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FROM A_{m-1} TRIGONOMETRIC S-MATRICES TO THE THERMODYNAMIC BETHE ANSATZ

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ABSTRACT

The thermodynamic Bethe Ansatz equations that have been proposed to describe massive integrable deformations of the coset conformal field theories $g_k \times g_l / g_{k+l}$ are shown to result directly by applying the usual thermodynamic Bethe Ansatz arguments to the trigonometric S -matrices for the algebras $g = a_{m-1}$.

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1. Introduction

Thermodynamic Bethe Ansatz (TBA) systems of equations have been conjectured to describe the renormalization group trajectories issuing in various directions from certain coset conformal field theories. Although perfectly reasonable, these conjectures, in most cases, have not been shown to arise from a consistent factorizable S -matrix theory; the exceptions being when the particles carry no internal quantum numbers in which case the S -matrix is purely elastic [1], and some examples involving the algebra a_1 [2,3]. In particular, TBA systems have been written down [4] which should describe the massive deformation of the coset $g_k \times g_l / g_{k+l}$, for a simply-laced algebra g (the subscripts denoting the level) by the relevant operator known conventionally as $\phi_{1,1,\text{adj}}$ having conformal dimension $\Delta = (k+l)/(\tilde{h} + k + l)$, where $\tilde{h}$ is the dual Coxeter number of g .

For the a_{m-1} algebras strong support for the proposed TBA systems comes in an entirely different direction from the *restricted solid-on-solid* (RSOS) integrable lattice models [5], that have been solved using Bethe Ansatz techniques [6], and which in the scaling limit are thought to be identical to the quantum field theories describing the deformations of the cosets discussed above [6]. In particular, [7] concluded that the TBA systems resulting from the lattice models should be associated to integrable quantum field theories with factorizable S -matrices whose elements are proportional to RSOS Boltzmann weights. The proposed S -matrices are the *trigonometric* S -matrices associated to a_{m-1} . The S -matrix elements for the elementary states were written down in [7,8] (see also [9]), however, the bootstrap was solved in [10] and the full S -matrix written down explicitly in [11].

With the expression for the complete S -matrix it is now possible to complete the picture and verify that the TBA systems from the lattice models, and hence those conjectured to describe the integrable deformations of the cosets, do actually come directly from a consistent factorizable S -matrix. Unfortunately, as with most other calculations involving the Bethe Ansatz, it is not possible to be completely rigorous and some reasonable conjectures must be made; for instance, the use of the string hypothesis.

2. The trigonometric S-matrices

In this section we write down the trigonometric S -matrices associated to the Lie algebra a_{m-1} . A full discussion of these S -matrices and a proof of the fact that they satisfy all the necessary requirements may be found in [10,11].

The particles transform in representations $V_a \otimes V_a$, $a = 1, 2, \dots, m-1$, of the tensor product $a_{m-1} \times a_{m-1}$ (V_a are the fundamental representations of a_{m-1} whose highest

weights are the fundamental weights ω_a). The masses of the multiplets are

$$m_a = m_0 \sin\left(\frac{\pi a}{m}\right), \quad a = 1, 2, \dots, m-1, \quad (2.1)$$

where m_0 is some overall mass scale. The S -matrix depends upon two additional parameters k and l which in this paper will be taken to be natural numbers but more generally can be thought of as coupling constants.

The two-body S -matrix element between the a^{th} and b^{th} multiplet has the following factored structure:

$$S_{(k,l)}^{ab}(\theta) = X^{ab}(\theta) \tilde{S}_{(k)}^{ab}(\theta) \otimes \tilde{S}_{(l)}^{ab}(\theta), \quad (2.2)$$

where θ is the rapidity difference of the incoming particles. We now define the terms appearing in (2.2).

