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Feedback System for Elimination of the Transverse Mode Coupling Instability

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Abstract

This report is devoted to a theoretical analysis of the effect of transverse feedback systems on the threshold of the Transverse Mode Coupling (TMC) instability. In the framework of a simplified model, a principle of TMC elimination by means of a special transverse feedback system is outlined. A reciprocal spectral property of the coupled mode system is established. Then a general approach including a many-mode system coupled to feedback oscillators is presented and the determination of the appropriate feedback parameters is reduced to an algorithm.

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1 Introduction

The transverse mode coupling (TMC) instability is one of the major limitations of the single bunch current in large storage rings. The recently proposed description of bunch collective motion in the space-time domain [1,2] leads to a set of equations, which give the eigenvalues of the head-tail modes and hence answer the question of their stability.

This report is devoted to a theoretical analysis of the effect of transverse feedback systems on TMC thresholds. Section 2 presents the space-time formalism for the hollow beam model and time-independent collective interaction. Then, in the framework of a simplified model a principle of eliminating the TMC by means of a special transverse feedback system is outlined in Section 3.

The “hollow beam” formalism is further developed in Section 4 for a practically important case of localized impedances. In Section 5 a given distribution over the synchrotron oscillation amplitudes is introduced, non-linearity of the synchrotron motion is included, and a numerical procedure is constructed for evaluating the radial modes, which appear in any realistic beam model.

An important reciprocal property of the eigenvalues for the TMC modes is proved in Section 6. Finally, a general algorithm for elimination of the TMC effect by a special arrangement of a transverse feedback system is developed in Section 7 for the general problem.

2 “Hollow Beam” Equations

Let us consider a bunch model (often called a “hollow beam model”) in which all particles have equal synchrotron oscillation amplitude a_s and are uniformly distributed over the synchrotron oscillation phase ϕ_s .

To simplify the following equations of transverse motion, the normalized betatron displacement $x/\sqrt{\beta}$ and its “time-derivative” are taken as dynamical variables, while the role of independent time-like variable t is given to the Floquet phase divided by the betatron frequency ω_β : $dt = ds_0/\omega_\beta\beta$, instead of the machine azimuth s_0 .

In these variables the transfer matrix takes a very simple form:

$$\begin{bmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{bmatrix}$$

here

$$\mu = \int \frac{ds_0}{\beta} = \omega_\beta \int dt$$

is the betatron phase advance and integration runs over the length of the lattice section.

2.1 Dipole Moments in Forward and Backward Fluxes

All the particles in the bunch rotate around a circle of radius a_s in the synchrotron phase plane, their position with respect to the equilibrium particle being given by the relative longitudinal coordinate $s = a_s \cos \phi_s$. We introduce the variables $D(s, t)$, $D'(s, t)$ which denote the sum of dipole moments and the sum of their derivatives of those particles which

have, at the time t , the same longitudinal displacement s . Particles belonging to two different synchrotron phases: $\pm\phi_s$ contribute to this sum.

It is also useful to define the variables $\overline{D}(s, t), \overline{D}'(s, t)$ as the difference of dipole moments of the particles belonging to $+\phi_s$ and $-\phi_s$ and the difference of their derivatives.

In terms of these variables the betatron motion with the frequency ω_β obeys the standard set of oscillator equations:

$$\begin{aligned} dD/dt &= \omega_\beta D' & d\overline{D}/dt &= \omega_\beta \overline{D}' \\ dD'/dt &= -\omega_\beta D & d\overline{D}'/dt &= -\omega_\beta \overline{D} \end{aligned}$$

In what follows we will use a single longitudinal variable ϕ (instead of s) in order to describe the particles with both positive and negative synchrotron phases: $\phi = |\phi_s|$. Note, that the synchrotron phase itself is inconvenient in such a role because it increases from the head to the tail of the bunch in the upper half of the “synchrotron circle” and decreases in its lower half.

We rewrite the total time derivatives in terms of the partial derivatives, and observe the distinction between the “+” and “−” particles (the upper and lower ones)¹:

$$\frac{dD}{dt} = \frac{dD^+}{dt} + \frac{dD^-}{dt} = \frac{\partial D}{\partial t} + \omega_s \left(\frac{\partial D^+}{\partial \phi} - \frac{\partial D^-}{\partial \phi} \right) = \frac{\partial D}{\partial t} + \omega_s \frac{\partial \overline{D}}{\partial \phi},$$

Here the minus sign appears from $\partial\phi/\partial t = -\omega_s$ for the lower particles with $\phi_s < 0$. Transforming all the total time derivatives like this we obtain:

$$\begin{aligned} \frac{\partial D}{\partial t} + \omega_s \frac{\partial \overline{D}}{\partial \phi} &= \omega_\beta D' \\ \frac{\partial D'}{\partial t} + \omega_s \frac{\partial \overline{D}'}{\partial \phi} &= -\omega_\beta D \\ \frac{\partial \overline{D}}{\partial t} + \omega_s \frac{\partial D}{\partial \phi} &= \omega_\beta \overline{D}' \\ \frac{\partial \overline{D}'}{\partial t} + \omega_s \frac{\partial D'}{\partial \phi} &= -\omega_\beta \overline{D} \end{aligned} \tag{1}$$

2.2 Structure of Eigenfunctions at Zero Intensity

Our next step is the construction of the spatial eigenfunctions, and they are sought in the form of a Fourier series in the interval $0 \leq \phi \leq \pi$. Since $\overline{D}$ is the difference of dipole moments for the “+” and “−” particles, we have $\overline{D}|_{\phi=0,\pi} = 0$ at the return points of the synchrotron oscillation, *i.e.* at the ends of the interval, occupied by the “hollow beam” bunch. This boundary condition for the set of equations (1) clearly determines the form of the desired harmonics as $\overline{D}, \overline{D}' \propto \sin m\phi$, and $D, D' \propto \cos m\phi$.

¹Above transition the lower part of the hollow beam forms a *forward flux* in the synchrotron motion, while its upper part forms a *backward flux*, *cf.* [1,2].

Substituting $D = d \cos m\phi$, $D' = d' \cos m\phi$, etc., and eliminating the spatial functions in (1), we obtain for the amplitudes $d, d', \bar{d}, \bar{d}'$ of the m th harmonic:

$$\begin{aligned}\frac{\partial d}{\partial t} + m\omega_s \bar{d} &= \omega_\beta d' \\ \frac{\partial d'}{\partial t} + m\omega_s \bar{d}' &= -\omega_\beta d \\ \frac{\partial \bar{d}}{\partial t} - m\omega_s d &= \omega_\beta \bar{d}' \\ \frac{\partial \bar{d}'}{\partial t} - m\omega_s d' &= -\omega_\beta \bar{d},\end{aligned}\tag{2}$$

Rearrangement of these equations reveals the eigenfrequencies for the modal combinations $X_m = d + \bar{d}'$, $X'_m = d' - \bar{d}$:

$$\begin{aligned}\frac{\partial X_m}{\partial t} &= (\omega_\beta + m\omega_s) X'_m \\ \frac{\partial X'_m}{\partial t} &= -(\omega_\beta + m\omega_s) X_m\end{aligned}\tag{3}$$

Similarly, for the two complementary modal combinations, $Y_m = d - \bar{d}'$, $Y'_m = d' + \bar{d}$, we have:

$$\begin{aligned}\frac{\partial Y_m}{\partial t} &= (\omega_\beta - m\omega_s) Y'_m \\ \frac{\partial Y'_m}{\partial t} &= -(\omega_\beta - m\omega_s) Y_m\end{aligned}\tag{4}$$

Thus, X_m and Y_m exhibit the structure of the eigenfunctions at zero intensity. At $m = 0$ they do not differ, $X_0 = Y_0$, at $m \neq 0$ it is convenient to introduce negative numbers: $X_{-m} = Y_m$ in order to simplify the notation.

2.3 Equations Including A Collective Force

The set of equations (1) can be modified to include a collective force F in the dipole approximation:

$$\begin{aligned}\frac{\partial D}{\partial t} + \omega_s \frac{\partial \bar{D}}{\partial \phi} &= \omega_\beta D' \\ \frac{\partial D'}{\partial t} + \omega_s \frac{\partial \bar{D}'}{\partial \phi} &= -\omega_\beta D + 2F \\ \frac{\partial \bar{D}}{\partial t} + \omega_s \frac{\partial D}{\partial \phi} &= \omega_\beta \bar{D}' \\ \frac{\partial \bar{D}'}{\partial t} + \omega_s \frac{\partial D'}{\partial \phi} &= -\omega_\beta \bar{D}\end{aligned}\tag{5}$$

Note that the R.H.S. of the fourth equation does not involve the dipole collective force: the difference of dipole moments (for $\pm\phi_s$) in this equation results in a cancellation of collective

forces because this *dipole* force is uniform for all the test particles at given s (unlike the lattice focusing force, it does not depend on test particle displacements).

