Differential Calculi on h-deformed Bosonic and Fermionic Quantum Planes

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Abstract

We study differential calculus on h-deformed bosonic and fermionic quantum space. It is shown that the fermionic quantum space involves a parafermionic variable as well as a classical fermionic one. Further we construct the classical $su(2)$ algebra on the fermionic quantum space and discuss a mapping between the classical $su(2)$ and the h-deformed $su(2)$ algebras.
1. Introduction

Quantum groups and quantum algebras [1-4] have attracted much attention in theoretical physics and mathematics, such as statistical models, integrable models, conformal field theories, knot theory and so on [5-8]. Quantum space was introduced to represent quantum groups [9]. Differential calculus on the quantum space was also studied [10-15]. It is intriguing also from the viewpoint of non-commutative geometry. The quantum differential calculus is closely related with q-oscillators, which have been applied to various fields [16-18]. Further, we can obtain quantum groups $Sp_q(2n)$ and $SO_q(2n)$ using the q-deformed phase space which is defined through the differential calculus on the quantum space for $SL_q(n)$ [19]. Furthermore, on the quantum space quantum deformed algebras have been constructed, e.g., q-deformed Lorentz, Poincaré, conformal and superconformal algebras [20-25]. These analyses imply that the quantum space is interesting as applications of quantum groups and useful to show some aspects of quantum groups.

Recently other than the conventional q-deformation and its multi-parametric extension, another deformation was discovered in [26-29] and is called h-deformation. Further, differential calculus on an h-deformed quantum bosonic space has been considered in [30, 31]. The purpose of this article is to investigate in detail bosonic and fermionic differential calculus on h-deformed quantum planes, extending the above analyses on the q-deformed differential calculus to the h-deformed case. The differential calculus on the bosonic quantum space is studied and a comment is obtained from the viewpoint of a constrained system. In addition, we discuss a differential calculus on an h-deformed fermionic space. Ref. [25] shows that q-deformed fermionic coordinates and derivatives represent $su_q(2)$ algebra, which is related with Drinfeld-Jimbo basis by some mapping. Following the approach, we investigate algebra which is constructed in terms of the h-deformed fermionic elements.

This paper is organized as follows. In section two we review on general non-commutative differential calculus including the quantum space. Then using a new solution of the Yang-Baxter equation, we derive h-deformed bosonic differential calculus. Further some comments on the h-deformed space are given. In section three, similarly we derive h-deformed differential calculus on the fermionic quantum space. It is shown that we can represent the classical Lie algebra $su(2)$, using their fermionic
variables and derivatives. Also we obtain a mapping between the classical algebra \( su(2) \) and the \( h \)-deformed algebra \( su_h(2) \). Further we discuss an \( h \)-deformed \( SO(4) \) group. The last section is devoted to conclusion and discussion.

2. Differential calculus on \( h \)-deformed bosonic space

A quantum space is a non-commutative space representing the corresponding quantum group. Ref. [11] clarified general non-commutative differential calculus including the quantum space and its analysis was extended to superspaces in [15]. We set up commutation relations between coordinates \( x^i \) and derivatives \( \partial_{x^i} \) as follows,

\[
x^i x^j = B^{ij}_{kl} x^k x^l, \quad \partial_{x^i} x^j = \delta^j_i + C^{ij}_{kl} x^k \partial_{x^j}.
\]  

These matrices should satisfied the following relations:

\[
B^{ij}_{pr} B^{rk}_{ln} B^{pt}_{on} = B^{jk}_{pr} B^{ip}_{rt} B^{tr}_{mn}, \quad (\delta^i_k \delta^j_\ell - B^{ij}_{k\ell})(\delta^k_m \delta^\ell_n + C^{k\ell}_{mn}) = 0.
\]

The former equation is called Yang-Baxter equation. We consider the case where \( B^{ij}_{k\ell} \) is proportional to \( C^{ij}_{k\ell} \). In this case we can write commutation relations of derivatives by \( B \)-matrix as follows,

\[
\partial_{x^i} \partial_{x^j} = B^{ij}_{k\ell} \partial_{x^k} \partial_{x^\ell}.
\]