The scalar factor $X^{ab}(\theta)$ is the minimal S -matrix associated to a_{m-1} [12]. It is usually expressed as

$$X^{ab}(\theta) = \prod_{\substack{j=|a-b|+1 \\ \text{step } 2}}^{b+a-1} \frac{\sinh\left(\frac{\theta}{2} + i\frac{\pi(j+1)}{2m}\right) \sinh\left(\frac{\theta}{2} + i\frac{\pi(j-1)}{2m}\right)}{\sinh\left(\frac{\theta}{2} - i\frac{\pi(j+1)}{2m}\right) \sinh\left(\frac{\theta}{2} - i\frac{\pi(j-1)}{2m}\right)}, \quad (2.3)$$

however for our purposes it is more useful to use the following integral expression which is valid for physical rapidities (i.e. $\text{Im}(\theta) = 0$):

$$X^{ab}(\theta) = \exp\left\{i\pi\delta_{ab} - \int_{-\infty}^{\infty} \frac{dx}{x} e^{im\theta x/\pi} \left[\hat{A}_{ab}^{(m)}(x) - \delta_{ab}\right]\right\}, \quad (2.4)$$

where the kernel, symmetric in its indices, is given by

$$\hat{A}_{ab}^{(m)}(x) = \frac{2 \sinh(ax) \sinh[(m-b)x] \cosh x}{\sinh(mx) \sinh x}, \quad b \geq a. \quad (2.5)$$

The other factors in (2.2) specify the non-trivial exchange of colour between the multiplets. They are proportional to the RSOS R -matrices for the quantum group associated to a_{m-1} with deformation parameters equal to roots of unity: $-\exp(-i\pi/(m+k))$ and $-\exp(-i\pi/(m+l))$, respectively. The S -matrices describe a set of kinks interpolating between a set of vacua which lie in the product of two highest weights of a_{m-1} subject to $\mu \cdot \vartheta \leq k$ and $\mu' \cdot \vartheta \leq l$, respectively, where ϑ is the highest root. We shall not require the explicit form of the S -matrix elements but only their spectral decompositions.

If we choose the convention that $\tilde{S}_{(k)}^{ab}(\theta)$ maps $V_a \otimes V_b$ to $V_a \otimes V_b$ (so with no permutation of the out-going vector spaces) then the spectral decomposition of the S -matrix element for $b \geq a$ is

$$\tilde{S}_{(k)}^{ab}(\theta) = \sigma_{(k)}^{ab}(\theta) \sum_{j=0}^{\min(m-b,a)} \rho_j^{ab}(\theta) \mathbb{P}_j, \quad (2.6)$$

where $\mathbb{P}_j$ is the quantum group projector onto the representation with highest weight $\omega_{a-j} + \omega_{b+j}$ that appears in the tensor product $V_a \otimes V_b$ with the understanding that the projector is zero if the highest weight does not satisfy $\mu \cdot \vartheta \leq k$. In the above, $\sigma_{(k)}^{ab}(\theta)$ and $\rho_j^{ab}(\theta)$ are scalar factors, symmetric in a and b , and with the normalization $\rho_0^{ab}(\theta) = 1$, which can be extracted from [11]. For physical values of rapidity and $b \geq a$ one can express them as

$$\sigma_{(k)}^{ab}(\theta) = \exp \int_{-\infty}^{\infty} \frac{dx}{x} e^{im\theta x/\pi} \frac{\sinh(ax) \sinh[(m+k-1)x] \sinh[(m-b)x]}{\sinh[(m+k)x] \sinh(mx) \sinh x}, \quad (2.7)$$

and for $j \geq 1$

$$\rho_j^{ab}(\theta) = c_j^{ab} \prod_{p=1}^j \frac{\sinh \left[\frac{m}{2(m+k)} \left(\theta + i \frac{\pi}{m} (2p + b - a) \right) \right]}{\sinh \left[\frac{m}{2(m+k)} \left(\theta - i \frac{\pi}{m} (2p + b - a) \right) \right]}, \quad (2.8)$$

where c_j^{ab} is a constant whose exact form we shall not require.

The diagonal S -matrix elements have the following important property:

$$S^{aa}(0) = -P, \quad (2.9)$$

where P is the permutation operator of the incoming states. This means that the particles are of “fermion-type” to use the terminology of [1].

3. The thermodynamic Bethe Ansatz

The goal of the TBA is to extract information about a integrable field theory from its factorizable S -matrix [1]. In particular one can probe the ultra-violet behaviour of the theory, which will be some massless theory and hence conformally invariant. In practice, it turns out that the central charge of the conformal field theory is the most accessible piece of information. The technique of the TBA has been dealt with at length elsewhere (see [1] and references therein) and so our *modus operandi* will be to only dwell on the details that are peculiar to the problem of dealing with non-diagonal S -matrices.