Substituting again

$$D, D' = \sum_{m=-\infty}^{\infty} d_m, d'_m \cos m\phi ,$$

and

$$\overline{D}, \overline{D}' = \sum_{m=-\infty}^{\infty} \overline{d}_m, \overline{d}'_m \sin m\phi ,$$

multiplying the 1st and 3rd equations in (5) by $\cos m\phi$, the 2nd and 4th equations by $\sin m\phi$, integrating from 0 to π and rearranging (5) similarly as in the previous section (in order to express dipole moments in terms of the eigenfunctions X_m, Y_m), we obtain a set of coupled equations

$$\ddot{X}_m + (\omega_\beta + m\omega_s)^2 X_m = \frac{2}{\pi} \int_0^\pi F \cos m\phi d\phi . \quad (6)$$

These equations are to be solved for the eigenfunctions of the finite intensity problem, as expressed via the basic set of X_m .

2.4 Example: Time-Independent Collective Force

For the sake of simplicity we take a time-independent collective force (such as appears in a beam pipe of uniform cross-section). By its definition, the dipole force exerted by a leading particle at s' on a test trailing particle at s is given by the convolution: $e^2 \cdot x(s') * W(s - s')$, where x is the leading particle displacement and W stands for the transverse wake function. Then for the collective force we have:

$$F(\phi) = k \int_0^\phi D(\phi') W(a \cos \phi - a \cos \phi') d\phi' \quad (7)$$

where the intensity parameter

$$k = \frac{N e^2}{\pi \gamma m_0}$$

combines the bunch intensity N , the Lorentz factor γ and the particle rest mass m_0 and includes π which appeared on the R.H.S. of (6).

For this model, namely a hollow beam and an evenly distributed impedance, one can illustrate the idea of elimination of the TMC instability and draw a number of conclusions which will also be useful in more general cases.

From the relations of the eigenfunctions to the dipole moments $X_m = d + \overline{d}'$ and $Y_m = X_{-m} = d - \overline{d}'$ we have:

$$D = \frac{1}{2} \sum_{m'=-\infty}^{\infty} X_{m'} \cos m'\phi .$$

Using the explicit expression for the force we rewrite (6):

$$\ddot{X}_m + (\omega_\beta + m\omega_s)^2 X_m = k \int_0^\pi \cos m\phi d\phi \int_0^\phi \sum_{m'=-\infty}^{\infty} X_{m'} \cos m'\phi' W(a \cos \phi - a \cos \phi') d\phi'$$

Taking the Fourier sums out of the integrals and introducing the mode interaction matrix $K_{mm'}$ on the R.H.S.

$$K_{mm'} = k \int_0^\pi \cos m\phi d\phi \int_0^\phi \cos m'\phi' W(a \cos \phi - a \cos \phi') d\phi' , \quad (8)$$

we obtain a compact form of the “coupled modes” equation:

$$\ddot{X}_m + (\omega_\beta + m\omega_s)^2 X_m = \sum_{m'=-\infty}^{\infty} K_{mm'} X_{m'} . \quad (9)$$

Note (from the properties of the mode interaction matrix given in Section 6) that $K_{mm'}$ is not symmetric with respect to interchange of m and m' :

$$K_{mm'} = (-1)^{m+m'} K_{m'm} .$$

This is a manifestation of the non-Hamiltonian nature of such a dynamical system: it is impossible to obtain all the components of such a (linear) force by differentiation of a unique function, *i.e.* the Hamiltonian, with respect to the relevant variables X_m, X'_m . This asymmetry allows a physical interpretation of the synchrobetatron mode coupling in which modes merge and imaginary mode frequencies emerge above a certain threshold. That would look unnatural in a Hamiltonian system, where the interaction between modes usually results in a larger separation of their eigenfrequencies, unless some of the modes have negative effective masses,² see the Hamiltonian in the next subsection.

We see that modes subjected to coupling obey a certain selection rule ($m + m'$ must be an odd number), which means that their interaction is antisymmetric, for example the neighbouring $m = 0$ and $m = -1$ modes can couple.

2.5 Eigenvalues in Reciprocal Pairs

The absolute values of the eigenvalues $\lambda = e^{i\omega t}$ are of particular interest, and the relevant imaginary parts of the eigenfrequencies ω are desired to provide stability of all modes. However, the systems in question appear to have reciprocal pairs of eigenvalues, similarly to symplectic maps, *i.e.*, for any eigenvalue λ , there exists an eigenvalue equal to $1/\lambda$ in the spectrum of this problem.

The following simple example illustrates this property. Consider a two-dimensional oscillator with an antisymmetric coupling term:

$$\begin{aligned} \dot{U} &= P_u \\ \dot{P}_u + \omega_u^2 U &= -kV \\ \dot{V} &= P_v \\ \dot{P}_v + \omega_v^2 V &= kU \end{aligned} \quad (10)$$

Here U, P_u and V, P_v stand for the canonically conjugate variables of any two coupled head-tail modes involved in (9), and k is the relevant matrix element of the mode interaction matrix $K_{mm'}$, responsible for coupling these two modes (and proportional to the beam intensity).

²This last remark is due to B. Zotter.

If we perform on (10) a non-canonical transformation which changes U for $U = -\bar{U}$ and leaves P_u unaltered, we can obtain a Hamiltonian:

$$\mathcal{H} = \frac{1}{2}(\omega_v^2 V^2 + P_v^2) - \frac{1}{2}(\omega_u^2 \bar{U}^2 + P_u^2) + kV\bar{U}$$

which is non-trivial in that it involves the two oscillatory parts with different signs: the transformation reversed the sign of the U -oscillator's effective mass. If we use $\mathcal{H}$ to derive the Hamiltonian equations and then make an inverse transformation $\bar{U} = -U$, we come back to the initial set of equations (10). Therefore such a system is (formally) reducible to a Hamiltonian form. The non-canonical transformation employed is equivalent to a similarity transformation which must preserve eigenvalues. Thus, our example of a system with mode coupling must have the eigenvalues in reciprocal pairs.

The non-canonical transformation employed in the above two-mode example can be extended to the multi-mode system where such a change of variables should be done in all odd-numbered (or in all even-numbered) terms in order to provide the Hamiltonian form of the coupled equations, see Section 6. Hence the conclusion on reciprocal pairs holds generally.

Of course, this conclusion for a uniformly distributed interaction force can be simply drawn from the real form of the coefficients in these equations. The above reasoning however will be generalized for localized interaction with an arbitrary azimuthal distribution in Sections 4,6.

3 Principle of Elimination of the TMC Instability

Feedback systems give the most universal cure for collective instabilities. Usually they treat multi-bunch and/or multi-turn instabilities where short bunches are considered as rigid and internal motion is disregarded. A feedback system, which cannot resolve the intra-bunch structure due to its limited bandwidth, and acts equally on all particles in a short bunch, is apparently capable of curing the rigid bunch effects.

The TMC instability is of quite another nature, its essential feature is the ultra-relativistic “causality” of the interaction in the bunch: its leading particles are free from interaction with the collective field, while the trailing particles are affected by the intensity-dependent force³. Compensation of such collective forces by means of a feedback system seems to be a complicated problem and is not feasible for very short bunches.

In what follows the feedback system will act equally on all the particles in the bunch, so that it couples only with the harmonic X_0 , since all other zero-intensity modes in the basic set X_m have zero dipole moment.

To begin with, we consider the same example (10) with only two antisymmetrically coupled modes. For the sake of simplicity their self-action as well as coupling to all other modes are neglected. The coupling term k in (10) is proportional to the bunch intensity and ω_u , ω_v are then the zero-intensity mode frequencies. The eigenfrequencies are readily obtained:

$$\omega^2 = \frac{1}{2}[\omega_u^2 + \omega_v^2 \pm \sqrt{(\omega_u^2 - \omega_v^2)^2 - 4k^2}] ,$$

³*c.f.* the alternative name of this instability: a strong head-tail effect.

and the threshold at $|k| = |\omega_u^2 - \omega_v^2|/2$ is evident.

Disregarding the details of the transverse feedback system design and operation, we note that in principle its response function (we mean the transfer function between the pick-up signal and the kicker action) can have a rather general form. Here the feedback representation with an oscillator (or an LC-circuit) coupled to the $m = 0$ mode, named U , will suffice. We take the feedback oscillator's coupling with the $m = 0$ mode equal to k , but put it symmetrically into the set of equations, *i.e.*, in a Hamiltonian manner. The feedback frequency is set equal to that of the mode V (say, with $m = -1$). According to what is stated above, the feedback oscillator Z is not coupled to the mode V . Then we have:

$$\begin{aligned}
\dot{U} &= P_u \\
\dot{P}_u + \omega_u^2 U &= -kV + kZ \\
\dot{V} &= P_v \\
\dot{P}_v + \omega_v^2 V &= kU \\
\dot{Z} &= P_z \\
\dot{P}_z + \omega_v^2 Z &= kU
\end{aligned} \tag{11}$$

These equations correspond to the ones with small synchrotron and betatron phase advances over one revolution and derived in averaged complex form in [9].

Simple calculations of the eigenvalues show that here the mode frequencies stay equal to their unperturbed values independently of k and hence do not depend on the beam current! Mode coupling is not totally eliminated: it is inherent in the current dependence of the eigenvectors. However, the mode frequencies can be made current-independent, hence they remain always real, which means elimination of the TMC instability.

For a multi-mode system (no matter how many modes coupled with the $m = 0$ mode there are) the principle shown in this example also works, with the exception of some degenerate cases.

