
action (2.5) including the conformal anomaly effect is precisely the condition for the vanishing of the energy-momentum tensor.

We define the canonically conjugate momenta \( \Pi_\chi = -\frac{\epsilon}{4\pi} \partial_0 \hat{\chi}, \Pi_\Omega = \frac{\epsilon}{4\pi} \partial_0 \hat{\Omega}, \) and \( \Pi_j = \frac{1}{4\pi} \partial_0 f^j \) for \( \hat{\chi}, \hat{\Omega}, \) and \( f^j \) respectively. We obtain \( T_{--} \) in terms of these canonical variables

\[
T_{--} = -\frac{\epsilon}{8}(4\pi\epsilon \Pi_\chi + \partial_1 \hat{\chi})^2 - \frac{\sqrt{|\kappa|}}{2}\epsilon \partial_1(4\pi\epsilon \Pi_\chi + \partial_1 \hat{\chi})
+ \frac{\epsilon}{8}(4\pi\epsilon \Pi_\Omega - \partial_1 \hat{\Omega})^2 + \frac{1}{8} \sum_{j=1}^{N}(4\pi \Pi_j - \partial_1 f^j)^2
+ \Lambda e^{\sqrt{|\kappa|}(-\hat{\chi}+\hat{\Omega})} - \frac{\kappa}{2}.
\] (3.4)

We have added the c-number term \(-\frac{\kappa}{2}\) in the energy-momentum tensor \( T_{--} \) because of the following consideration [14], [25], [26]. Since we are interested in the manifold with a boundary corresponding to the macroscopic loop, we use a coordinate system appropriate to the geometry of cylinder. Therefore we have to supplement the energy-momentum tensor by adding the Schwarzian derivative term arising from the coordinate transformation from a disk to a cylinder. In our case, the Schwarzian derivative turns out to give \(-\kappa/2\). We can expand the canonical fields in terms of oscillator modes. For instance, \( \hat{\chi} \) and its conjugate momentum \( \Pi_\chi \) can be expanded as [25]

\[
\hat{\chi} = \hat{\chi}_0 + i \sum_{n \neq 0} \frac{1}{n}(\alpha_n^\chi(x^0) e^{inx^1} + \tilde{\alpha}_n^\chi(x^0) e^{-inx^1}),
\]

\[
\Pi_\chi = -\frac{\epsilon}{4\pi} \partial_0 \hat{\chi} = -\frac{\epsilon}{4\pi} \left[ 2p_0^\chi + \sum_{n \neq 0} (\alpha_n^\chi(x^0) e^{inx^1} + \tilde{\alpha}_n^\chi(x^0) e^{-inx^1}) \right].
\] (3.5)

It is convenient to define \( p_0^\chi = \alpha_0^\chi = \tilde{\alpha}_0^\chi \). The mode expansions for other fields can be done similarly. The zero-th moment \( L_0 \) of the energy-momentum tensor \( T_{--} \) is given by

\[
L_0 = \epsilon \left[ -\frac{1}{2}(p_0^\chi)^2 - \sum_{m=1}^{\infty} \alpha_m^\chi \alpha_m^\chi + \frac{1}{2}(p_0^\Omega)^2 + \sum_{m=1}^{\infty} \alpha_m^\Omega \alpha_m^\Omega \right] - \frac{\kappa}{2}
+ \frac{\Lambda}{2\pi} \int_0^{2\pi} dx^1 e^{\sqrt{|\kappa|}(-\hat{\chi}+\hat{\Omega})} + \frac{1}{2} \sum_{j=1}^{N}(p_j^j)^2 + \sum_{m=1}^{\infty} \alpha_m^j \alpha_m^j.
\] (3.6)

Since the metric is integrated in quantum gravity, the energy-momentum tensor has to be treated as a constraint. The wave function \( |\Psi\rangle \) representing a macroscopic
loop amplitude should satisfy the following constraint equation which is called the Wheeler-DeWitt equation

\[(L_0 - 1) |\Psi\rangle = 0. \quad (3.7)\]

Here we have assumed that the ghost part is not excited at all. Consequently the only remnant of the ghost part is the constant term $-1$ in the Wheeler-DeWitt equation [26].

In the mini-superspace approximation, we ignore the dependence of the fields on the spatial coordinate $x^1$. Namely we discard all the oscillator modes of these fields. The zero-th moment of the energy-momentum tensor becomes in the mini-superspace approximation

\[
L_0 = \frac{\epsilon}{2}(p_0^\chi)^2 + \frac{\epsilon}{2}(p_0^\Omega)^2 + \frac{1}{2} \sum_{j=1}^{N} (p_j^\chi)^2 - \frac{\kappa}{2} + \Lambda e^{\sqrt{|\kappa|}(-\hat{\chi}_0 + \hat{\Omega}_0)}
\]

\[
= \epsilon \left[ \frac{1}{2} \frac{\partial^2}{\partial \hat{\chi}_0^2} - \frac{1}{2} \frac{\partial^2}{\partial \hat{\Omega}_0^2} \right] - \frac{\kappa}{2} - \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial f_j^2} + \Lambda e^{\sqrt{|\kappa|}(-\hat{\chi}_0 + \hat{\Omega}_0)} . \quad (3.8)
\]