Since particles are neither created nor destroyed it makes sense to consider the thermodynamics of the one-dimensional gas of N particles on a circle with circumference L . Suppose that the j^{th} particle has rapidity θ_j and transforms in the representation a_j and hence has mass $m_j = m_{a_j}$.

The asymptotic wavefunction of the system, which is valid if the particles are far apart apart compared with the interaction length ($\sim m_0^{-1}$) has the familiar Bethe-Ansatz-type form

$$\Psi(x_1, \dots, x_N) = \exp \left(i \sum_{j=1}^N x_j m_j \sinh \theta_j \right) \sum_{Q \in S_N} \Theta(x_Q) \zeta(Q), \quad (3.1)$$

where the sum is over the permutations $Q = \{Q_1, \dots, Q_N\}$ of $\{1, 2, \dots, N\}$ with

$$\Theta(x_Q) = \begin{cases} 1 & \text{if } x_{Q_1} < x_{Q_2} < \dots < x_{Q_N} \\ 0 & \text{otherwise,} \end{cases} \quad (3.2)$$

and $\zeta(Q)$ is an element of the product of two copies of the tensor product $V_{a_1} \otimes V_{a_2} \otimes \dots \otimes V_{a_N}$. The coefficients $\zeta(Q)$ are related via the S -matrix in such a way that if Q differs from Q' only by a permutation of the i^{th} and j^{th} particles then

$$\zeta(Q') = S_{(k,l)}^{a_i a_j}(\theta_i - \theta_j) \zeta(Q), \quad i < j. \quad (3.3)$$

These relations determine all the $\zeta(Q)$ in terms of a reference value.

We now impose periodic boundary conditions on the asymptotic wavefunction:

$$\Psi(\dots, x_j, \dots) = \Psi(\dots, x_j + L, \dots), \quad \text{for } j = 1, 2, \dots, N. \quad (3.4)$$

This implies that the wavefunction must be a simultaneous eigenvector of the *transfer matrix* for bringing the j^{th} particle through the rest and back to its starting point:

$$\exp(im_j L \sinh \theta_j) T(\theta_j | \theta_{j+1}, \dots, \theta_N, \theta_1, \dots, \theta_{j-1}) \Psi = \Psi, \quad j = 1, \dots, N. \quad (3.5)$$

The transfer matrix $T(\theta_j | \theta_{j+1}, \dots, \theta_N, \theta_1, \dots, \theta_{j-1})$ is constructed out the S -matrix in the following way [3]. Consider the product of S -matrix elements involving an auxiliary particle with rapidity θ and colour index a :

$$T^a(\theta) = S^{a a_1}(\theta - \theta_1) S^{a a_2}(\theta - \theta_2) \dots S^{a a_N}(\theta - \theta_N), \quad (3.6)$$

where the labels $\{a_j\}$ and $\{\theta_j\}$ are understood. The transfer matrices are now given by

$$T(\theta_j | \theta_{j+1}, \dots, \theta_N, \theta_1, \dots, \theta_{j-1}) = -\text{tr}_0 T^{a=a_j}(\theta = \theta_j), \quad (3.7)$$

where the trace is over the representation a of the auxiliary particle. In order to prove (3.7) one uses the fact (2.9) that the diagonal S -matrix element at zero rapidity difference is minus the permutation. The advantage of writing the transfer matrix in this way is that the standard techniques pioneered in the field of integrable lattice models may be wheeled-in to analyze its spectrum. The only difference in the present context is that there are local inhomogeneities specified by the rapidities $\{\theta_j\}$ and the particles carry varying colour indices $\{a_j\}$.