The above consideration concerned the hollow beam. General cases have to be treated with much more complicated techniques and require a more general formalism based on [1,2] and developed in the following sections. Finally we will see that the idea of current-independent eigenfrequencies, achieved with a special transverse feedback system, can be extended to the general case.

The present section was deliberately restricted to systems based on simple models (hollow beam, time-independent collective force), and to simple examples, truncating multi-mode systems to a few modes, in order to clarify the idea of this approach to counteracting the TMC instability with a specially arranged feedback system. The conclusions above, obtained for uniformly distributed collective forces, are validated by a computer code which treats impedances as localized and azimuthally spaced, and constructs the relevant single-turn mapping of the collective motion to seek the mode frequencies. The necessary formalism will be described in the following sections.

4 Single Turn Mapping with Circulant Matrices

The particles of a hollow beam occupy a ring in synchrotron phase space as shown in Fig. 1. Let us divide this ring into $2k + 1$ mesh elements with the n th one centred at the synchrotron

Figure 1: Division of the longitudinal phase space into mesh elements for the hollow beam model. Here $N = 2k + 1 = 5$.

phase ϕ_n :

$$\phi_n = \frac{2\pi n}{2k+1} - \frac{\pi}{2k+1} \quad (1 \leq n \leq 2k+1).$$

Each mesh element comprises a group of real particles which are at this moment in this position of phase space (and have the total dipole moment D_n and angle D'_n).

The division of the ring implies that all mesh elements with the numbers n and $2k+2-n$ have equal synchrotron coordinates and opposite synchrotron “velocities” $\frac{\Delta E}{E}$, with the exception of the element on the bunch tail; it has no counterpart (see “3” in Fig.1). Such a paired disposition allows the introduction of the sum and difference variables $D, \bar{D}$ for the dipole moments at each point. A numerical solution procedure can thereby be constructed in the most natural way.

4.1 Representation via Fourier Harmonics

In Section 2.1 we introduced the set of dipole moment variables $D, D', \bar{D}, \bar{D}'$ whose dependence on the longitudinal coordinates can be simply expressed through the Fourier harmonics with the numbers $\pm m$ in terms of the phase ϕ (which is an equivalent of the longitudinal coordinate):

$$\begin{aligned} D(\phi)(D'(\phi)) &= d(d') \cos(m\phi) \\ \bar{D}(\phi)(\bar{D}'(\phi)) &= \bar{d}(\bar{d}') \sin(m\phi), \end{aligned}$$

where the amplitudes $d, d', \bar{d}, \bar{d}'$ depend only on time. They are related to each other as far as they are combined in Section 2.2 into the zero-intensity eigenfunctions X_m, Y_m so that

the latter obey the uncoupled oscillatory equations (3,4). We have:

$$X_m = d + \bar{d}' , \quad X'_m = d' - \bar{d}$$

for the eigenfrequency $\omega_\beta + m\omega_s$, and:

$$Y_m = d - \bar{d}' , \quad Y'_m = d' + \bar{d}$$

for the eigenfrequency $\omega_\beta - m\omega_s$.

On the basis of those exact formulas we now construct four $2(2k+1)$ -dimensional real vectors, whose components D_n, D'_n are composed in such a manner that the vectors will represent the unperturbed modes X_m, X'_m, Y_m, Y'_m on the mesh nodes. They should coincide with the exact functions at $2k+1$ division points, specified at the phases ϕ_n . For X_m, X'_m we take $d = d' = 1$, then from $Y_m, Y'_m = 0$ we have $\bar{d} = -d'$, $\bar{d}' = d$, hence the vector components are:

$$X_m \rightarrow D_n = \cos m\phi_n, D'_n = \sin m\phi_n \quad X'_m \rightarrow D_n = -\sin m\phi_n, D'_n = \cos m\phi_n$$

for the eigenfrequency $\omega_\beta + m\omega_s$, and similarly for Y_m, Y'_m :

$$Y_m \rightarrow D_n = \cos m\phi_n, D'_n = -\sin m\phi_n \quad Y'_m \rightarrow D_n = \sin m\phi_n, D'_n = \cos m\phi_n$$

for the eigenfrequency $\omega_\beta - m\omega_s$.

Now it is easy to check that, at the points specified by the numbers n or $2k+2-n$, we will have for the dipole moments the sum $D \propto \cos m\phi$ and the difference $\bar{D} = D(n) - D(2k+2-n) \propto \sin m\phi$, while the relation of the signs and amplitudes is suitable to compose the desired approximation for the modal combinations X_m, Y_m .

The transfer matrix in this constructed basic set is (by its definition):

$$\text{diag} \left\{ \dots, \begin{bmatrix} \cos \alpha_m & \sin \alpha_m \\ -\sin \alpha_m & \cos \alpha_m \end{bmatrix}, \dots \right\} ,$$

α_m being the eigenfrequency related to each vector pair.

The total number of divisions $2k+1$ determines the maximum number of available independent vector pairs. Therefore we can only describe a finite set of coupled modes, truncated in order to retain only the lower ones, whose unperturbed frequencies are:

$$\omega_\beta - k\omega_s, \dots, \omega_\beta, \dots, \omega_\beta + k\omega_s .$$

4.2 Localized Kick from Collective Force

The *transfer matrix* in the basic set, as constructed in the previous section, has a block-diagonal form with the above written 2×2 blocks. To account for interaction with a localized impedance we have to multiply it by a *kick matrix*, which should only add angles to the modal variables. The kick strength corresponds to the interaction matrix $K_{mm'}$ (8), which previously appeared in the distributed force equations:

$$\Delta X'_m = \Delta Y'_m = k_0 \sum_{m'} (X_{m'} + Y_{m'}) \int_0^\pi \cos m\phi d\phi \int_0^\phi \cos m'\phi' W(a \cos \phi - a \cos \phi') d\phi'$$

The localized force is determined by the wake function $W(s - s')$ of this lumped impedance component, and the interaction parameter (*c.f.* k in (7)) now involves the local value of the β -function:

$$k_0 = \frac{e^2 N \omega_\beta}{\pi \gamma m_0 c} \beta$$

For a localized kick we deal with the change of the transverse momentum of the trailing particle $\Delta p_\perp = e^2 D(s') W(s - s')$ instead of the continuous collective force (7) for a distributed impedance.

If there are several localized impedances placed over a machine circumference, one should construct a kick matrix for each impedance and a transfer matrix for each azimuthal spacing so as to account for the mode phase advance in between the kicks. Then sequential multiplication of these matrices will result in a single-turn matrix which contains all the information on the stability of the $2k + 1$ modes in question, neglecting their coupling with the truncated higher modes. From experience with the numerical computation, a good approximation for the eigenvalues is available with only few lower modes.

4.3 Space-Time Representation of the Single Turn Matrix

In practice a space-time representation of the matrices is convenient, rather than the Fourier harmonic basic set, which we used to clarify the concept of the one-turn mapping for the TMC modes. Now we will use the dipole moments and the angles of the $2k + 1$ mesh elements D_n, D'_n , rather than the modal combinations X_m, Y_m , in order to construct the basic set from the appropriate $2(2k + 1)$ st order vectors. The advantage is in the simplification of the localized kick matrix form. On the passage through a localized impedance, the angle of a trailing particle is changed by

$$\Delta D'_n = \frac{Q_{n'} D_{n'} W(s_n - s_{n'})}{\gamma m_0 c},$$

where $D_{n'}, Q_{n'}$ are the dipole moment and charge of the leading particle, and $s_{n'}, s_n$ are the longitudinal coordinates of the leading and trailing particles, respectively. This expression is employed in construction of a $2(2k + 1)$ st-order kick matrix, similar to a thin lens matrix in its form.

However, the space-time representation complicates the form of the transfer matrix. Here we must keep in mind that the divisional sectors of the “synchrotron motion circle”, should not be loosely associated with (macro)particles, because they do not physically rotate in the synchrotron motion phase plane, as a real particles do⁴. So, variation of the dipole moment, ascribed to each sector, is actually effected by the transport of real particles along the synchrotron ring, due to their synchrotron oscillations, and accompanied with their variable betatron phase advance. The needed transformation is performed by a *block circulant matrix* [3]. We will construct it on the basis of the above obtained solutions.

According to usual rules of linear algebra, we have to transform the obtained block-diagonal matrix in the Fourier harmonic basic set to the new basic set, using the similarity transformation with the matrix, composed of the columns of the eigenvectors.

⁴Actually these sectors form a mesh for a numerical solution of the integro-differential equations, introduced in [1,2].

We introduce a convenient numbering of the rows and columns, *i.e.* of the eigenvectors. We will take them in pairs X_m, X'_m , and arrange their order in the matrix columns so that the zeroth mode pair is in the first place, followed by the modes $m = -1$ and $m = +1$ pairs, then those of the modes $m = -2$ and $m = +2$, etc. Thus the rows will sequentially accommodate the dipole moments and their derivatives for the sectors as numbered counter-clockwise around the circle, accordingly to their synchrotron rotation sense. The eigenvectors, corresponding to the unperturbed mode basic set X_m, X'_m, Y_m, Y'_m , have already been found, pairs of their components D_n, D'_n , ($1 \leq n \leq 2k + 1$) are:

$$X_m \rightarrow D_n = \cos m\phi_n, D'_n = \sin m\phi_n \quad X'_m \rightarrow D_n = -\sin m\phi_n, D'_n = \cos m\phi_n$$

for the eigenfrequency $\omega_\beta + m\omega_s$, and

$$Y_m \rightarrow D_n = \cos m\phi_n, D'_n = -\sin m\phi_n \quad Y'_m \rightarrow D_n = \sin m\phi_n, D'_n = \cos m\phi_n$$

for the eigenfrequency $\omega_\beta - m\omega_s$.