Since the energy-momentum tensor is the sum of contributions from the matter $f^j$ and the fields $\chi$ and $\Omega$, we can consider the wave function in (3.7) as a tensor product of matter part and the rest

\[
|\Psi\rangle = |\Psi\rangle_{\hat{\chi}, \hat{\Omega}} \bigotimes |\Psi\rangle_{\text{matter}} . \quad (3.9)
\]

If we define the eigenvalue of $L_0$ of the matter part as $\Delta_0$

\[-\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial f_j^2} |\Psi\rangle_{\text{matter}} = \frac{1}{2} \sum_{j=1}^{N} (p_j^\chi)^2 |\Psi\rangle_{\text{matter}} \equiv \Delta_0 |\Psi\rangle_{\text{matter}} , \quad (3.10)\]

we can rewrite the Wheeler-DeWitt equation into the following form

\[
\left[ \frac{\epsilon}{2} \left( \frac{\partial^2}{\partial \hat{\chi}_0^2} - \frac{\partial^2}{\partial \hat{\Omega}_0^2} \right) - \left( 1 + \frac{\kappa}{2} - \Delta_0 \right) + \Lambda e^{\sqrt{|\kappa|}(-\hat{\chi}_0 + \hat{\Omega}_0)} \right] \Psi(\hat{\chi}_0, \hat{\Omega}_0) = 0. \quad (3.11)
\]

We now solve the Wheeler-DeWitt equation (3.11). It is convenient to change variables

\[
X_0 = \frac{1}{\sqrt{2}}(\hat{\Omega}_0 - \hat{\chi}_0), \quad Y_0 = \frac{1}{\sqrt{2}}(\hat{\Omega}_0 + \hat{\chi}_0) . \quad (3.12)
\]
Depending on the choice of the physical metric, the definition of the length $l$ of the loop may not be unique. By regarding the cosmological constant term as the square root of the physical metric, we shall choose the following definition of $l$ in solving the Wheeler-DeWitt equation,

$$ l = \int_0^{2\pi} \frac{dx^1}{2\pi} \sqrt{-\hat{g}} e^{\frac{1}{\sqrt{|\kappa|}(\frac{1}{2} - \Omega_0)}} = e^{\frac{1}{\sqrt{2|\kappa|}}} X_0, \quad (3.13) $$

where we denote the spatial coordinate along the boundary as $x^1$. We find the final form of the Wheeler-DeWitt equation for $\kappa \neq 0$ in terms of the length $l$ and the zero mode of the $Y$ field

$$ \left[ \frac{\partial}{\partial l} \frac{\partial}{\partial Y_0} \right] + \epsilon \sqrt{2|\kappa|} \left( \frac{\kappa}{2} + 1 - \Delta_0 - \Lambda l^2 \right) \Psi(l, Y_0) = 0. \quad (3.14) $$

The general solution of (3.14) can be given by a Laplace transformed form in terms of the momentum $\hat{\beta}_Y$ for $Y$

$$ \Psi(l, Y_0) = \int d\hat{\beta}_Y e^{-\hat{\beta}_Y Y_0} \tilde{\Psi}(l, \hat{\beta}_Y), $$

$$ \tilde{\Psi}(l, \hat{\beta}_Y) = C(\hat{\beta}_Y) l^{\frac{1}{\hat{\beta}_Y} \sqrt{2|\kappa|} (1 + \frac{\kappa}{2} - \Delta_0)} e^{-\frac{\epsilon \sqrt{2|\kappa|}}{2\hat{\beta}_Y \Lambda} l^2}. \quad (3.15) $$

We have assumed that the momentum $\hat{\beta}_Y \neq 0$. If $\hat{\beta}_Y$ vanishes, we can have an exceptional solution proportional to the delta function for the length $\delta(l^2 - \frac{1}{\Lambda}(1 + \frac{\kappa}{2} - \Delta_0))$. If we think of this exceptional solution not just an artifact, we have a difficulty in applying the scaling argument in the next section. We shall discuss this in Sect.5 briefly. A peculiar feature of the solution of the Wheeler-DeWitt equation for the dilaton gravity is the appearance of the $Y$ momentum in the denominator of the exponent. This is due to the different structure of the equation compared to the Liouville gravity case [21].