Recently a new solution of the Yang-Baxter equation was discovered in [27] as follows,

\[
\hat{R} = \begin{pmatrix}
1 & -h' & h' & hh' \\
0 & 0 & 1 & h \\
0 & 1 & 0 & -h \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Also in Ref. [26] and [28], the same type of the \( \hat{R} \)-matrix was discovered in the case where \( h = h' \) and \( h = h' = -1 \), respectively. New type of a quantum group \( SU_h(2) \) was studied in [26-28] and the corresponding deformed algebra \( su_h(2) \) has been constructed in [29]. We identify \( \hat{R} \) with \( B \) in order to obtain \( h \)-deformed differential calculus. The \( \hat{R} \)-matrix has \( \pm 1 \) as eigenvalues. This fact leads to a condition that we
should identify $C$ with $\tilde{R}$, too. Namely we hereafter study the case where

$$B^{ij}_{k\ell} = C^{ij}_{k\ell} = \tilde{R}^{ij}_{k\ell}. \quad (2.5)$$

We restrict ourselves to the case where $h = h'$. It is easy to extend the following analysis to the case with two independent parameters, $h$ and $h'$. We substitute eq.(2.4) into (2.1) and (2.3), so that we obtain the following commutation relations;

$$[x^1, x^2] = h(x^2)^2, \quad [\partial_{x^1}, \partial_{x^2}] = h(\partial_{x^1})^2,$$

$$[\partial_{x^2}, x^1] = h x^1 \partial_{x^2} + h x^2 \partial_{x^2} + h^2 x^2 \partial_{x^1}, \quad (2.6)$$

$$[x^i \partial_{x^i}, x^j] = 1 - h x^2 \partial_{x^2}, \quad [\partial_{x^i}, x^2] = 0.$$

The above algebra becomes classical in the limit, i.e., $h \to 0$.

We can introduce h-deformed quantum group $T^i_j$ such that the elements transform the coordinates and the derivatives like $T^i_j x^j$. Covariance of the commutation relations (2.6) under the transformation requires the following commutation relation of $T^i_j$;

$$\tilde{R}^{ij}_{k\ell} T^k_m T^\ell_n = T^i_k T^j_\ell \tilde{R}^{k\ell}_{m\ell} \quad (2.7)$$

Next we consider conjugation consistent with (2.6). Suppose that $h$ is pure imaginary, i.e., $\overline{h} = -h$. Then we obtain the consistent conjugation as follows,

$$\overline{x^1} = x^1 + h x^2, \quad \overline{x^2} = x^2,$$

$$\overline{\partial_{x^1}} = -\partial_{x^2}, \quad \overline{\partial_{x^2}} = -\partial_{x^2} - h \partial_{x^1}. \quad (2.8)$$

Through the conjugation we can define real coordinates and momenta as follows,

$$\hat{x}^1 = x^1 + \frac{h}{2} x^2, \quad \hat{x}^2 = x^2,$$

$$\hat{p}_1 = -i \partial_{x^2}, \quad \hat{p}_2 = -i (\partial_{x^2} + \frac{h}{2} \partial_{x^1}). \quad (2.9)$$

They satisfy the following commutation relations;

$$[\hat{x}^1, \hat{x}^2] = h(\hat{x}^2)^2, \quad [\hat{p}_1, \hat{p}_2] = h(\hat{p}_1)^2,$$

$$[\hat{p}_j, \hat{x}^l] = -i - h \hat{x}^2 \hat{p}_1, \quad [\hat{p}_1, \hat{x}^2] = 0, \quad (2.10)$$
\[ [\hat{p}_2, \hat{x}^1] = \hbar (-i + \hat{x}^1 \hat{p}_1 + \hat{x}^2 \hat{p}_2 - \hbar \hat{x}^2 \hat{p}_1). \]

The similar phase space algebra as well as (2.6) has been obtained in [30].

These h-deformed coordinates and momenta can be represented in terms of classical ones \( \hat{x} \) and \( \hat{p} \). Suppose that \( \hat{x}^2 = \hat{x}^2 \) and \( \hat{p}_1 = \hat{p}_1 \), then we have

\[ \hat{x}^1 = \hat{x}^1 (1 - i \hbar \hat{p}_1 \hat{x}^2) + i \hbar (\hat{x}^2)^2 \hat{p}_2, \quad \hat{p}_2 = \hat{p}_2 (1 - i \hbar \hat{p}_1 \hat{x}^2) + i \hbar (\hat{p}_1)^2 \hat{x}^2. \quad (2.11) \]

The deformed phase space algebra such as eq.(2.10) might remind us of Dirac brackets of a constrained system [32]. Actually we can find a “constraint” which we can make identically equal to zero through the above commutation relations (2.10), as follows,

\[ f = 1 - 2 i \hbar \hat{p}_1 \hat{x}^2. \quad (2.12) \]

The function \( f \) commutes with \( \hat{x}^2 \) and \( \hat{p}_1 \) and satisfies commutation relations with the other elements as follows,

\[ [f, \hat{x}^1] = i \hbar \hat{x}^2 f, \quad [\hat{p}_2, f] = i \hbar f. \quad (2.13) \]

These relations seems to be somewhat different from the Dirac bracket, whose right hand side vanishes completely. But we can make \( f \) equal to zero identically. The above fact seems to show that \( SU_h(2) \) is a ‘group’ which transforms the “constraint” \( f \) and its relations (2.13) covariantly.