The resulting form of the transformation matrix, composed from the eigenvectors, is:

$$M = \begin{pmatrix} 1 & 0 & \dots & \cos m\phi_1 & \sin m\phi_1 & \cos m\phi_1 & -\sin m\phi_1 & \dots \\ 0 & 1 & & -\sin m\phi_1 & \cos m\phi_1 & \sin m\phi_1 & \cos m\phi_1 & \\ 1 & 0 & & \cos m\phi_2 & \sin m\phi_2 & \cos m\phi_2 & -\sin m\phi_2 & \\ 0 & 1 & & -\sin m\phi_2 & \cos m\phi_2 & \sin m\phi_2 & \cos m\phi_2 & \\ 1 & 0 & & \cos m\phi_3 & \sin m\phi_3 & \cos m\phi_3 & -\sin m\phi_3 & \\ 0 & 1 & & -\sin m\phi_3 & \cos m\phi_3 & \sin m\phi_3 & \cos m\phi_3 & \\ \dots & & & & & & & \dots \end{pmatrix}$$

where

$$\phi_n = \frac{2\pi n}{2k+1} - \frac{\pi}{2k+1}.$$

The former transfer matrix reads:

$$T = \begin{pmatrix} \cos \Omega_0 T & \sin \Omega_0 T & \dots & 0 & & & \dots \\ -\sin \Omega_0 T & \cos \Omega_0 T & & & & & \\ 0 & & \dots & \cos \Omega_{-m} T & \sin \Omega_{-m} T & 0 & 0 \\ \dots & & & -\sin \Omega_{-m} T & \cos \Omega_{-m} T & 0 & 0 \\ & & & 0 & 0 & \cos \Omega_m T & \sin \Omega_m T \\ & & & 0 & 0 & -\sin \Omega_m T & \cos \Omega_m T \\ & & & & & & \dots \end{pmatrix}$$

where $\Omega_m = \omega_\beta + m\omega_s$, and T is the “time” interval for the transfer section (free from impedances).

Thus, the desired block circulant matrix C_B results from

$$C_B = MTM^{-1},$$

and can be rewritten as a direct product:

$$C_B = C \otimes T_2,$$

where

$$T_2 = \begin{pmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{pmatrix},$$

and μ is the betatron phase advance. The calculated matrix elements of the circulant are:

$$C_{ij} = \frac{\sin(2k+1)\varphi_{ij}}{(2k+1)\sin \varphi_{ij}}$$

with

$$\varphi_{ij} = \frac{1}{2}(\omega_s T - (2k+1-i+j)\frac{2\pi}{2k+1})$$

Such is the $2(2k+1)$ -dimensional block circulant C_B which performs mapping of the $2k+1$ pairs D_n, D'_n on passage of an impedance-free section. It is now expressed in the space-time basic set, in terms of dipole moments, ascribed to divisional sectors of the hollow beam (see Fig.1).

After multiplication of the circulants by the kick matrices due to localized impedances in the order they are placed over the machine circumference, we will obtain a single-turn mapping matrix, whose eigenvalues are responsible for stability of the head-tail modes. Note, that we made no assumptions as to the size of the impedance kick, or the transverse wake form. In addition, the velocity-dependent wakes (such as those arising in the long transmission lines) can also be accounted for.

5 Generalization of the Hollow Beam Equations, Including Non-Linear Synchrotron Oscillations

Eqs. (5-6) have been derived for a “hollow beam” bunch and linear synchrotron motion. For this case Section 4 presents a procedure for evaluating the eigenvalues for localized collective interaction either in the harmonic representation, or in the space-time representation by means of block circulants. Here we will derive general equations for localized collective interactions, allowing for an arbitrary distribution over synchrotron oscillation amplitudes, an arbitrary shape of longitudinal potential, and with no limitation on the interaction strength. The TMC instability due to localized structures and synchrobetatron resonances was under investigation earlier (See, for example [8, 4]). For a nonlinear potential well some analytical and simulation results are presented in [6, 7]. Most of these references treat the TMC effect in the frequency domain, while the present report is devoted to the alternative formulation of the TMC theory in the space-time domain.

5.1 A New Phase Variable

For a given longitudinal potential $U(s)$ the Hamiltonian of synchrotron motion reads:

$$H = \frac{\alpha}{2}\delta^2 + U(S),$$

where the dynamical variables are:

$$S = \frac{cs}{\Pi^2} \quad \text{and} \quad \delta = \frac{\Delta E}{E},$$

with α the momentum compaction, Π the orbit circumference, E and ΔE the equilibrium energy and the energy offset, respectively. We can introduce a generalized phase ϕ for the non-linear case:

$$\phi = \frac{2\pi}{T(H)} \int_0^S \frac{dS'}{\sqrt{2\alpha(H - U(S'))}} , \quad (12)$$

so that

$$\frac{\partial \phi}{\partial t} = \frac{2\pi}{T(H)} = \omega_s(H) .$$

One can see that the L.H.S. of eqs (5-6) after replacement of ϕ by this generalized phase and $\omega_s \rightarrow \omega_s(H)$ will describe the variation of the dipole moments and their derivatives in a non-linear potential well without a collective force. The latter should now be rewritten so as to include integration over the synchrotron amplitude distribution $g(H)$:

$$F(s) = k \int_s^\infty ds' W(s - s') \int_{U(\frac{cs'}{\Pi^2})}^\infty D(s', H) \frac{g(H) dH}{\sqrt{2\alpha[H - U(S)]}} \quad (13)$$

where the distribution is normalized to unity:

$$\int_{-\infty}^\infty g(H) ds d\delta = 1 .$$

These modifications in eqs. (5-6) (and in corresponding equations for the eigenfunctions $D \propto e^{i\omega t}$) result in a serious complication: now these equations become two-dimensional and must hold for any value of H .

The special case of these equations with a rectangular longitudinal potential and $W \propto \delta(s - s')$ has been solved in [2] by means of Landau integration, actually corresponding to an initial value problem.

The analytical solvability of these equations for the general case is problematic. However, they can be solved numerically on the basis of the procedure for the hollow beam and linear synchrotron motion given in Section 4.

5.2 Two-Dimensional Mesh

We will use the space-time representation for this construction. The block circulant matrix was used in Section 4 to perform variation of dipole moments, ascribed to stationary sectors of the hollow beam, resulting from its division into sectorial mesh elements, which was done uniformly over the synchrotron phase ϕ_s . Similarly to the case of linear synchrotron oscillation we now divide each ring with a given value of H into $2k + 1$ sectors, which are equal with respect to the new phase ϕ in the non-linear longitudinal potential. The centre of the n th sector will again be placed at

$$\phi(H) = \frac{2\pi n}{2k + 1} - \frac{\pi}{2k + 1} \quad (1 \leq n \leq 2k + 1).$$

The single-turn variation of dipole moments, contained in these sectors, is performed by the same circulant with the synchrotron frequency ω_s which is now dependent on the

Figure 2: The mesh elements for radial modes.

amplitude variable H . In a special case where the synchrotron tune is a rational number $Q_s = l/(2k + 1)$, the dipole moments would move over exactly the l positions during one revolution.⁵ So, only in such a case the circulant matrix is band-patterned, with a single nonnull 2×2 block, and performs the transfer of the dipole moment (and its derivative) around the sector array, *i.e.* a simple circular permutation [4, 5].

In the general case with irrational synchrotron tunes (especially important for the non-linear synchrotron oscillations, where the frequency continuously varies with the amplitude), the construction of a general circulant, as shown in Section 4, leaves no null blocks, and results in a generalized transfer matrix of $2k + 1$ dipole moments of the mesh.

Since the mapping of each “hollow beam” ring is now clear, we divide the longitudinal phase space of the beam into rings with a given value of the synchrotron amplitude variable H , which means a definite synchrotron frequency $\omega(H)$. Apparently the thinner the rings are, the better is our approximation to the desired eigenvalues of the two-dimensional system (and the more is their number, corresponding to different “radial” modes).

In the absence of the collective force the new transfer matrix with the account of L “radial” divisions is block-diagonal in the newly introduced radial dimension. Its blocks constitute the appropriate circulant matrices of order $L \times 2(2k + 1)$, accordingly to the $2k + 1$ sectorial divisions in each ring (L is the number of the rings), and involve the relevant synchrotron frequency $\omega(H_r)$ of the r th ring as a parameter. A sketch of the two-dimensional mesh is shown in Figure 2.

⁵Note, that sectors themselves do not rotate with the synchrotron frequency, since they are actually the mesh divisions, rather than macroparticles.

