

INTRODUCTION

1.1

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The MIT bag [9] is typical of this general class of models. In the case of the heavy quark-antiquark $(Q, \bar{Q})$ system described above, the magnitude of the color-electric field is determined by a balance between the pressure generated by the field and an external "bag" pressure $B$ [1]. \( \frac{1}{2} E_\|^2 = B \). It follows that $A_\| \propto Q$, and the resulting string tension $\sigma_\|$ also scales as the square root of the Casimir, $\sigma_\| \propto \sqrt{(j(j+1))}$.

In fact, the phenomenological models discussed above are not compatible with lattice simulations, which have shown that the string tension actually scales to an excellent approximation like the Casimir of the representation. This has been observed for both SU(2) and SU(3) gauge groups [12]. In the context of the flux-tube model, this suggests that the cross-section $A_\|$ is independent of representation.

Direct measurements of the flux-tube cross-section for heavy quark-antiquark $(Q, \bar{Q})$ sources are obtained here in three representations of three-dimensional SU(2) lattice gauge theory: fundamental $(j = 1/2)$, adjoint $(j = 1)$, and quartet $(j = 3/2)$. A more thorough check of scaling and finite volume effects is achieved by working in three dimensions than would be obtained (with the same computing power) in four dimensions. We think that our results are relevant to the problem of confinement in four-dimensional QCD. In particular, previous lattice studies have shown that the string tension scales like the Casimir of the representation in three dimensions [13, 14], as well as in four dimensions [12]. Moreover, the flux-tube picture of confinement is qualitatively the same in both three and four dimensions. It is therefore reasonable to expect that the qualitative features of flux-tubes reported here would be reproduced in four dimensions. Of course, such a calculation can and should be done.

We find $A_\| \approx \text{constant}$ for the three representations, to within about 10% (a rough estimate of the overall quality of our data). This is consistent with the flux-tube picture, given that the string tension scales like the Casimir of the representation (which is confirmed here up to the quartet representation). Several additional qualitative features of the flux-tube picture are also verified.

These results suggest a connection between confinement in QCD and the physics of a dual superconductor. Indeed, if a multiply-charged monopole would be inserted into an ordinary (type II) superconductor, all the quanta of magnetic flux would be carried by a single flux-tube, whose diameter is fixed by the penetration depth [15]. A pair of monopoles of opposite sign therefore would be confined, with a string tension that would scale like the squared-charge.

It is well-known that dual superconductivity (magnetic monopole condensation) results in confinement of electric charges in compact QED in three-dimensions (QED$_3$) [16]. A simple extension of the analytical calculation of Ref. [17] in the Villain approximation to the Wilson action, to include Wilson loops for multiply-charged sources, demonstrates that the string tension scales like the squared-charge.

We have performed lattice simulations of singly- and doubly-charged Wilson loops in compact QED$_3$, and our results confirm the expected scaling properties of the string tension and flux-tube cross-section. The potential is found to scale like the squared-charge to within a few percent, and the flux-tubes in the two cases have the same cross-section to within about 10%. The results of our three-dimensional SU(2) and U(1) simulations taken together lend some support, albeit indirectly, to the dual superconductor picture of confinement in four-dimensional QCD [18].

II. METHOD

To begin with, we consider the three-dimensional SU(2) lattice theory. Wilson loops are used to introduce static $(Q, \bar{Q})$ sources. Lattice measurements of the color-electric and -magnetic fields generated by these sources are obtained from correlators $F_{\mu\nu}^W$ of plaquettes with a Wilson loop

$$ F_{\mu\nu}^W(x) \equiv \beta \langle \frac{1}{a^4} \left( \frac{1}{W_j} \text{Tr} U_{\mu\nu}(x) \right) \rangle - \frac{1}{2} \text{Tr} U_{\mu\nu} \rangle , $$

(2)

where $U_{\mu\nu}(x)$ is the plaquette located at $x$ (measured relative to the center of the Wilson loop), and $W_j$ is the normalized trace of the Wilson loop in the $j$-th representation:

$$ W_j \equiv \frac{1}{2j+1} \text{Tr} \left( \prod_{i \in L} D_j[U_i] \right) . $$

(3)

$D_j[U_i]$ denotes an appropriate irreducible representation of the link $U_i$, and $L$ the closed loop. $\beta \equiv 4/(g^2 a)$, where the coupling constant $g$ has dimensions of (mass)$^{1/2}$ in three dimensions.

In the continuum limit, the trace of a 1 x 1 plaquette is by construction independent of representation (up to overall normalizations). As in several previous lattice calculations of higher representation Wilson loops (cf. Refs. [12-14]), we use the action expressed in terms of links in the fundamental representation to perform simulations at arbitrary $\beta$. The trace of the plaquette $U_{\mu\nu}$ in the fundamental representation is also used to compute the correlators of Eq. (2).