The fact that the S -matrix satisfies the Yang-Baxter equation guarantees that the $\text{tr}_0 T^a(\theta)$ commute for different values of θ and a ; hence the wavefunction can indeed be a simultaneous eigenvector of the transfer matrices for different j . The diagonalization of the $\text{tr}_0 T^{a_j}(\theta_j)$ can be accomplished via standard Bethe Ansatz techniques. Recall that $\text{tr}_0 T^{a_j}(\theta_j)$ acts on two copies of $V_{a_1} \otimes \dots \otimes V_{a_N}$ (reflecting the product form of the S -matrix) and is furthermore commutant to the action of the quantum group implying

that the eigenvectors correspond to certain irreducible subspaces of $V_{a_1} \otimes \cdots \otimes V_{a_N}$. The eigenvalues are a sum of terms and depend upon a set of parameters which satisfy the Bethe Ansatz equations [6,13]. Fortunately, we do not require the exact expression for the eigenvalues (which are only available in limited cases) rather only the dominant term in the thermodynamic limit ($N \rightarrow \infty$). Even so, this problem has not been solved in general and we must rely on a conjecture [6,13] which reproduces all the known results.

The conjecture implies that the dominant contribution to the eigenvalues of $\text{tr}_0 T^{a_j}(\theta_j)$ is expressed as

$$\begin{aligned} & \epsilon_j \prod_{\substack{i=1 \\ \neq j}}^N \left[X^{a_j a_i}(\theta_j - \theta_i) \sigma_{(k)}^{a_j a_i}(\theta_j - \theta_i) \sigma_{(l)}^{a_j a_i}(\theta_j - \theta_i) \right] \\ & \times \prod_{\alpha=1}^{M_{a_j}} \frac{\sinh \left[\frac{m}{2(m+k)} \left(\theta_j - u_{\alpha}^{(a_j)} + i \frac{\pi}{m} \right) \right]}{\sinh \left[\frac{m}{2(m+k)} \left(\theta_j - u_{\alpha}^{(a_j)} - i \frac{\pi}{m} \right) \right]} \prod_{\alpha=1}^{M'_{a_j}} \frac{\sinh \left[\frac{m}{2(m+l)} \left(\theta_j - v_{\alpha}^{(a_j)} + i \frac{\pi}{m} \right) \right]}{\sinh \left[\frac{m}{2(m+l)} \left(\theta_j - v_{\alpha}^{(a_j)} - i \frac{\pi}{m} \right) \right]}, \end{aligned} \quad (3.8)$$

where ϵ_j is a constant whose explicit form we shall not require. The eigenvalues, for given whole numbers M_a and M'_a , $a = 1, 2, \dots, m-1$ (which are subject to constraints to be discussed shortly), are then determined by the parameters $u_{\alpha}^{(a)}$ and $v_{\alpha}^{(a)}$ which are separately solutions of a set Bethe Ansatz equations:

$$\begin{aligned} & \prod_{j=1}^N \frac{\sinh \left[\frac{m}{2(m+k)} \left(u_{\alpha}^{(a)} - \theta_j + i \frac{\pi}{m} \omega_{a_j} \cdot \alpha_a \right) \right]}{\sinh \left[\frac{m}{2(m+k)} \left(u_{\alpha}^{(a)} - \theta_j - i \frac{\pi}{m} \omega_{a_j} \cdot \alpha_a \right) \right]} \\ & = \Omega_{\alpha}^{(a)} \prod_{b=1}^{m-1} \prod_{\beta=1}^{M_b} \frac{\sinh \left[\frac{m}{2(m+k)} \left(u_{\alpha}^{(a)} - u_{\beta}^{(b)} + i \frac{\pi}{m} \alpha_a \cdot \alpha_b \right) \right]}{\sinh \left[\frac{m}{2(m+k)} \left(u_{\alpha}^{(a)} - u_{\beta}^{(b)} - i \frac{\pi}{m} \alpha_a \cdot \alpha_b \right) \right]}, \end{aligned} \quad (3.9)$$

where $\Omega_{\alpha}^{(a)}$ is a phase appropriate to the RSOS Bethe Ansatz, whose explicit form we shall not require. The $v_{\alpha}^{(a)}$ satisfy an analogous set of equations with k replaced by l and M_a by M'_a . In (3.9) the α_a 's are the simple roots of a_{m-1} dual to the fundamental weights: $\alpha_a \cdot \omega_b = \delta_{ab}$.