5.3 Collective Force in the Non-Linear Case

In the case of a short bunch we account for a localized collective force (13) by simply multiplying the above generalized transfer matrix by a kick matrix. The latter is a “thin lens matrix” in its construction, while the kicks from each leading sector $\{r', n'\}$ with the longitudinal position s' to each trailing one $\{r, n\}$ at s are evaluated from $W(s - s') \cdot g(H_{r'})$, where $g(H_{r'})$ is the weight factor of the leading sector, known from the beam distribution function over the synchrotron oscillation amplitudes.

The only (minor) difficulty is in the determination of the longitudinal coordinates s_{rn} relevant to the centres $\{H_r, \phi_n(H)\}$ of all divisional sectors. To this end we should use eq. (12) which relates S to $\phi(H)$, for finding s_{rn} .

5.4 Discussion of Spectrum

For a complete analysis we should obtain the single-turn mapping matrix, whose eigenvalues and eigenvectors yield an approximation to the mode frequencies and mode functions of the TMC problem with localized impedances and radial modes. To this end we collect all the sections around the machine circumference by sequential multiplication of the localized impedance kick matrices and the transfer matrices (circulants) in between the kicks.

It is apparent from the physical sense of the construction, that the smaller are mesh divisions (and the greater is their number), the more accurate is the approximation to the precise spectral problem. A proof of completeness of the spectral set is, in principle, available from general statements of linear algebra concerning linear transformations and completeness of the eigenvectors of the transformation matrix.

An important property of the spectrum is that the eigenvalues occur only in reciprocal pairs (Section 6), provided that the lattice chromaticity is zero and the bunch is short enough in order to neglect the betatron phase advance over the bunch length. This means, that the modes of our problem with non-linear longitudinal motion correspond to the Van Kampen collective modes [VK] in plasma physics (See [7]), which also come in pairs of reciprocal eigenvalues, in contrast to the Landau-damped solution of the initial value problem. An exhaustive discussion of the relation between these two solutions is available elsewhere [10].

A few more comments are noteworthy. In case the collective interaction is distributed over a finite length (or if the bunch is not short, and the betatron phase advance over the bunch length must be accounted for), we can apply the same technique after dividing the interaction length into sufficiently short steps.

At non-zero chromaticity the development of the numerical procedure is readily obtained [1,2] by substituting $D \rightarrow De^{i\xi}$, where ξ is the chromatic phase. As a result of this substitution the left-hand sides of the equations retain their form. The collective force on the R.H.S. will however be modified. The imaginary factor therein implies a part of this force, depending on the derivative of the leading particles' dipole moment, in addition to the part, dependent on their dipole moment itself. Only this latter part has been treated in the present paper throughout. So, introduction of a finite chromaticity would immediately upset the pairwise reciprocal property of eigenvalues resulting in unpaired imaginary parts of the mode frequencies.

In conclusion we note that interaction of the modes with a transverse feedback system, characterized by any given response function, can be fairly well described in terms of coupling

to a set of oscillators with appropriate frequencies and damping rates. Introduction of an oscillator, coupled to the mode system, is straightforward in the proposed formalism: we should add two rows and two columns to the matrix of our system to account for the variables x, p_x of each oscillator. At the feedback pick-up and kicker azimuths we have to multiply by the kick matrices, firstly incrementing the oscillator's momentum p_x by a value proportional to the total dipole moment of the beam (as measured by the pick-up) and secondly adding to the values D' in all the beam sectors the same increment $\propto x$. A time delay in the kicker action can be accounted for by recalculation of the values D, D', x, p_x for desired previous moments of time. More details are given in the Section 7 dealing with feedback.

The numerical procedure as presented in Sections 4,5 is implemented in a computer code. Its output provided the illustrations to the present paper.

6 Reciprocal Spectral Property of the TMC Mode System

The mode interaction matrix $K = ||K_{mm'}||$ in (8) involves a general form of the wake function, nevertheless there are some relations between its elements, which will lead to physically important conclusions, concerning the mode behaviour.

6.1 Chess Board Symmetry of the Mode Coupling Matrix

Let us begin with the hollow beam model. The integration in (8) is done over a triangular domain: $0 \leq \phi' \leq \phi$, $0 \leq \phi \leq \pi$, according to the causal nature of the wake function (for $s > s'$ $W(s - s') = 0$):

$$K_{mm'} = k \int_0^\pi \cos m\phi d\phi \int_0^\phi \cos m'\phi' W(a \cos \phi - a \cos \phi') d\phi'.$$

Interchanging the order of integration:

$$K_{mm'} = k \int_0^\pi \cos m'\phi' d\phi' \int_{\phi'}^\pi \cos m\phi W(a \cos \phi - a \cos \phi') d\phi$$

and introducing new variables $\Phi = \pi - \phi$, $\Phi' = \pi - \phi'$:

$$\begin{aligned} K_{mm'} &= k \int_0^\pi \cos(m'\pi - m'\Phi') d\Phi' \int_0^{\Phi'} \cos(m\pi - m\Phi) W(a \cos(\pi - \Phi) - a \cos(\pi - \Phi')) d\Phi \\ &= (-1)^{m+m'} k \int_0^\pi \cos m'\Phi' d\Phi' \int_0^{\Phi'} \cos m\Phi W(a \cos \Phi' - a \cos \Phi) d\Phi = (-1)^{m+m'} K_{m'm}. \end{aligned}$$

we come to a relation of “chess board symmetry” between the matrix elements:

$$K_{mm'} = (-1)^{m+m'} K_{m'm}. \quad (14)$$

Considering the radial mode construction, we should write a set of equations, similar to (9) for coupling of two hollow beams with different radii a and A ($a < A$) and different populations q and Q . The interaction between the two beams is given by the matrix

$$\tilde{K} = \begin{pmatrix} qK_{mm'}^{aa} & QK_{mm'}^{aA} \\ qK_{mm'}^{Aa} & QK_{mm'}^{AA} \end{pmatrix}. \quad (15)$$

Now the integrals for $K_{mm'}^{aA}$ and $K_{mm'}^{Aa}$ are more complicated. The correspondence between longitudinal coordinates in the two rings is shown in Figure 3a. The integration domain for $K_{mm'}^{aA}$: $0 \leq \phi' \leq \arccos(\frac{a}{A} \cos \phi)$, $0 \leq \phi \leq \pi$, shown in Figure 3b, accounts for wakes of the larger ring, acting on the smaller one:

$$K_{mm'}^{aA} = k \int_0^\pi \cos m\phi d\phi \int_0^{\arccos(\frac{a}{A} \cos \phi)} \cos m'\phi' W(a \cos \phi - A \cos \phi') d\phi'.$$

Again we interchange the order of integration and denote $\phi'_1 = \arccos(\frac{a}{A})$:

$$\begin{aligned} K_{mm'}^{aA} &= k \int_0^{\phi'_1} \cos m'\phi' d\phi' \int_0^\pi \cos m\phi W(a \cos \phi - A \cos \phi') d\phi \\ &+ k \int_{\phi'_1}^{\pi-\phi'_1} \cos m'\phi' d\phi' \int_{\arccos(\frac{A}{a} \cos \phi')}^\pi \cos m\phi W(a \cos \phi - A \cos \phi') d\phi. \end{aligned}$$

We obtain a sum of two terms. On the other hand, action of the smaller ring on the larger one, and reversal of the indices m, m' , also splits the integral into two terms, according to the modification of the integration domain in Figure 3c:

$$\begin{aligned} K_{m'm}^{Aa} &= k \int_{\phi'_1}^{\pi-\phi'_1} \cos m'\phi d\phi \int_0^{\arccos(\frac{A}{a} \cos \phi)} \cos m\phi' W(A \cos \phi - a \cos \phi') d\phi' \\ &+ k \int_{\pi-\phi'_1}^\pi \cos m'\phi d\phi \int_0^\pi \cos m\phi' W(A \cos \phi - a \cos \phi') d\phi'. \end{aligned}$$

The same transformation of variables $\Phi = \pi - \phi$, $\Phi' = \pi - \phi'$ on the R.H.S. of the matrix elements reveals the factor $(-1)^{m+m'}$:

$$K_{m'm}^{aA} = (-1)^{m+m'} K_{m'm}^{Aa}, \quad (16)$$

hence the chess board symmetry relation holds for the matrix K , however the interaction matrix $\tilde{K}$ is complicated by the different weight factors q, Q of the two rings, according to difference in their population.

Note, that in these transformations we used only the symmetry of $\arccos(\frac{a}{A} \cos \phi)$ with respect to $\pi/2$, while the form of the function was not important. Therefore all the properties of the interaction matrices hold in any symmetric non-linear longitudinal potential well (with a single minimum).

Figure 3: a) correspondence between longitudinal coordinates in the two rings: $A \cos \phi' = a \cos \phi = s$; b),c) integration domains for K_{nm}^{aA} and $K_{m'm}^{Aa}$.

6.2 Generalized Symplecticity

In contrast to the symmetry of the interaction matrix of a linear Hamiltonian system, the above obtained relations (14,16) for the mode coupling matrix manifest the non-Hamiltonian character of our system which results from the (causal) asymmetry of the wakes. Nevertheless, we will be able to draw certain conclusions on the eigenvalues from these relations, and find similarities with Hamiltonian systems, where the time evolution is performed by symplectic maps.