3. Differential calculus on h-deformed fermionic space

In this section we discuss differential calculus on an h-deformed quantum fermionic space. Commutation relations of h-deformed coordinates \( \theta^a \) and derivatives \( \partial_a \) are obtained by replacing \( \hat{R} \) in (2.1) and (2.3) in terms of \( -\hat{R} \) as follows,

\[ \theta^a \theta^b = -\hat{R}^{a \beta}_{\mu \nu} \theta^\mu \theta^\nu, \quad \partial_a \partial^a = -\hat{R}^{a \beta}_{\mu \nu} \partial^\mu \partial^\nu, \]

\[ \partial_a \theta^\alpha = \delta^a_\mu - \hat{R}^{a \beta}_{\mu \nu} \theta^\nu \partial^\beta. \quad (3.1) \]

These relations are written explicitly as follows,

\[ (\theta^1)^2 = \hbar \theta^1 \theta^2, \quad \{\theta^1, \theta^2\} = (\theta^2)^2 = 0, \]

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\[(\partial_1)^2 = \{\partial_1, \partial_2\} = 0, \quad (\partial_2)^2 = h\partial_1\partial_2, \]
\[
\{\partial_2, \theta^\alpha\} = 1 + h\theta^2\partial_1, \quad \{\partial_1, \theta^2\} = 0, \quad (3.2)
\]
\[
\{\partial_2, \theta^1\} = -h\theta^1\partial_1 - h\theta^2\partial_2 - h^2\theta^2\partial_1.
\]

Further eq.(3.2) leads to the following trilinear relations;

\[(\theta^1)^3 = (\partial_2)^3 = 0. \quad (3.3)\]

The coordinate \(\theta^1\) and the derivative \(\partial_2\) are parafermionic [33], while its derivatives \(\partial_1\) and its coordinates \(\theta^2\) are nilpotent.

As discussed in the previous section, we can introduce h-deformed quantum group elements \(T^\alpha_\beta\) which transform \(\theta^\alpha\) into \(\theta^\alpha = T^\alpha_\beta\theta^\beta\). The elements satisfy the same commutation relation as (2.7). This h-deformed quantum group is interesting also from the aspect that it transforms fermionic and parafermionic variables, i.e., \(\theta^1\) and \(\theta^2\). Further, we can define a determinant of \(T^\alpha_\beta\) using the h-deformed fermionic space as follows,

\[\theta^1\theta^2 = \det T \cdot \theta^1\theta^2. \quad (3.4)\]

The definition leads to

\[\det T = T^1_1T^2_2 - T^1_2T^2_1 + hT^1_1T^2_1. \quad (3.5)\]

Eq.(3.5) coïnside with the definition of the determinant in [28].

Ref.[25] shows that q-deformed fermionic coordinates and derivatives can represent q-deformed quantum algebra, e.g., \(su_q(2)\). Here we apply the similar analysis to the h-deformed case. In the similar way to [25], we define the following generators;

\[L_+ = \theta^2\partial_1, \quad L_- = \theta^1\partial_2. \quad (3.6)\]

Using their commutation relation, we introduce a Cartan generator as follows;

\[L_0 = [L_+, L_-] = \theta^2\partial_2 - \theta^1\partial_1. \quad (3.7)\]

In addition to (3.7), these generators satisfy the following relations;

\[[L_0, L_\pm] = \pm 2L_\pm. \quad (3.8)\]
Eqs. (3.7) and (3.8) are nothing but the classical $su(2)$ algebra up to the normalization factor, although the coordinates and the derivatives are deformed. The generators act on the coordinates and the derivatives as follows,

$$[L_0, \theta^a] = (-1)^a \theta^a, \quad [L_0, \partial_a] = -(-1)^a \partial_a,$$

$$[L_+, \theta^1] = \theta^2,$$  
$$[L_+, \partial_1] = -\partial_1,$$  
$$[L_+, \theta^2] = [L_+, \partial_2] = 0,$$  
$$[L_+, \theta^3] = -\hbar \theta^1 L_0 - \hbar^2 \theta^2 L_+,$$  
$$[L_+, \partial_1] = 0,$$  
$$[L_+, \partial_2] = -h \partial_2 L_0 - 3h^2 \partial_1 + 3h^2 \partial_2 L_+.$$  

(3.9)

The actions of $L_+$ and $L_0$ are never deformed, while those of $L_-$ are deformed.