Next we discuss the case of $\kappa = 0$. We can obtain the energy-momentum tensor from the action (2.7). The equations of motion for the $\chi_{\pm}$ fields guarantee that the trace part of the energy-momentum tensor $T_{++}$ vanishes. The energy-momentum tensor $T_{--}$ is given in light-cone coordinates

$$ T_{--} = \frac{1}{8} (4\pi \Pi_+ - \partial_1 \chi^+)^2 - \frac{Q}{4} \partial_1 (4\pi \Pi_+ - \partial_1 \chi^+) $$

$$ - \frac{1}{8} (4\pi \Pi_- + \partial_1 \chi^-)^2 - \frac{Q}{4} \partial_1 (4\pi \Pi_- + \partial_1 \chi^-) $$
\[ + \frac{1}{8} \sum_{j=1}^{N} (4 \pi \Pi_j + \partial_i f^j)^2 + \Lambda \, e^{-\frac{1}{Q}(\chi^+ + \chi^-)}, \quad (3.16) \]

where \( \Pi_+ = \frac{1}{4 \pi} \partial_0 \chi^+ \), \( \Pi_- = -\frac{1}{4 \pi} \partial_0 \chi^- \) and \( \Pi_j = \frac{1}{4 \pi} \partial_0 f^j \) are canonically conjugate momenta for \( \chi^+, \chi^- \) and \( f^j \) respectively. Using mode expansions of the canonical fields and the mini-superspace approximation, we find the zero-th moment of the energy-momentum tensor \( T_{\ldots} \)

\[ L_0 = \frac{1}{2} (p_0^+)^2 - \frac{1}{2} (p_0^-)^2 + \frac{1}{2} \sum_{j=1}^{24} (p_j^0)^2 + \Lambda \, e^{-\frac{1}{Q}(\chi_0^+ + \chi_0^-)} \]
\[ = -\frac{1}{2} \left( \frac{\partial^2}{\partial \chi_0^+} - \frac{\partial^2}{\partial \chi_0^-} \right) - \frac{1}{2} \sum_{j=1}^{24} \frac{\partial^2}{\partial f_j^0} + \Lambda \, e^{-\frac{1}{Q}(\chi_0^+ + \chi_0^-)} . \quad (3.17) \]

Here the notations for the zero-modes of the canonical fields are similar to the case of \( \kappa \neq 0 \). By defining the length \( l \) of the macroscopic loop as

\[ l = e^{-\frac{1}{\sqrt{2}\Lambda} \Xi_0^+}, \quad \Xi_0^\pm = \frac{1}{\sqrt{2}} (\chi_0^- \pm \chi_0^+), \quad (3.18) \]

we obtain the Wheeler-DeWitt equation for \( \kappa = 0 \) in the mini-superspace approximation

\[ \left[ l \frac{\partial}{\partial l} + \sqrt{2} Q \left( 1 - \Delta_0 - \Lambda l^2 \right) \right] \Psi(l, \Xi_0^-) = 0 . \quad (3.19) \]

The solution for this case is similar to the \( \kappa \neq 0 \) case (3.15)

\[ \Psi(l, \Xi_0^-) = \int d\tilde{\gamma}^- \, e^{-\gamma^- \Xi_0^-} \tilde{\Psi}(l, \tilde{\gamma}^-) , \]
\[ \tilde{\Psi}(l, \gamma^-) = C(\gamma^-) l^{\frac{\gamma^- \sqrt{2} Q (1-\Delta_0)}{\sqrt{8}}} \, e^{-\frac{1}{\sqrt{8} \gamma^-} \Lambda l^2} . \quad (3.20) \]

We have assumed that the momentum \( \tilde{\gamma}^- \neq 0 \). If \( \tilde{\gamma}^- \) vanishes, we can have an exceptional solution proportional to the delta function for the length \( \delta(l^2 - \frac{1}{\Lambda} (1 - \Delta_0)) \).

### 4 The scaling argument in the dilaton gravity

The Wheeler-Dewitt equation determines the wave function as a function of the length \( l \) of the macroscopic loop. However, we are interested in the dependence not
only on the length $l$ but also on the cosmological constant $\Lambda$. In order to discriminate the dependence on $l$ and $\Lambda$, it is useful to consider the scaling argument \cite{12}, \cite{21}.

In order to use the scaling argument, it is more convenient to employ the Euclidean signature metric in this section. It is necessary to distinguish three cases with respect to the number of matter fields $N$.

4.1 $N \neq 0, 24$ case

We define the field $X$ and $Y$ as linear combinations of the fields $\hat{\chi}$ and $\hat{\Omega}$

$$ X = \frac{1}{\sqrt{2}}(\hat{\Omega} - \hat{\chi}), \quad Y = \frac{1}{\sqrt{2}}(\hat{\Omega} + \hat{\chi}). \quad (4.1) $$

The momenta for $X$ and $Y$ are related to those for $\hat{\chi}$ and $\hat{\Omega}$ as

$$ \beta_X = \frac{1}{\sqrt{2}}(\beta_{\Omega} - \beta_{\chi}), \quad \beta_Y = \frac{1}{\sqrt{2}}(\beta_{\Omega} + \beta_{\chi}). \quad (4.2) $$