Let us review the important properties of a linear Hamiltonian system. For convenience we rearrange the oscillator variables of n modes in this section, so that the vector of dynamical variables X now becomes:

$$X = \text{column}\{p_1, \dots, p_n, x_1, \dots, x_n\}$$

The equations of motion:

$$\dot{X} = S \cdot H \cdot X$$

are written with the help of the unitary symplectic matrix S . This matrix is partitioned into “ p ” and “ x ” blocks of the n th-order:

$$S = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

$$S^2 = -I, \quad S^T = S^{-1} = -S$$

where I is the identity matrix and H is the Hamiltonian matrix:

$$H = \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix},$$

The interaction matrix G , given by,

$$G = \begin{pmatrix} \omega_1^2 & g_{12} & \dots & g_{1n} \\ g_{21} & \omega_2^2 & \dots & g_{2n} \\ \dots & \dots & \dots & \dots \\ g_{n1} & g_{n2} & \dots & \omega_n^2 \end{pmatrix}, \quad G^T = G$$

is symmetric in a Hamiltonian system, as is the Hamiltonian matrix: $H^T = H$. We can then construct a Hamiltonian function in a quadratic form $\mathcal{H} = \frac{1}{2}X^T H X$, and

$$\dot{\mathcal{H}} - \frac{1}{2}X^T \dot{H} X = \frac{1}{2}(X^T H^T S^T H X + X^T H S H X) = 0$$

The most important property is an invariance of the S-product of two arbitrary solutions X_1, X_2 :

$$(X_1^T S X_2)^\cdot = X_1^T H^T S^T S X_2 + X_1^T S S H X_2 = 0$$

If, on the other hand, the time evolution of the solutions is expressed with the help of the transfer matrix M through the initial vectors X_{01}, X_{02} :

$$X_1 = M X_{01};, \quad X_2 = M X_{02},$$

then the invariance of the S-product

$$(X_1^T S X_2) = (X_{01}^T M^T S M X_{02}) = (X_{01}^T S X_{02})$$

yields the symplectic condition for these transfer matrices:

$$M^T S M = S.$$

We therefore have $n(2n - 1)$ constraints on otherwise arbitrary $(2n)^2$ matrix elements of M , one of which is $\det M = 1$.

In addition to the simple fact that the eigenvalues of any real matrix, being the roots of its characteristic equation, are pairwise complex conjugates, we have here another important spectral property: the eigenvalues of a symplectic matrix come in reciprocal pairs. We modify the proof of this theorem in [11] in order to emphasize that neither of the properties $S^2 = -I$, $\det M^T = \det M$, $\det S = \det M = 1$ is necessary, only the symplectic property $M = S^{-1}(M^T)^{-1}S$ which relates any symplectic matrix to its transpose is essential.

We can perform a chain of transformations on the characteristic polynomial:

$$|M - \lambda I| = |S^{-1}(M^T)^{-1}S - \lambda I| = |(M^T)^{-1} - \lambda I| = |I - \lambda M| = \lambda^{2n} |M - \frac{1}{\lambda} I|$$

whence we see that $1/\lambda$ is always the eigenvalue of a symplectic matrix, provided that λ is its eigenvalue, *Q.E.D.* The reciprocal property of the roots is equivalent to the symmetry of the coefficients a_i in the characteristic polynomial: $a_{2n} = a_0, a_{2n-1} = a_1$, etc.

We now turn back to the coupled mode system whose equations of motion (9) (written now in the matrix form)

$$\dot{X} = \begin{pmatrix} 0 & -F \\ I & 0 \end{pmatrix} X, \quad F = K + \text{diag}\{\dots, \omega_m^2, \dots\}$$

differ from the Hamiltonian form only in that $F^T \neq F$ (since $K^T \neq K$). We notice that relation (14) can be modified in order to recover the symmetry of the mode coupling matrix K :

$$(-1)^{m'} K_{mm'} = (-1)^m K_{m'm}.$$

This allows us to obtain a *symmetric* interaction matrix G , instead of F :

$$G_{mm'} = (\omega_m^2 \delta_{mm'} + K_{mm'}) (-1)^{m'}$$

by means of a special transformation of variables $Y = TX$. This transformation alternates the signs of the coordinates and has the form:

$$T = \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix},$$

where

$$P = \text{diag}\{\dots, (-1)^j \dots\}, \quad (1 \leq j \leq n)$$

is the parity matrix. Indeed, its action on the equation of motion:

$$\dot{Y} = T \dot{X} = T \begin{pmatrix} 0 & -F \\ I & 0 \end{pmatrix} T^{-1}(TX) = \begin{pmatrix} 0 & -FP \\ P & 0 \end{pmatrix} Y = S \begin{pmatrix} P & 0 \\ 0 & FP \end{pmatrix} = S \tilde{H} Y,$$

brings it into a Hamiltonian form with a symmetric $G = FP$ (since $F^T = PFP$) at the expense of an indefinite signature of the resulting Hamiltonian quadratic form

$$\tilde{\mathcal{H}} = \frac{1}{2} Y^T \tilde{H} Y.$$

The transformation is symmetric, unitary, but non-canonical:

$$T^T S T = \begin{pmatrix} 0 & -P \\ P & 0 \end{pmatrix} = R$$

since $R \neq S$. However, with a symmetric Hamiltonian matrix $\tilde{H}$, we can obtain all the symplectic properties in terms of Y . Therefore the mapping matrices M for our coupled mode system in terms of X obey a general symplectic condition:

$$M^T R M = R,$$

where matrix R satisfies all the conditions of the above theorem on the eigenvalues in reciprocal pairs.

Since the chess board symmetry is valid for the radial modes, the only complication in (15) coming from different weight factors q, Q of different coupled rings can be hidden in the weight matrix W ($\det W \equiv 1$):

$$W = \text{diag}\{\dots, q, Q, \dots\}$$

to be introduced into the transformation T :

$$T = \begin{pmatrix} I & 0 \\ 0 & PW \end{pmatrix},$$

Although this modification complicates the form of R , it does not affect any of its essential properties concerning the R-symplecticity. In particular, the characteristic polynomial of any mode mapping matrix remains reciprocal.⁶

Thus we have proved in general the R-symplecticity of the coupled mode system and extended the results of Section 3 to the general case. The proof was given in the harmonic representation (see Section 4). The form of R is much more complicated in the space-time representation. The space-time expression for R is available (if needed) by means of the transformation from harmonics to the space-time mesh given in Section 4.

In conclusion for this section we note that an oscillator acting on any of the modes similarly to the action of other modes (*e.g.* an oscillator simulated by the transverse feedback system and coupled to the mode $m = 0$) would not upset the reciprocal property, if the coupling to the feedback is pointwise and similar to a (skew) thin lens: the momentum variables of the mode and the oscillator are both kicked proportionally to the coordinate variables of each other. If the pick-up and kicker are separated by an impedance-free section, the case can be reduced to the previous situation. Otherwise the impedance elements in between them couple the modes and prevent us from representing the delayed feedback action in the form of the above R-symplectic matrices; therefore the reciprocal property of the eigenvalues in general will fail.

7 General Technique for Elimination of Current Dependency of the TMC Mode Frequencies.

In Section 3 the possibility of obtaining current-independent frequencies of the synchrotron modes by means of a transverse feedback was demonstrated on the basis of a simple model: only two modes and one feedback oscillator were considered with distributed interaction. Here we first present an algorithm for an arbitrary number of modes with the same interaction, then we generalize the technique in order to account for a localized collective force (and feedback) action.

7.1 Time-Independent Interaction

To simplify the presentation of the algorithm we first suppose the collective force and the feedback to be time-constants, *i.e.* independent of the machine azimuth [9].

⁶The reciprocal spectral property of the synchrotron modes has been noticed previously [4] for some particular cases.

Let us take the $N + 1$ unperturbed modes of a hollow beam as a basic set (see X_m , Y_m introduced in Section 2.2). They comprise the mode $m = 0$ and N higher modes, and their interaction is given by the set of equations (9), where the collective force is expressed via the coupling coefficients $K_{mm'}$ (8).

The transverse feedback can interact only with the unperturbed mode X_0 , the only one with non-zero dipole moment. The feedback model is represented by a set of L undamped oscillators, which are coupled to the mode X_0 , and by a single-turn reactive feedback system intended for stabilizing the current-dependent tune of the $m = 0$ mode.

Let us demonstrate, that $L = N$ oscillators will suffice for the problem of eliminating the current-dependency in all the mode tunes. When we couple our mode system with L additional oscillators, each brings in 2 unknown parameters: its frequency and the coupling coefficient to the zeroth mode. For solvability with respect to the $2L + 1$ unknowns (one comes from the single-turn feedback) we dispose $L + N + 1$ equations for (squared) eigenfrequencies ω^2 of the oscillator system; hence $2L + 1 = L + N + 1$, or $L = N$, *Q.E.D.* Note that the eigenfrequencies can have *a priori* assigned values.

The characteristic equation $\det B = 0$ for the mode system $\omega_0, \omega_1, \dots$ without feedback results from (9), whence the $(N + 1)$ st order square matrix B is:

$$B = \begin{pmatrix} \omega_0^2 - \omega^2 - K_{00} & -K_{01} & \dots & -K_{0N} \\ -K_{10} & & & \\ \dots & M & - & \omega^2 E \\ -K_{N0} & & & \end{pmatrix}. \quad (17)$$

Here matrix M describes interaction of N higher modes.