The eigenvalue (3.8) is associated to a representation of the product $a_{m-1} \times a_{m-1}$ with highest weight

$$(\mu, \mu') = \left(\sum_{i=1}^N \omega_{a_i} - \sum_{a=1}^{m-1} M_a \alpha_a, \sum_{i=1}^N \omega_{a_i} - \sum_{a=1}^{m-1} M'_a \alpha_a \right). \quad (3.10)$$

The fact that μ and μ' have to be highest weights and furthermore $\mu \cdot \vartheta \leq k$ and $\mu' \cdot \vartheta \leq l$ imposes restrictions on the possible choices for M_a and M'_a .

In the thermodynamic limit ($N \rightarrow \infty$ and $L \rightarrow \infty$) the number of particles becomes large and the separation between the states in rapidity-space becomes small, so that it is meaningful to discuss the density of particles in rapidity-space (per unit length of real

space) with colour index a , a quantity we denote $\sigma^a(\theta)$. As usual we also introduce $\tilde{\sigma}^a(\theta)$ the density of unoccupied one-particle states (or holes) states so that the density of one-particle states with colour index a is then $\sigma^a(\theta) + \tilde{\sigma}^a(\theta)$. In taking the thermodynamic limit of the Bethe Ansatz equations (3.9) we have to employ the standard conjecture concerning the nature of the solutions which dominate in the thermodynamic limit [6]. This is the string hypothesis that maintains that the dominating solutions are ‘ p -strings’ having the form

$$u^{(a)} = u_0^{(a)} + i\frac{\pi}{m}(p+1-2j), \quad j = 1, 2, \dots, p, \quad (3.11)$$

where the centre of the string $u_0^{(a)} \in \mathbb{R}$ and $p = 1, 2, \dots, k$. We then introduce the densities of the centres of occupied p -strings and their associated holes $r_p^a(\theta)$ and $\tilde{r}_p^a(\theta)$, respectively. A similar picture holds for the Bethe Ansatz equations for the $v^{(a)}$, requiring the introduction of p -string densities $s_p^a(\theta)$ and $\tilde{s}_p^a(\theta)$, where in this case $p = 1, 2, \dots, l$. The r and s densities are often called *magnon* densities.

The next step is to take the logarithm of the eigenvalue equation (3.5) and then differentiate with respect to θ_j . In the thermodynamic limit the discrete sums are replaced by integrals over the densities that we introduced above and the resulting integral equations just depends on the colour index a_j and not on j itself. In taking the thermodynamic limit it is important that the particles are of fermionic-type. With this in mind, the eigenvalue equations (3.5) become the coupled integral equations

$$\frac{m_a}{2\pi} \cosh \theta + \gamma_{(k,l)}^{ab} * \sigma^b(\theta) - a_p^{(m+k)} * r_p^a(\theta) - a_p^{(m+l)} * s_p^a(\theta) = \sigma^a(\theta) + \tilde{\sigma}^a(\theta), \quad (3.12)$$

for $a = 1, 2, \dots, m-1$ (repeated indices are summed) where $*$ denotes the convolution $f * g(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta')g(\theta')$, and the kernels are found to be after some algebra

$$\begin{aligned} \gamma_{(k,l)}^{ab}(\theta) &= \frac{1}{2\pi i} \frac{d}{d\theta} \log \left[X^{ab}(\theta) \sigma_{(k)}^{ab}(\theta) \sigma_{(l)}^{ab}(\theta) \right] \\ &= \delta(\theta) \delta_{ab} - \frac{m}{\pi} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{im\theta x/\pi} \frac{\sinh(ax) \sinh[(m-b)x] \sinh[(2m+k+l)x]}{\sinh(mx) \sinh[(m+k)x] \sinh[(m+l)x]} \\ &= \delta(\theta) \delta_{ab} - \left[A_{m+l, m+l}^{(2m+k+l)} \right]^{-1} * A_{ab}^{(m)}(\theta), \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} a_p^{(m+k)}(\theta) &= \frac{1}{2\pi i} \frac{d}{d\theta} \sum_{j=1}^p \log \frac{\sinh \left[\frac{m}{2(m+k)} \left(\theta - i\frac{\pi}{m}(p+2-2j) \right) \right]}{\sinh \left[\frac{m}{2(m+k)} \left(\theta + i\frac{\pi}{m}(p+2-2j) \right) \right]} \\ &= \frac{m}{\pi} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{im\theta x/\pi} \frac{\sinh[(m+k-p)x]}{\sinh[(m+k)x]}, \end{aligned} \quad (3.14)$$