The on-shell condition for tachyon operators is given in terms of these variables as

$$ -\epsilon(\beta_X^{(k)} + \frac{e}{2}\sqrt{2|\kappa|})(\beta_Y^{(k)} - \frac{e}{2}\sqrt{2|\kappa|}) = 1 + \frac{\kappa}{2} - \Delta_0^{(k)}, $$

$$ \Delta_0^{(k)} = \frac{1}{2} \sum_{j=1}^{N} (p_j^{(k)})^2, \quad (k = 1, \ldots, n). \quad (4.3) $$

We consider a manifold $\Sigma$ with $h$ handles and a boundary together with the insertion of $n - 1$ tachyon operators. The length of the boundary is specified as $l$ and the zero mode of the $Y$ and $f^j$ fields are denoted as $Y_0$ and $f_0^j$ respectively

$$ l = \int_{0}^{2\pi} \frac{dx}{2\pi} \sqrt{\hat{g}} e^{\frac{1}{4\sqrt{2|\kappa|}}} X. \quad (4.4) $$

With these boundary conditions, the macroscopic amplitude $Z$ is given by

$$ Z[\Lambda, l, Y_0, f_0^j; q_1, \ldots, q_{n-1}] = \int_{l, Y_0, f_0^j} \mathcal{D}X \mathcal{D}Y \prod_{i=1}^{N}(\mathcal{D}f^i) e^{-S[X, Y, f^i; \Lambda]} \prod_{k=1}^{n-1} \int_{\Sigma} d^2 x_k \sqrt{\hat{g}(x_k)} e^{\beta_X^{(k)} X + \beta_Y^{(k)} Y + i \sum_j p_j^{(k)} f^j}, $$
\begin{equation}
S = \frac{1}{8\pi} \int d^2x \sqrt{\hat{g}} \left( 2\epsilon \hat{g}^{\mu\nu} \partial_\mu X \partial_\nu Y + \epsilon \sqrt{2|\kappa|} \hat{R}(Y - X) + \sum_{j=1}^{N} \hat{g}^{\mu\nu} \partial_\mu f^j \partial_\nu f^j \right) \\
+ \frac{1}{4\pi} \epsilon \sqrt{2|\kappa|} \oint_{\partial\Sigma} ds \sqrt{\hat{g}} \hat{g}(Y - X) + \frac{\Lambda}{\pi} \int d^2x \sqrt{\hat{g}} \sqrt{\kappa} e^{\sqrt{2|\kappa|} X},
\end{equation}

where the momenta of the tachyon operators are denoted as \( q^k \equiv ( -i\beta^{(k)}_X, -i\beta^{(k)}_Y, \rho^{(k)}_j ) \), \( (k = 1, \ldots, n - 1 \text{ and } j = 1, \ldots, N) \). The integration over \( X \) and \( Y \) has to be performed with the boundary condition: the length \( l \) and the zero mode \( Y_0 \) are fixed on the boundary. Our macroscopic loop amplitude depends on the momenta of the \( n - 1 \) tachyon operators, zero mode \( Y_0 \) of the \( Y \) field and the length \( l \) of the loop.

Let us first note that the dependence on the zero mode \( Y_0 \) and \( f^j_0 \) of the macroscopic loop amplitude (4.5) is given by

\[ Z[\Lambda, l, Y_0, f^j_0, q_1, \ldots, q_{n-1}] \propto \exp(-\hat{\beta}_Y Y_0 + i \sum_{k=1}^{n-1} \rho^{(k)}_j f^j_0), \]

\[ \hat{\beta}_Y = - \left( \sum_{k=1}^{n-1} \beta^{(k)}_Y - \frac{\epsilon}{2} \sqrt{2|\kappa|} (1 - 2h) \right). \]  

(4.6)

This implies that the macroscopic loop carries the momentum \(- \sum_{k=1}^{n-1} \rho^{(k)}_j \) for the matter fields and the momentum \( \hat{\beta}_Y \) for the \( Y \) field as dictated by the conservation law. Here we assigned the inward convention for the momentum of the loop.

In order to apply the scaling argument, we should make a shift of the zero modes \( X_0 \) and/or \( Y_0 \). Any linear combination of them can be used as long as it contains \( X_0 \), since such a combination does correspond to a change of scale. It is easy to see that any linear combination of \( X_0 \) and \( Y_0 \) will give the same result because of the momentum conservation. Let us change the scale by performing a shift of \( X_0 \)

\[ X \to X' = X + \sqrt{\frac{|\kappa|}{2}} \rho, \quad Y : \text{fixed}, \quad l \to l' = e^{-\hat{\beta}_Y} l. \]  

(4.7)