Let z_l be variables of the feedback oscillators, $\bar{\omega}_l$ their frequencies and q_l , $\bar{q}_l$ the two coefficients⁷ of their coupling to the $m = 0$ mode ($1 \leq l \leq N$). Adding L feedback oscillators to (9) will leave the N higher mode equations unchanged, and only the equation for the zeroth mode X_0 will be altered:

$$X_0'' + \omega_0^2 X_0 + \Delta X_0 = \sum_{i=0}^N K_{0i} X_i + \sum_{l=1}^N q_l z_l. \quad (18)$$

Here Δ is the coefficient of the single turn feedback, which is just the shift of the tune of “zero” mode (see below). The set of equations for the feedback oscillators take the following form:

$$\begin{aligned} z_1'' + \bar{\omega}_1^2 z_1 &= \bar{q}_1 X_0 \\ &\dots \\ z_N'' + \bar{\omega}_N^2 z_N &= \bar{q}_N X_0. \end{aligned} \quad (19)$$

Searching for the eigenfrequencies ω , we substitute in (19)

$$X_i = A_i \exp(i\omega t)$$

⁷In fact we can always scale the values $z_l \rightarrow q_l \cdot z_l$ so that these two coefficients (proportional to the beam intensity), merge into one parameter, effectively, into their product, which would scale as the beam intensity squared.

$$z_i = Z_i \exp(i\omega t)$$

and obtain for the zeroth mode:

$$A_0(\omega_0^2 - \omega^2 + \Delta - K_{00}) = \sum_{i=1}^N K_{0i} A_i + \sum_{l=1}^N q_l Z_l \quad (20)$$

and for the feedback oscillators:

$$\begin{aligned} Z_1(\bar{\omega}_1^2 - \omega^2) &= \bar{q}_1 A_0 \\ &\dots \\ Z_N(\bar{\omega}_N^2 - \omega^2) &= \bar{q}_N A_0 . \end{aligned} \quad (21)$$

We can eliminate the feedback oscillators' amplitudes Z_l and thus come to the following equation:

$$A_0(\omega_0^2 - \omega^2 + \Delta - K_{00} - \sum_{l=1}^N \frac{q_l \bar{q}_l}{\bar{\omega}_l^2 - \omega^2}) = \sum_{i=1}^N K_{0i} A_i . \quad (22)$$

Note that the sum over the higher modes on the R.H.S. is left unaltered by the feedback oscillators, as is the mode interaction matrix M . Therefore the old matrix B without feedback is only modified in its b_{00} element by addition of the sum involving the feedback oscillators' parameters:

$$\tilde{b}_{00} = b_{00} + \Delta - \sum_{l=1}^N \frac{q_l \bar{q}_l}{\bar{\omega}_l^2 - \omega^2} .$$

Thus we arrive at a new characteristic equation including the feedback: $\det \tilde{B} = 0$.⁸ It involves q_l and $\bar{q}_l$ only as a product, constituting a single parameter $y_l = q_l \bar{q}_l$. Hence together with N frequencies ω_l^2 (and one parameter for the single turn reactive feedback), we have $2N + 1$ unknowns, as was expected.

The explicit form of the new characteristic equation $\det \tilde{B} = 0$ is given by:

$$\det \tilde{B} = \begin{vmatrix} \Psi + \omega_0^2 - \omega^2 - K_{00} & -K_{01} & \dots & -K_{0N} \\ -K_{10} & & & \\ \dots & M & - & \omega^2 E \\ -K_{N0} & & & \end{vmatrix} = 0 \quad (23)$$

where

$$\Psi = \Delta - \sum_{l=1}^N \frac{q_l \bar{q}_l}{\bar{\omega}_l^2 - \omega^2} .$$

The collected form of Ψ is:

$$\Psi = \Delta + \frac{F(\omega^2)}{\prod_{l=1}^N (\bar{\omega}_l^2 - \omega^2)} ,$$

⁸Of course, the above procedure for eliminating the feedback amplitudes is simply due to the partitioned determinant of the system. This saves in calculation effort when compared to the straightforward evaluation of this doubled determinant.

where $F(\omega^2)$ is an N th order polynomial. The whole determinant can be developed in terms of the 1st row (M_i being the minors of the elements K_{0i}):

$$(\Psi + \omega_0^2 - \omega^2 - K_{00})|M - \omega^2 E| + \sum_{i=1}^N K_{0i} M_i(\omega^2) .$$

Collecting all the terms we reduce the determinant to the form:

$$\frac{H(\omega^2)}{\prod_{l=1}^N (\bar{\omega}_l^2 - \omega^2)} ,$$

where H is a polynomial of the order $2N + 1$.

For there to be $2N + 1$ roots Ω_i^2 of this equation, H must have the form:

$$H(\omega^2) = \prod_{i=1}^{2N+1} (\Omega_i^2 - \omega^2) .$$

In this case the characteristic equation (23) can be rewritten as:

$$(\Psi + \omega_0^2 - \omega^2 - K_{00})|M - \omega^2 E| + \sum_{i=1}^N K_{0i} M_i(\omega^2) = \frac{H(\omega^2)}{\prod_{l=1}^N (\bar{\omega}_l^2 - \omega^2)}$$

or:

$$(\Psi + \omega_0^2 - \omega^2 - K_{00})|M - \omega^2 E| = \frac{H(\omega^2)}{\prod_{l=1}^N (\bar{\omega}_l^2 - \omega^2)} - \sum_{i=1}^N K_{0i} M_i(\omega^2)$$

or, finally:

$$(\Psi + \omega_0^2 - \omega^2 - K_{00}) = \frac{H(\omega^2) - \sum_{i=1}^N K_{0i} M_i(\omega^2) \cdot \prod_{l=1}^N (\bar{\omega}_l^2 - \omega^2)}{|M - \omega^2 E| \prod_{l=1}^N (\bar{\omega}_l^2 - \omega^2)} . \quad (24)$$

From the form of Ψ we conclude that the last equation holds, if and only if, the numerator on its R.H.S. is divisible by the N th order polynomial $|M - \omega^2 I|$. Let x_k be all the roots of this polynomial. Then, having in mind the above stated divisibility, the same x_k must be the roots of the numerator. Hence we can obtain a separate set of N nonlinear equations for the N unknowns ω_l .

Indeed, on the R.H.S. of eq.(24) we have a symmetric polynomial with respect to ω_l^2 :

$$\prod_{l=1}^N (x_k - \bar{\omega}_l^2) = \frac{H(x_k)}{\sum_{i=1}^N K_{0i} M_i(x_k)} ,$$

where $k = 1, \dots, N$. The product can be expanded:

$$\prod_{l=1}^N (x_k - \bar{\omega}_l^2) = x_k^N - \sigma_1 x_k^{N-1} \dots + (-1)^N \sigma_N$$

so that $\sigma_1, \sigma_2, \dots, \sigma_N$ are the elementary symmetric functions of ω_l^2 :

$$\begin{aligned} \sigma_1 &= \bar{\omega}_1^2 + \dots + \bar{\omega}_N^2 \\ \sigma_2 &= \bar{\omega}_1^2 \bar{\omega}_2^2 + \dots + \bar{\omega}_{N-1}^2 \bar{\omega}_N^2 \\ &\dots \\ \sigma_N &= \bar{\omega}_1^2 \cdot \dots \cdot \bar{\omega}_N^2 . \end{aligned}$$

In case that all the N roots x_k are different, we obtain N *linear* equations to be solved for the N unknowns $\sigma_1, \sigma_2, \dots, \sigma_N$. By the definition of the elementary symmetric functions the desired solutions for the feedback oscillators' frequencies $\bar{\omega}_1^2, \dots, \bar{\omega}_N^2$ are found as N roots of the following N th order equation:

$$z^N - \sigma_1 z^{N-1} \dots + (-1)^N \sigma_N = 0 .$$

Thus the N unknowns from the total set of $2N + 1$ are found separately, by solving this N th-order algebraic equation for the complete set of its roots.

Since

$$\Psi = \Delta - \sum_{l=1}^N \frac{q_l \bar{q}_l}{\bar{\omega}_l^2 - \omega^2}$$

the remaining $N + 1$ unknowns, namely, the coupling coefficients y_l and the single-turn feedback parameter Δ are to be found from the set of $N + 1$ *linear* equations, which appear when we substitute $N + 1$ *a priori* assigned values of ω into (23). Alternatively, we can simply find them by equating polynomial coefficients on the L.H.S. and R.H.S. of (24).