In the continuum limit the correlator $F_{\mu\nu}^W$ corresponds to the expectation value of the square of the Euclidean field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g \epsilon_{\mu\nu} A_\rho A^\rho$, $\lim_{\beta \to 0} F_{\mu\nu}^W = \frac{1}{2} \left( \sum_{a} \langle F_{\mu\nu}^a \rangle \right)_{Q, \bar{Q}} - \frac{1}{4} \left( \sum_{a} \langle F_{\mu\nu}^a \rangle \right)_0$

(4)

where the expectation value $\langle \ldots \rangle_{Q, \bar{Q}}$ is taken in a state with external sources in the $j$-th representation, and $\langle \ldots \rangle_0$ is the vacuum expectation value.

To compute the energy density, the Euclidean 3-axis is identified with a temporal side of the Wilson loop, and the 1-axis with a radial side. We separate contributions to the total energy density $E_{\text{tot}}$ corresponding to the two spatial components of the color-electric field (in the directions parallel and perpendicular to the line joining the quarks), and the color-magnetic field (a scalar in three dimensions):

$$ E_{\text{tot}}(x) = E_{\|}^W(x) + E_{\perp}^W(x) + E_{\perp}^W(x) , $$

(5)
For example, the value of the function at \( x = 0 \) is given by the integral

\[
\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}
\]

which is evaluated using the Gaussian integral. For \( x \) in the range \( -\infty < x < \infty \), the function is continuous and differentiable.

When \( x \) is the sum of the components of \( x \), the equation becomes

\[
\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}
\]

where the integral is over the product space of the components of \( x \).
The trace of an element of the group in the \( j \)-th representation can be expressed in terms of its trace in the fundamental representation using trigonometric expressions among the group characters. In the case of the adjoint and quartet representations \[21]\):

\[
W_{3/2} = 2W_{1/2}^3 - W_{1/2},
\]

\[
W_1 = (4W_{1/2}^3 - 1)/3.
\]

Hence one need only compute the Wilson loop in the fundamental representation, using the “unreduced” links \( U_i \) or the “reduced” elements \( V_i, V_i' \) of Eqs. (8) and (10), as the case may be. The Wilson loops in higher representations then follow from Eq. (13). The Bessel functions for the analytical integrations are tabulated separately for the three representations.

III. RESULTS AND DISCUSSION

Our main results were obtained on a 32\(^3\) lattice at \( \beta = 10 \) (which is well within the scaling region for the string tension on a lattice of this size \[14]\]). Wilson loops and plaquette correlators were calculated in the three representations \( j = 1/2, 1, \) and \( 3/2 \) for all loops of sizes \( T \times R \) from \( 3 \times 4 \) to \( 8 \times 8 \) (these observables were measured in groups in several separate runs). Some additional data was taken at \( \beta = 14 \) in order to check for scaling of the physical flux-tube dimensions. A standard heat-bath algorithm was employed. More than 10,000 sweeps were typically used for thermalization. 2,000 measurements were made, taking 20 sweeps between measurements. The resulting integrated autocorrelation times \( \tau_{\text{int}} \) for the Wilson loops generally satisfy \( \tau_{\text{int}} \lesssim 1 \), consistent with the results of a systematic study made in Ref. \[14]\). Estimates of the statistical errors were obtained using the jackknife method. However, measurements of different observables (and of a given observable in the three representations) tend to be strongly correlated, since many Wilson loops and plaquette correlators were measured simultaneously on a given lattice.

The quartet representation is much more difficult to measure than the two lower representations, due to the exponential suppression of the Wilson loop with the \( Q_i\bar{Q}_i \) potential, which is found to scale with the Casimir of the representation. Energy density measurements in the quartet case obtained from loops larger than about \( 6 \times 6 \) are of poor quality, although these data are consistent with conclusions drawn from results obtained from smaller loops.

Representative data for Wilson loops in the three representations are shown in Fig. 1. Earlier studies have shown that the potentials scale with the Casimir of the representation at essentially all lengths scales \( R \) \[13,14]\). This is made evident in Fig. 1, where the logarithms of the Wilson loops are scaled by a ratio of Casimirs,

\[
e_j = \frac{3/4}{j(j + 1)}.
\]

The quantity \(-\ln(W_j(T, R))/T\), which extrapolates to the \( Q_i\bar{Q}_i \) potential \( V_j(R) \) in the limit \( T \to \infty \), is found to scale as \( j(j + 1) \) to within a few tenths of a percent at all \( T \) and \( R \) considered here. A simple extrapolation of the data using \( V_j(R) \approx \ln(W_j(T_{\text{max}} - 1, R))/W_j(T_{\text{max}}, R) \), where \( T_{\text{max}} \) is the largest \( T \) value in the data set, gives agreement to a few tenths of a percent with the results of a careful statistical analysis of fundamental and adjoint Wilson loops reported in Ref. \[14]\).