where $A_{ab}^{(m)}(\theta)$ is defined in terms of $\hat{A}_{ab}^{(m)}(x)$ in (2.5) using the general definition of the Fourier transform

$$f(\theta) = \frac{m}{\pi} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{im\theta x/\pi} \hat{f}(x), \quad (3.15)$$

and we define

$$[f]^{-1}(\theta) = \frac{m}{\pi} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{im\theta x/\pi} \frac{1}{\hat{f}(x)}. \quad (3.16)$$

In a similar way, the Bethe Ansatz equations yield in the thermodynamic limit [6]

$$\begin{aligned} \tilde{r}_p^a(\theta) + A_{pq}^{(m+k)} * K_{ab}^{(m)} * r_q^b(\theta) &= a_p^{(m+k)} * \sigma^a(\theta), \quad p = 1, 2, \dots, k \\ \tilde{s}_p^a(\theta) + A_{pq}^{(m+l)} * K_{ab}^{(m)} * s_q^b(\theta) &= a_p^{(m+l)} * \sigma^a(\theta), \quad p = 1, 2, \dots, l, \end{aligned} \quad (3.17)$$

where

$$K_{ab}^{(m)}(\theta) = \frac{m}{\pi} \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{im\theta x/\pi} \left(\delta_{ab} - \frac{1}{2 \cosh x} I_{ab}^{(m)} \right), \quad (3.18)$$

and $I^{(m)}$ is the incidence matrix of the a_{m-1} Dynkin diagram. It can be shown [7] that $K_{ab}^{(m)}(\theta)$ is actually the inverse of $A_{ab}^{(m)}(\theta)$ in the sense that $K_{ab}^{(m)} * A_{bc}^{(m)}(\theta) = \delta(\theta) \delta_{ac}$.

Using (3.13) equation (3.12) can be rewritten as

$$\tilde{\sigma}^a(\theta) + \left[A_{m+l, m+l}^{(2m+k+l)} \right]^{-1} * A_{ab}^{(m)} * \sigma^b(\theta) + a_p^{(m+k)} * r_p^a(\theta) + a_p^{(m+l)} * s_p^a(\theta) = \frac{m_a}{2\pi} \cosh \theta. \quad (3.19)$$

The equations can be simplified because we shall argue that the k -strings in the r -system and the l -strings in the s -system do not contribute in the thermodynamic limit. Consider the zero-mode of the r -system (3.17) at $p = k$:

$$\hat{r}_k^a(0) + \sum_{b=1}^{m-1} \sum_{q=1}^k \frac{qm}{m+k} C_{ab} \hat{r}_q^b(0) = \frac{m}{m+k} \hat{\sigma}^a(0), \quad (3.20)$$

where C_{ab} is the Cartan matrix of a_{m-1} . In the thermodynamic limit the highest weight (3.10) of the eigenvector becomes

$$\mu = L \sum_{a=1}^{m-1} \left(\hat{\sigma}^a(0) \omega_a - \sum_{q=1}^k q \hat{r}_q^a(0) \alpha_a \right) = L \frac{m+k}{m} \sum_{a=1}^{m-1} \hat{r}_k^a(0) \omega_a, \quad (3.21)$$

by (3.20). But we know that μ is bounded by $\mu \cdot \vartheta \leq k$ and so the number of k -string holes of $u^{(a)}$ goes to zero in the thermodynamic limit and so we can safely take $\hat{r}_k^a(\theta) = 0$ and similarly $\tilde{s}_l^a(\theta) = 0$ in (3.17) and (3.19), for $a = 1, 2, \dots, m-1$. With $\hat{r}_k^a(\theta) = 0$ one can eliminate $r_k^a(\theta)$ via

$$r_k^a(\theta) = - \left[A_{kk}^{(m+k)} \right]^{-1} * A_{kp}^{(m+k)} * r_p^a(\theta) + \left[A_{kk}^{(m+k)} \right]^{-1} * a_k^{(m+k)} * A_{ab}^{(m)} * \sigma^b(\theta), \quad (3.22)$$