Applying this zero-mode shift to the path integral (4.5), we find that

\[ Z[\Lambda, l, Y_0, f^j_0, q_1, \ldots, q_{n-1}] \]  

\[ = \exp \left[ \left( \frac{K}{2} (2 - 2h - 1) + \sqrt{\frac{|\kappa|}{2}} \sum_{k=1}^{n-1} \beta^{(k)}_X \rho \right) \right] Z[e^\rho \Lambda, e^{-\frac{\epsilon}{2}} l, Y_0, f^j_0, q_1, \ldots, q_{n-1}] \]  

\[ = \Lambda^{\frac{K}{2}(1-2h)-\sqrt{\frac{|\kappa|}{2}} \sum_{k=1}^{n-1} \beta^{(k)}_X} e^{-\hat{\beta}_Y Y_0} e^i \sum_{j=1}^{n-1} f^j_0 F(\sqrt{\Lambda} l). \]  

(4.8)
The scaling argument gives information on powers of dimensionful quantities such as \( l \) and \( \Lambda \). Since this is precisely the role played by the scaling argument, one should expect that the arbitrary function \( F \) of the dimensionless combination \( \sqrt{\Lambda} l \) cannot be determined.

Additional important information can be obtained by examining the limit of small length \( l \to 0 \). In this limit, we should obtain local operators with the momentum dictated by the conservation law. Possible local operators corresponding to physical states have been classified by means of the BRST cohomology [10]. Since we have assumed the mini-superspace approximation for the Wheeler-DeWitt equation, we correspondingly expect local operators without oscillator excitations. Namely a local tachyon operator should appear in the small length limit. A similar assumption has been used in the case of the Liouville gravity to give the correct result in agreement with the matrix model [21]. We expect that the same assumption should also be valid for the \( N = 0 \) case, since there is no physical local operator other than the tachyon for generic values of momenta. For \( N \neq 0 \), however, we should consider also other operators in the small length limit, if we abandon the mini-superspace approximation. In accordance with the mini-superspace approximation, we assume here that the macroscopic loop in the small length limit can be replaced by a local tachyon operator with the momentum

\[
q_n \equiv (-i\beta_X^{(n)}, -i\beta_Y^{(n)}, p_j^{(n)}), \quad (j = 1, \ldots, N).
\]

We leave the power \( k \) of the length \( l \) in the coefficient to be determined by the scaling argument

\[
Z[\Lambda, l, Y_0, f_0; q_1, \ldots, q_{n-1}] \overset{l \to 0}{\sim} l^k e^{-\beta_Y Y_0} e^{i\sum_{k=1}^{n-1} p_j^{(k)} f_0} \langle O_{q_1} \cdots O_{q_n} \rangle (\Lambda). \tag{4.9}
\]

The \( n \)-point function \( \langle O_{q_1} \cdots O_{q_n} \rangle \) denotes the correlation function of local tachyon operators after the momentum conservation delta functions are factored out

\[
\langle O_{q_1} \cdots O_{q_n} \rangle (\Lambda)
\]

\[
= \int D X D\tilde{Y} \prod_{i=1}^{N} (D\tilde{f}_i) e^{-\tilde{S}[X, \tilde{Y}, \tilde{f}; \Lambda]} \prod_{k=1}^{n} \int_{\Sigma} d^2 x_k \sqrt{\tilde{g}(x_k)} e^{\beta_X^k X + \beta_Y^k \tilde{Y} + i\sum_j p_j^{(k)} \tilde{f}_j},
\]

\[
\tilde{S} = \frac{1}{8\pi} \int_{\Sigma} d^2 x \sqrt{\tilde{g}} \left( 2\epsilon \tilde{g}^{\mu\nu} \partial_\mu X \partial_\nu \tilde{Y} + \epsilon \sqrt{2|\kappa|} \hat{R}(\tilde{Y} - X) + \sum_{j=1}^{N} \tilde{g}^{\mu\nu} \partial_\mu \tilde{f}_j \partial_\nu \tilde{f}_j \right)
\]

\[
+ \frac{\Lambda}{\pi} \int_{\Sigma} d^2 x \sqrt{\tilde{g}} e^{\sqrt{2|\kappa|} X}. \tag{4.10}
\]

We have implicitly used the coordinate system appropriate for the cylinder geometry to describe the macroscopic amplitude near the boundary. Since the Liouville field
has an anomalous transformation property, we need to take into account the relation between the momentum $\beta_Y$ for the disk geometry and the momentum $\beta_{Y'}^{cyl}$ for the cylinder geometry [26]

\[ \beta_{Y'}^{cyl} = \beta_Y - \frac{\epsilon}{2} \sqrt{2|\kappa|}. \]  

The momentum $\hat{\beta}_Y$ obtained in Eq.(4.6) corresponds to the momentum on the cylinder. The momentum $q_n = (-i\beta_X^{(n)}, -i\beta_Y^{(n)}, p_j^{(n)})$ of the local tachyon operator corresponding to the shrunken loop is then given by the conservation law as shown in Eq.(4.6)

\[
p_j^{(n)} = -\sum_{k=1}^{n-1} p_j^{(k)}, \quad \beta_Y^{(n)} = \hat{\beta}_Y^{(n)} + \frac{\epsilon}{2} \sqrt{2|\kappa|} = -\left(\sum_{k=1}^{n-1} \beta_Y^{(k)} - \frac{\epsilon}{2} \sqrt{2|\kappa|}(2 - 2h)\right). \]  