Several attributes of the plaquette correlators were measured. To begin with, results for the fundamental and adjoint representations are presented. The correlators were measured over a range of distances \( x_\perp \) from the center of the Wilson loop, in the direction normal to the plane of the loop. Results for the \( T \times R = 8 \times 6 \) loop are shown in Fig. 2. The cross-sections of the fundamental and adjoint representation flux-tubes are indistinguishable within statistical errors. This is true for all Wilson loops that were considered. For example, the \( T \) evolution of \( \xi_j^R \) for \( R = 6 \) Wilson loops is illustrated in Fig. 3. As observed in Refs. \[4,6\], the plaquette correlators are more sensitive to higher states than the Wilson loop. Our data are consistent with a one-excited-state parameterization given in Ref. \[6\].

Figure 2 demonstrates that the component of the color-electric field parallel to the line joining the charges dominates the energy, as assumed in the flux-tube model. The magnetic energy turns out to be negative, which has also been observed in four-dimensional SU(2) lattice theory \[5\]. The formation of a well-defined flux-tube is demonstrated by measurements of \( \xi_j^R \) in the plane of the Wilson loop. Figure 4 shows \( \xi_j^R \) for the \( T \times R = 6 \times 8 \) loop as a function of the longitudinal distance \( x_\parallel \) of the plaquette centroid from the center of the loop. Notice the approximate symmetry of the energy density about the center of the loop. The formation of the flux-tube is further illustrated in Fig. 5, where \( \xi_j^R \) is shown as a function of the radial separation \( R \) of the Wilson loop (for fixed \( T = 6 \)).

A stringent test of energy density calculations using Eq. \( (7) \) is provided by a sum rule derived by Michael \[22\]

\[
a^2 \sum_x \xi_j^{\text{int}}(x) = V_j(R).
\]

The analogous sum rule in four-dimensional SU(2) was studied in detail by Haymaker and Woseik \[5\]. The flux-tube picture suggests a related sum rule that is much simpler to measure. If the interaction energy is dominated by a constant color-electric field along the line joining the charges (as expected in the limit of quark separations much greater than the flux-tube thickness), then the integral of the energy density along one transverse “slice” of the flux-tube should equal the string tension \( [\text{cf. } \sigma_j = \lim_{a \to 0}(V_j(R) - V_j(R - a))/a] \):

\[
a \sum_{x_\perp} \xi_j^R(x_\perp, x_\parallel = \text{fixed}) \approx \sigma_j,
\]

where the sum is taken over positive and negative distances \( x_\perp \) from the plane of the Wilson loop.

Our results are in good agreement with Eq. \( (16) \). Figure 6 shows the left-handside of this equation for the \( T \times R = 8 \times 6 \) loop in the fundamental and adjoint
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finement is due to a bulk property of the QCD vacuum (such as a vacuum pressure) predict a sufficiently rapid increase in $A_t$ with representation as to be incompatible with the results obtained from our lattice simulations.

We also made flux-tube measurements in compact QEDs, which exhibits electric confinement due to magnetic monopole condensation. We considered singly- and doubly-charged Wilson loops. The string tension was found to scale like the squared-charge, and the flux-tube cross-section was found to be independent of the charge, to a good approximation. The results of our three-dimensional SU(2) and U(1) simulations taken together lend some support, albeit indirectly, to a conjecture that the dual superconductor mechanism underlies confinement in compact gauge theories in both three and four dimensions. This conclusion is also supported by the results of a recent study of dual Abrikosov vortices in an Abelian projection of SU(2) lattice gauge theory in four dimensions [25]. Flux-tube measurements in four-dimensional SU(2) gauge theory similar to those reported here should be made in order to further explore this possibility.

ACKNOWLEDGMENTS

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

* Permanent address.


Figure Captions

Fig. 1. $T$ evolution of Wilson loops in three representations of SU(2) lattice gauge theory: $j = 1/2(\psi)$, $j = 1(\Omega)$, and $j = 3/2(\Delta)$. $c_j$ is a ratio of Casimir, defined in Eq. (14). The quantity $-\ln(W_j(T, R))/T$ extrapolates to the QCD potential in the limit $T \to \infty$.

Fig. 2. Energy density profiles transverse to the plane of the $T \times R = 8 \times 6$ Wilson loop [$j = 1/2(\psi)$, and $j = 1(\Omega)$]. These results are an average over plaquettes with centroids at distances $z_{\perp}$ transverse to the plane of the loop.

Fig. 3. $T$ evolution of $\xi_{\perp}(x_{\perp} = 0, 2a)$ for Wilson loops with $R = 6$ [$j = 1/2(\psi)$, $j = 1(\Omega)$].

Fig. 4. Energy density profile in the plane of the $T \times R = 6 \times 8$ Wilson loop [$j = 1/2(\psi)$, $j = 1(\Omega)$]. $x_{\parallel}$ is the distance of the centroid of the plaquette from the center of the Wilson loop. The radial sides of the Wilson loop are located at $x_{\parallel} = \pm 4a$. Plaquettes with a side touching the Wilson loop cannot be measured using the variance reduction of Eq. (8), and are not shown.

Fig. 5. Energy density $\xi_{\perp}(x_{\perp} = 0, 2a)$ as a function of $R$, for fixed $T = 6$ [$j = 1/2(\psi)$, $j = 1(\Omega)$].
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