and similarly for $s_l^a(\theta)$. Substituting for $r_k^a(\theta)$ and $s_l^a(\theta)$ in (3.19) and after some algebra (in particular the identities in appendix A of [7] are helpful) leads finally to

$$\tilde{\sigma}^a(\theta) + \left[A_{ll}^{(k+l)} \right]^{-1} * A_{ab}^{(m)} * \sigma^b(\theta) + a_p^{(k)} * r_p^a(\theta) + a_p^{(l)} * s_p^a(\theta) = \frac{m_a}{2\pi} \cosh \theta. \quad (3.23)$$

In a similar way (3.17) becomes

$$\begin{aligned}\tilde{r}_p^a(\theta) + A_{pq}^{(k)} * K_{ab}^{(m)} * r_q^b(\theta) &= a_p^{(k)} * \sigma^a(\theta) \\ \tilde{s}_p^a(\theta) + A_{pq}^{(l)} * K_{ab}^{(m)} * r_q^b(\theta) &= a_p^{(l)} * \sigma^a(\theta),\end{aligned}\tag{3.24}$$

where the string labels on r now run from 1 to $k-1$ and those on s run from 1 to $l-1$. Notice that (3.23) and (3.24) are identical to the Bethe Ansatz equations for the integrable lattice model in regime I of [6,7].

The equations can be amalgamated by defining

$$\rho_p^a(\theta) = \begin{cases} \tilde{s}_{l-p}^a(\theta) & p = 1, 2, \dots, l-1 \\ \sigma^a(\theta) & p = l \\ \tilde{r}_{p-l}^a(\theta) & p = l+1, l+2, \dots, l+k-1, \end{cases}\tag{3.25}$$

and

$$\tilde{\rho}_p^a(\theta) = \begin{cases} s_{l-p}^a(\theta) & p = 1, 2, \dots, l-1 \\ \tilde{\sigma}^a(\theta) & p = l \\ r_{p-l}^a(\theta) & p = l+1, l+2, \dots, l+k-1. \end{cases}\tag{3.26}$$

After some algebra, and using the fact that $K_{pq}^{(k+l)}(\theta)$ is the inverse of $A_{pq}^{(k+l)}(\theta)$, it follows that (3.23) and (3.24) are subsumed by

$$\tilde{\rho}_p^a(\theta) + K_{pq}^{(k+l)} * A_{ab}^{(m)} * \rho_q^b(\theta) = \delta_{pl} \frac{m_a}{2\pi} \cosh \theta,\tag{3.27}$$

where the string label on ρ runs from 1 to $k+l-1$.

We are now in a position to discuss the thermodynamics of the system at a temperature T . To this end we note that the energy per unit length of a solution to the equations is

$$\mathcal{E} = \int_{-\infty}^{\infty} d\theta \sigma^a(\theta) m_a \cosh \theta \equiv \int_{-\infty}^{\infty} d\theta \rho_l^a(\theta) m_a \cosh \theta,\tag{3.28}$$

where as usual there is an implied sum over the repeated index. The entropy per unit length of a solution is given by

$$\mathcal{S} = \int_{-\infty}^{\infty} d\theta \{ (\rho_p^a + \tilde{\rho}_p^a) \log (\rho_p^a + \tilde{\rho}_p^a) - \rho_p^a \log \rho_p^a - \tilde{\rho}_p^a \log \tilde{\rho}_p^a \},\tag{3.29}$$

which is the form appropriate to a set of particles of fermionic-type. The next stage is to minimize the free energy per unit length $\mathcal{F} = \mathcal{E} - T\mathcal{S}$ subject to the constraint (3.27). The resulting equation for the densities is

$$\frac{1}{T} \delta_{pl} m_a \cosh \theta = \epsilon_p^a(\theta) + L_p^a(\theta) - K_{pq}^{(k+l)} * A_{ab}^{(m)} * L_q^b(\theta),\tag{3.30}$$

where we used the fact that $K_{pq}^{(k+l)}(\theta)$ is the inverse of $A_{pq}^{(k+l)}(\theta)$ and we have defined

$$\frac{\tilde{\rho}_p^a(\theta)}{\rho_p^a(\theta)} = \exp \epsilon_p^a(\theta), \quad L_p^a(\theta) = \log \left(1 + e^{-\epsilon_p^a(\theta)} \right).\tag{3.31}$$