The on-shell condition for the tachyon $O_{q_n}$ determines the momentum $\beta_X^{(n)}$

\[-\epsilon (\beta_X^{(n)} + \frac{\epsilon}{2} \sqrt{2|\kappa|})(\beta_Y^{(n)} - \frac{\epsilon}{2} \sqrt{2|\kappa|}) = 1 + \frac{\kappa}{2} - \Delta_0^{(n)}, \]

\[
\Delta_0^{(n)} = \frac{1}{2} \sum_{j=1}^{N} (p_j^{(n)})^2.
\]  

By repeating the same scaling argument for the $n$-point function of the local tachyon operators (4.10), we find the following $\Lambda$ dependence

\[
\langle O_{q_1} \cdots O_{q_n} \rangle (\Lambda) = \Lambda^{-\frac{\kappa}{2}(2 - 2h)} \sqrt{|\kappa| \sum_{k=1}^{n} \beta_X^{(k)} \langle O_{q_1} \cdots O_{q_n} \rangle (\Lambda = 1)}
\]

\[= \Lambda^s \langle O_{q_1} \cdots O_{q_n} \rangle (\Lambda = 1), \]  

where the exponent $s$ is the same as the one that appeared in the calculation of the correlation function by the analytic continuation [10]

\[
s = -\frac{\kappa}{2} (2 - 2h) - \sqrt{\frac{|\kappa|}{2} \sum_{k=1}^{n} \beta_X^{(k)}}.
\]  

Combining (4.9) and (4.14), we find that the function $F$ in (4.8) at small $l$ is proportional to $(\sqrt{\Lambda} l)^k$ and that the exponent $k$ is given by

\[
k = -\kappa - \sqrt{2|\kappa|} \beta_X^{(n)} = \frac{\epsilon \sqrt{2|\kappa|}(1 + \frac{s}{2} - \Delta_0^{(n)})}{\beta_Y^{(n)} - \frac{\epsilon}{2} \sqrt{2|\kappa|}}.
\]
We see that our scaling argument is consistent with the solution (3.15) of the Wheeler-DeWitt equation in the mini-superspace approximation. Moreover we have determined the dependence of $l, \Lambda$, and $Y_0$ in the macroscopic loop amplitude (4.5) completely.

If we can write the path integral (4.5) by means of a macroscopic loop operator $W(l, \hat{\beta}_Y)$ similarly to the matrix model, we can interpret our result by a small length expansion of the macroscopic operator

$$W(l, \hat{\beta}_Y) \sim C l^k \mathcal{O}_{q_n} + \cdots .$$

where $\mathcal{O}_{q_n}$ is the local tachyon operator with the appropriate momentum.

### 4.2 $N = 24$ case

Without any additional assumption, we can repeat the scaling argument of the preceding subsection to the case of $\kappa = 0$. After changing variables to

$$\Xi^\pm = \frac{1}{\sqrt{2}}(\chi^- \pm \chi^+), \quad \gamma^\pm = \frac{1}{\sqrt{2}}(\beta_- \pm \beta_+),$$

we can write the macroscopic loop amplitude as follows

$$Z[\Lambda, l, \Xi^{-}; f^i_0; q_1, \cdots, q_{n-1}] = \int_{\{\Xi, f_0\}} \mathcal{D}\Xi^+ \mathcal{D}\Xi^- \sum_{i=1}^{24} (\mathcal{D}f^i) e^{-S[\Xi^+, \Xi^-, f^i, \Lambda]}$$

$$\times \prod_{k=1}^{n-1} \int_{\Sigma} d^2 x_k \sqrt{\hat{g}(x_k)} e^{\gamma^{(k)\Xi^+} + \gamma^{(k)\Xi^-} + \sum_j p^{(k)j}_j f^j},$$

$$S = \frac{1}{8\pi} \int_{\Sigma} d^2 x \sqrt{\hat{g}} \left( -2\hat{g}^{\mu\nu} \partial_\mu \Xi^+ \partial_\nu \Xi^- + \sqrt{2}Q \hat{R} \Xi^- + \sum_{j=1}^{24} \hat{g}^{\mu\nu} \partial_\mu f^j \partial_\nu f^j \right)$$

$$+ \frac{\sqrt{2}Q}{4\pi} \int_{\partial\Sigma} ds \sqrt{\hat{g}} \hat{k} \Xi^- + \frac{\Lambda}{\pi} \int_{\Sigma} d^2 x \sqrt{\hat{g}} e^{-\frac{\sqrt{2}Q}{4\pi} \Xi^+},$$