In Ref.[29] a $h$-deformed $su(2)$ algebra has been constructed as follows,

$$[H, X] = \frac{2 \sinh(h X)}{h},$$

$$[H, Y] = -\{Y, \cosh(h X)\},$$  
$$[X, Y] = H.$$  

(3.10)

The $h$-deformed algebra $su_h(2)$ can be related with the classical algebra $su(2)$ through the following mapping;

$$X = \frac{\log \hat{L}_+}{h},$$

$$H = (\hat{L}_+ - (\hat{L}_+)^{-1}) \hat{L}_0,$$  
$$Y = \frac{h}{2} (\hat{L}_+ - (\hat{L}_+)^{-1})(2 \hat{L}_+ \hat{L}_- - h \hat{L}_0),$$  

(3.11)

where $\hat{L}_0 = L_0/2$ and $\hat{L}_\pm = L_\pm / \sqrt{2}$.

In Ref.[19] Zumino derived quantum groups $Sp_q(2n)$ and $SO_q(2n)$ from $q$-deformed bosonic and fermionic phase space algebra for $SL_q(n)$. That was extended to the supersymmetric case in [15]. In the similar way, here we discuss $h$-deformation of $SO(4)$.  

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At first, we define h-deformed gamma matrices $\gamma^a$ as

$$\gamma^a = \partial_a, \quad \gamma^{\bar{a}} = \theta^a. \quad (3.12)$$

They satisfy the following h-deformed Clifford algebra;

$$\gamma^a \gamma^b + \hat{B}^{\alpha \beta}_{\mu \nu} \gamma^\mu \gamma^\nu = \eta^{\alpha \beta}, \quad (3.13)$$

where $\eta^{\alpha \beta}$ is an SO(4) metric. The matrix $\hat{B}$ is composed of $\hat{R}$ matrix (2.4) as follows,

$$\hat{B}^{\alpha \beta'}_{\mu \nu'} = -\hat{R}^{\beta \beta'}_{\mu \nu}, \quad \hat{B}^{\alpha \beta}_{\mu \nu} = -\hat{R}^\mu_{\beta \alpha}, \quad (3.14)$$

where $\alpha, \beta, \mu, \nu = 1, 2$ and $\alpha' = 5 - \alpha$. Ref.[15] shows that $\hat{B}$ constructed through the above procedure satisfy the Yang-Baxter equation, if $\hat{R}$ is the solution of the Yang-Baxter equation. We could introduce h-deformed SO(4) group which transform the relation (3.13) covariantly. Instead of deriving explicitly $SO_h(4)$, we here introduce h-deformed $SO(4)$ quantum space $X^i (i = 1 \sim 4)$ whose commutation relations are written as $X^i X^j = \hat{B}^{ij}_{kl} X^k X^l$. We have explicitly

$$[X^1, X^2] = h(X^1)^2, \quad [X^1, X^3] = 0,$$

$$[X^1, X^4] = [X^2, X^3] = -h X^3 X^1,$$

$$[X^2, X^4] = h X^4 X^1 + h X^3 X^2 + h^2 X^3 X^1, \quad [X^3, X^4] = -h (X^3)^2. \quad (3.15)$$

The algebra has a center element $C \equiv X^4 X^1 + X^3 X^2$. Differential calculus on the above h-deformed space $X^i$ could be similarly obtained. Their commutation relations are covariant under the $SO_h(4)$ transformation, as said the above. Elements of $SO_h(4)$ satisfy the same relation as (2.7) except $\hat{R}$-matrix replacing $\hat{B}$-matrix.

4. Conclusion

We have studied here the differential calculi on the h-deformed bosonic and fermionic spaces. We have constructed the classical $su(2)$ algebra on the h-deformed fermionic space. The algebra is related with the h-deformed algebra $suh(2)$. It is
shown that $SU_h(2)$ is a 'group' which transforms fermionic and parafermionic variables into each other. This fact is very interesting in applications to the parastatistics. Also $SO_h(4)$ was discussed. Supersymmetric extension is also intriguing. It is easy to introduce $h$-deformed oscillators in the similar way to the above $h$-deformed differential calculus. For example, we can define a sort of deformed oscillators by identifying the $h$-deformed coordinates and derivatives with deformed anihilation and creation operators, respectively. Their applications to various fields are very interesting.

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References