Once these equations are solved the free energy per unit length is given by

$$\mathcal{F} = -\frac{T}{2\pi} \int_{-\infty}^{\infty} d\theta L_l^a(\theta) m_a \cosh \theta. \quad (3.32)$$

This completes the derivation of the TBA system resulting from the trigonometric S -matrices.

To make contact with [4] one defines

$$\frac{1}{2\pi} \phi^{ab}(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \log X^{ab}(\theta) = \delta_{ab} \delta(\theta) - A_{ab}^{(m)}(\theta), \quad (3.33)$$

and $\psi^{ab}(\theta)$ via its Fourier transform:

$$\frac{1}{2\pi} \hat{\psi}^{ab}(x) = \frac{1}{2 \cosh x} \left(\delta_{ab} - \frac{1}{2\pi} \hat{\phi}^{ab}(x) \right). \quad (3.34)$$

In terms of these quantities (3.30) becomes precisely equation (2) of the first reference of [4] (except for the error in sign):

$$\frac{1}{T} \delta_{pl} m_a \cosh \theta = \epsilon_p^a(\theta) + \frac{1}{2\pi} \left(\phi^{ab} * L_p^b(\theta) + I_{pq}^{(k+l)} \psi^{ab} * L_q^b(\theta) \right), \quad (3.35)$$

where $I^{(k+l)}$ is the incidence matrix of the a_{k+l-1} Dynkin diagram.

In the limit $k, l \rightarrow \infty$ the TBA system is also identical to that proposed for the $SU(m)$ principle chiral model [14] reflecting the fact that the trigonometric S -matrix also reduces to the conjectured S -matrix of this model in that limit. Also when $k = l = 1$ (3.35) reduces to that TBA system for the a_{m-1} purely elastic scattering theory as it should since in this case the S -matrix collapses to the scalar factor $S_{(1,1)}^{ab}(\theta) = X^{ab}(\theta)$.

4. Discussion

We have shown how the TBA systems for the algebras a_{m-1} are associated to the trigonometric S -matrices. One point which is worth emphasizing is that in going from the S -matrices to the TBA equations one uses the expression for dominant contribution to the eigenvalues of the transfer matrices in thermodynamic limit. It seems, therefore, that it is not possible to reverse the process and deduce the S -matrix elements from the TBA equations. Unfortunately, this seems to preclude a deduction of the S -matrices associated to other algebras (only the full S -matrices of the a_{m-1} and c_m theories are known [11]) from the TBA equations.

Using the analysis of [4] we can now investigate the UV-limit of the scattering theory defined in section 2. The UV limit corresponds taking to $T \rightarrow \infty$. The free energy can be interpreted as R^{-1} times the Casimir energy of the quantum field theory on a periodic strip with period $R = 1/T$. It is known from the theory of finite-size effects that the Casimir

energy of the periodic strip in the UV-limit gives the central charge c of the resulting conformal field theory via

$$\lim_{R \rightarrow 0} R^2 \mathcal{F}(R) = -\frac{\pi c}{6}. \quad (4.1)$$

The central charge that one finds is that of the coset $g_k \times g_l / g_{k+l}$ where $g = a_{m-1}$.

The construction of the TBA system has been based on the S -matrix of (2.2). It is possible to write down an S -matrix having the more general form

$$S_{(k_1, k_2, \dots, k_s)}^{ab}(\theta) = X^{ab}(\theta) \tilde{S}_{(k_1)}^{ab}(\theta) \otimes \tilde{S}_{(k_2)}^{ab}(\theta) \otimes \dots \otimes \tilde{S}_{(k_s)}^{ab}(\theta), \quad (4.2)$$

for a set of natural numbers $\{k_1, k_2, \dots, k_s\}$, where the particles now transform in the representations $V_a^{\otimes s}$. It would be interesting to calculate the central charge of the UV-limit of this more general theory.

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