$$l = \int_0^{2\pi} \frac{dx}{2\pi} \sqrt{\hat{g}} e^{-\frac{1}{\sqrt{2}Q} \Xi^+}.$$
The on-shell conditions of the \( n - 1 \) tachyon operators are written in the form
\[
\gamma^{(k)}_{+} (\gamma^{(k)}_{-} - \frac{Q}{\sqrt{2}}) = 1 - \Delta^{(k)}_{0},
\]
\[
\Delta^{(k)}_{0} = \frac{1}{2} \sum_{j=1}^{24} (p^{(k)}_{j})^2, \quad (k = 1, \ldots, n - 1) . \tag{4.21}
\]
Evaluating the \( \Xi_{0}^{\pm} \) dependence we find the momentum for the macroscopic loop
\[
\hat{\gamma}^{\pm} = - \left( \sum_{k=1}^{n-1} \gamma^{(k)}_{-} - \frac{Q}{\sqrt{2}} (1 - 2h) \right) . \tag{4.22}
\]
By using the same argument as in the preceding subsection, the zero-mode shift is performed only on the \( \Xi^{+} \) field
\[
\Xi^{+} \rightarrow \Xi^{+\prime} = \Xi^{+} - \frac{Q}{\sqrt{2}} \rho, \quad \Xi^{-} \text{ : fixed}, \quad l \rightarrow l' = e^{-\frac{Q}{2l}} . \tag{4.23}
\]
We can perform the scaling argument as before and find
\[
Z[\Lambda, l, \Xi_{0}^{-} ; f_{0}^{j}; q_{1}, \ldots, q_{n-1}] \sim t^{k} \Lambda^{s} e^{-\hat{\gamma}_{-} \Xi_{0}^{-}} e^{i \sum_{m=1}^{n-1} p^{(m)}_{j} f_{0}^{j}} \langle O_{q_{1}} \cdots O_{q_{n}} \rangle (\Lambda = 1) , \tag{4.24}
\]
\[
\hat{\gamma}^{-} = - \left( \sum_{k=1}^{n-1} \gamma^{(k)}_{-} - \frac{Q}{\sqrt{2}} (1 - 2h) \right) \]
\[
= \gamma^{-}_{-} - \frac{Q}{\sqrt{2}} = \gamma^{-}_{-}, \text{cyl} . \tag{4.25}
\]
The momentum \( \hat{\gamma}^{-} \) is the \( \gamma^{-} \) momentum of the tachyon \( O_{q_{n}} \) in the cylindrical coordinate. Since the local tachyon operator \( O_{q_{n}} \) is assumed to appear in the small length limit \( l \rightarrow 0 \), it has the momentum \( q_{n} \equiv (-i\gamma^{(n)}_{+}, -\gamma^{(n)}_{-}, p^{(n)}_{j}) \), \( (j = 1, \ldots, 24) \) which is determined uniquely by the momentum conservations of the matter fields and \( \Xi^{-} \) field and the on-shell condition for the tachyon
\[
\gamma^{(n)}_{+} = \frac{1}{\hat{\gamma}^{-}} (1 - \Delta^{(n)}_{0}) . \tag{4.26}
\]
In this way we obtain the scaling exponents \( k \) and \( s \) in Eq.(4.24)

\[
k = \sqrt{2Q\gamma_+(n)} = \frac{1}{\gamma_-}\sqrt{2Q(1 - \Delta_0^{(n)})}.
\]

\[
s = \frac{Q}{\sqrt{2}} \sum_{k=1}^{n} \gamma_+^{(k)},
\]

Thus the complete dependence on the length \( l \) and the cosmological constant \( \Lambda \) is determined in this case too.

4.3 \( N = 0 \) case

Finally we consider the case of no matter fields. Though we included the \( N = 0 \) case in the case of \( N \neq 24 \) when discussing the Wheeler-DeWitt equation, we have two reasons to treat this case separately. First, we should introduce the notion of chirality similarly to the \( c = 1 \) Liouville gravity theory, since the on-shell condition for the tachyon operator is different from \( N \neq 0 \) cases. Second, we can study a more detailed form of the macroscopic loop amplitudes in this case, since we can calculate the correlation functions explicitly for an arbitrary number of local tachyon operators.

The on-shell condition in this case is given by Eq.(4.13)

\[
(\beta^{(k)}_X - 1)(\beta^{(k)}_Y + 1) = 0, \quad (k = 1, \ldots, n-1).
\]

Since the on-shell condition is factorized, the value of the momentum \( \beta_X \) does not specify the value of the other component \( \beta_Y \) and vice versa. This is in contrast to the on-shell condition in other cases (4.3) and (4.21). We define the chirality of the tachyon to be positive if \( \beta_X = 1 \) and negative if \( \beta_Y = -1 \). If both conditions are satisfied, the chirality is ill-defined. This definition is quite similar to the chirality for the tachyon in the case of the \( c = 1 \) Liouville gravity.

By repeating the scaling argument, we find the behavior of the macroscopic loop amplitude for \( N = 0 \) in the small length limit as follows

\[
Z[\Lambda, l, Y_0; q_1, \ldots, q_{n-1}] \sim \Lambda^s l^k e^{-\hat{\beta}_Y Y_0} f(q_1, \ldots, q_{n-1}),
\]

\[
\hat{\beta}_Y = \beta_Y^{(n)} + 1 \neq 0,
\]
This result means that the local tachyon operator corresponding to the small length limit of the macroscopic loop has positive chirality. The result (4.30) is consistent with the solution (3.15) of the Wheeler-DeWitt equation in the mini-superspace approximation.

For the cases in which the matters exist, we can study the scaling behavior of the macroscopic loop amplitude at small $l$, but we can evaluate the correlation functions explicitly only up to three local tachyon operators. On the other hand, for the $N = 0$ case in which no matter fields are present, we can evaluate the correlation functions explicitly for an arbitrary number of tachyons. Therefore we can obtain more detailed information of the macroscopic loop amplitude such as the dependence on the momenta of the other inserted tachyons.

There are two methods of computing the tachyon correlation functions with no matter fields. The first method is a direct evaluation of the path integral by means of an analytic continuation. In Ref.[10], we computed the correlation functions of a single negative chirality tachyon $\mathcal{O}_{q_1}$ and an arbitrary number of positive chirality tachyons $\mathcal{O}_{q_k}$, $(k = 2, \ldots, n - 1)$ for spherical topology

$$Z[\Lambda, l, Y_0; q_1, \ldots, q_n]_{h=0} \sim \pi^{n-3} e^{-p_1 Y_0} \Lambda^{2-n-p_1} \frac{\Gamma(n-2+p_1)}{\Gamma(-p_1)} \sum_{k=2}^{n} \Delta(1-p_k) \Delta(p_k),$$

where $\Delta(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$ and $\beta^{(1)} = 1 + p_1$, $\beta^{(k)} = -1 + p_k$ $(k = 2, \ldots, n)$. It is difficult to obtain the correlation functions for other chirality combinations by the analytic continuation [17].

The second method treats this problem as a scattering problem in a conformal field theory using the matrix model approach [20]. A generating function for tachyons with arbitrary momenta has been constructed in this method. Although it is originally derived for the $c = 1$ Liouville theory, the dilaton gravity for $N = 0$ can be regarded as a condensation of background tachyon in the $c = 1$ Liouville gravity. This background tachyon serves as the cosmological term (2.11) for the dilaton gravity. We can apply this method even for general chirality configurations. For the case of single negative chirality tachyon, it gives the same results as those by the analytic continuation.
5 Discussion

We have seen that the solution of the Wheeler-DeWitt equation is consistent with the result of the scaling argument. However, the Wheeler-DeWitt equation admits an exceptional solution if $\hat{\beta}_Y$ vanishes. The solution gives the fixed length for the macroscopic loop. We consider this exceptional solution as an artifact or a pathology of the Wheeler-DeWitt equation in the mini-superspace approximation. It seems unphysical that only a particular value is allowed for the one-dimensional universe. At the moment, however, we are not able to demonstrate that the exceptional solution should be excluded. One might need more information than the Wheeler-DeWitt equation. It is desirable to clarify this point.

In defining the loop length $l$ in Eqs.(3.13) or (3.18), we have used the square root of the operator defining the cosmological term (2.11). This definition is most convenient to solve the Wheeler-DeWitt equation. However, we can choose other metric such as $g_{\mu\nu} = e^{2\rho} \hat{g}_{\mu\nu}$ as the physical metric to define the loop length $l$. By changing variables to $\rho$ and $\phi$, we can obtain solutions of the Wheeler-DeWitt equation from our solution (3.15) or (3.20) in terms of any combinations of the Liouville and the dilaton fields as a conformal factor. Consequently, we obtain the solution as an integral over the unknown weight $C(\hat{\beta}_Y)$ or $C(\hat{\gamma}_-)$. We have considered amplitudes with only one macroscopic loop. It is possible to consider amplitudes with more than one macroscopic loop. As a first approximation, the Wheeler-DeWitt equation in the mini-superspace approximation can be applied to find the length dependence of each loop.

In quantizing the dilaton gravity, we have used the free field representation together with the translation invariant measure of these free fields. In particular, we have assumed the path integral region (values of the fields) to be infinite $(-\infty, \infty)$. However, the transformation from the original variables $\rho, \phi$ to free fields $\hat{\chi}, \hat{\Omega}$ is nonlinear and quite complex. The path integral region does not in general correspond to the entire real values of the original variables. It may be necessary to consider the restricted region of the integration for the transformed variables [9], [24]. The change of the integration region does not affect the Wheeler-DeWitt equation much, since it is a local equation. However, it is difficult to compute the correlation functions of local tachyon operators. It is quite possible that some of our arguments as well as our results need to be modified if we change the integration region.

We have chosen the present approach motivated by the success of the continuum
approach for the Liouville gravity. However, the lack of a more rigorous calculational tool like the matrix model seems to suggest that we need to consider the canonical quantization of the dilaton gravity step by step more carefully.

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