7.2 Example

We now show the implementation of the above algorithm. As a simple example we take the system considered in Section 3: two modes, one (unknown) coefficient of the single-turn reactive feedback Δ , and one feedback oscillator, with the unknown frequency $\bar{\omega}_1$ and coupling coefficient $y_1 = q_1 \bar{q}_1$. The characteristic equation is:

$$\begin{vmatrix} \Delta + \omega_0^2 - \omega^2 - K_{00} + \frac{y_1}{\omega^2 - \bar{\omega}_1^2} & -K_{01} \\ -K_{10} & \omega_1^2 - \omega^2 - K_{11} \end{vmatrix} = 0 . \quad (25)$$

Similar to the treatment in Section 3, we take $K_{00} = K_{11} = 0$ and $K_{10} = -K_{01} = q$ (to simplify the formulas), while the assigned values of eigenfrequencies will be $\omega_0, \omega_1, \omega_1$. Then we have:

$$\begin{vmatrix} \Delta + \omega_0^2 - \omega^2 + \frac{y_1}{\omega^2 - \bar{\omega}_1^2} & q \\ -q & \omega_1^2 - \omega^2 \end{vmatrix} = \frac{(\omega_0^2 - \omega^2)(\omega_1^2 - \omega^2)^2}{\bar{\omega}_1^2 - \omega^2} \quad (26)$$

or, explicitly:

$$(\Delta + \omega_0^2 - \omega^2 + \frac{y_1}{\omega^2 - \bar{\omega}_1^2})(\omega_1^2 - \omega^2) + q^2 = \frac{(\omega_0^2 - \omega^2)(\omega_1^2 - \omega^2)^2}{\bar{\omega}_1^2 - \omega^2}. \quad (27)$$

According to the algorithm in Section 7.1 the ratio

$$\frac{(\omega_0^2 - \omega^2)(\omega_1^2 - \omega^2)^2 - q^2(\bar{\omega}_1^2 - \omega^2)}{\bar{\omega}_1^2 - \omega^2}$$

must be divisible by $\omega_1^2 - \omega^2$. Therefore ω_1^2 should be the root of the numerator polynomial, giving $\bar{\omega}_1^2 = \omega_1^2$. We thus find that the unknown frequency of the feedback oscillator coincides with the unperturbed $m = 1$ mode frequency.

Substituting first $\omega = \omega_0$, then $\omega = \omega_1$ into (27), we obtain a pair of equations:

$$\Delta(\omega_1^2 - \omega_0^2) + y_1 + q^2 = 0$$

$$y_1 + q^2 = 0,$$

and therefore $\Delta = 0$ and $y_1 = -q^2$. These solutions are identical to the results of Section 3.

7.3 Generalization of the Algorithm for Localized Action of a Collective Force and Feedback

For extension of the algorithm presented in the previous sections, we will need some properties of the single-turn mapping matrix for N modes. We are interested, as before, in the case of zero chromaticity and a short bunch, so that the phase advance over the bunch length be negligibly small.

In Section 6 it was shown that this matrix has the eigenvalues in reciprocal pairs, and that the interaction of the modes with an additional feedback oscillator preserves this property, provided that its mutual coupling to the modes takes place at the same point of the machine azimuth. In practice this requires the close positioning of the feedback pick-up and kicker.

This property of the single-turn mapping matrix however will hold even when the pick-up and kicker are separated by certain advances of both betatron and synchrotron phases in the machine lattice. In this case we can reduce the mutual coupling to the single point by calculating the dipole moment at the kicker azimuth for each turn, assuming the dipole moment is measured by two upstream pick-ups and the machine optics functions are known.

A really serious restriction is that the *collective* phase advance coming from transverse impedances located in between the feedback pick-up and kicker should be small, otherwise it prevents us from evaluating the dipole moment at the kicker from its value(s) measured remotely at the pick-up(s). The reason is that “invisible” higher modes would disturb the transfer of the zeroth mode through the lattice optics.

Note, that the requirement for mutual coupling at the same point does not imply any problems with its physical feasibility, because at each turn the feedback oscillator kicks the beam proportionally to its *already known* coordinate. The reciprocal action from the beam results only in changing the oscillator’s momentum variable while its coordinate is left unchanged at the moment of interaction. It will be altered only after the next turn, so each

Figure 4: Mode frequencies Ω versus the interaction parameter Q . The parameter Q is scaled so that at $Q = 1$ the coherent tunes of the uncoupled mode $m = 0$ would be equal to the synchrotron tune Q_s .

kick of the feedback oscillator contains a cumulative effect of the of beam dipole moment on all the previous beam passages through the kicker, except the last passage.

The single-turn feedback intended for compensation of the mode $m = 0$ tune slope, which cannot be realized without a delay between the pick-up and kicker, will be accepted in the following extension of the algorithm as an exception. However, other possibilities with the feedback counteracting the TMC will be discussed in the end of this section, specifically, for this single-turn feedback and for an alternative compensation of the zeroth mode tune slope with an extra oscillator.

We introduce now the notation $A = ||a_{ij}||$ for the single-turn mapping matrix of $N + 1$ modes in question. Its order is $2(N + 1)$; its two first rows and columns correspond to transformation of the mode X_0 . In Figure 4 one can see the eigenvalues of this matrix vs. current. The mode system here is truncated to only three modes: $m = 0, +1, -1$. The eigenfrequencies exhibit the usual behavior without a feedback.

Let $y_1, \dots, y_N$ be the coupling coefficients between the modes and feedback oscillators, in the same sense as in Section 7.1, and Δ be the single-turn feedback coefficient.

We denote the real parts of eigenfrequencies of the feedback oscillators uncoupled with the modes' system as $\cos \alpha_1, \dots, \cos \alpha_N$, and let $\cos \mu_1, \dots, \cos \mu_{2N+1}$ correspond to the assigned eigenfrequencies of the united system of the modes and feedback oscillators.

Lastly, we denote

$$F(\lambda, \alpha_i) = \lambda^2 - 2\lambda \cos \alpha_i + 1$$

as an analogue of $\omega^2 - \omega_i^2$, used in Section 7.1, in the sense that equation $F(\lambda, \alpha_i) = 0$ gives the set of unperturbed eigenfrequencies of our system: $\lambda = e^{\pm i\alpha_i}$.

The $2(2N + 1)$ st order transfer matrix for $N + 1$ modes and N oscillators,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & 0 & \dots & & & \\ a_{21} & a_{22} & & & 0 & & & & \\ a_{31} & a_{32} & & & 0 & & & & \\ a_{41} & a_{42} & & & 0 & & & & \\ \dots & & & \dots & 0 & & & & \\ 0 & & & & \cos \alpha_1 & \sin \alpha_1 & 0 & 0 & \dots \\ 0 & & & & -\sin \alpha_1 & \cos \alpha_1 & 0 & 0 & \\ 0 & & & & 0 & 0 & \cos \alpha_2 & \sin \alpha_2 & \\ 0 & & & & 0 & 0 & -\sin \alpha_2 & \cos \alpha_2 & \\ \dots & & & & & & & & \dots \end{pmatrix}$$

should be right-multiplied by the interaction matrix ⁹

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \Delta & 1 & 0 & 0 & & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & & 0 & \dots & & \\ 0 & 0 & 0 & 1 & & 0 & & & \\ \dots & & & \dots & & 0 & & & \\ 0 & 0 & & & & 1 & 0 & 0 & 0 \dots \\ y_1 & 0 & & & & 0 & 1 & 0 & 0 \\ 0 & 0 & & & & 0 & 0 & 1 & 0 \\ y_2 & 0 & & & & 0 & 0 & 0 & 1 \\ \dots & & & & & & & & \dots \end{pmatrix}$$

in order to form the single-turn mapping matrix of the unified system B :

$$B = \begin{pmatrix} a_{11} + \Delta a_{12} & a_{12} & \dots & \dots & a_{12} & 0 & a_{12} & 0 & \dots \\ a_{21} + \Delta a_{22} & a_{22} & & & a_{22} & 0 & a_{22} & 0 & \\ a_{31} + \Delta a_{32} & a_{32} & & & 0 & \dots & & & \\ a_{41} + \Delta a_{42} & a_{42} & & & 0 & & & & \\ \dots & & & \dots & & & & & \\ y_1 \sin \alpha_1 & 0 & \dots & & \cos \alpha_1 & \sin \alpha_1 & 0 & 0 & \dots \\ y_1 \cos \alpha_1 & 0 & & & -\sin \alpha_1 & \cos \alpha_1 & 0 & 0 & \\ y_2 \sin \alpha_2 & 0 & & & 0 & 0 & \cos \alpha_2 & \sin \alpha_2 & \\ y_2 \cos \alpha_2 & 0 & & & 0 & 0 & -\sin \alpha_2 & \cos \alpha_2 & \\ \dots & & & & & & & & \dots \end{pmatrix}. \quad (28)$$

⁹Unities in the second row imply that the oscillator variables are already properly scaled; Δ stands for the thin lens, which only kicks the mode $m = 0$, representing the single-turn feedback; it takes the value of the total dipole moment and equally kicks all the particles.

Here we let the turn begin with the localized interaction followed by “free oscillations” of the feedback oscillators. The modes are coupled to each other via localized (and distributed) impedances.

Searching for the eigenvectors, taken in the form of a column:

$$(B - \lambda I) \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_{2(2N+1)} \end{pmatrix} = 0 \quad (29)$$

we can, similarly to Section 7.1, eliminate almost one half of the unknowns, if the relevant terms in the determinant are properly collected, thereby reducing the order of the system. To this end, we can substitute in the first equation of (29):

$$(a_{11} + \Delta a_{12} - \lambda)z_1 + \dots + a_{12}z_{2N+3} + a_{12}z_{2N+5} + \dots = 0 \quad (30)$$

the amplitude z_{2n+3} , expressed in terms of z_1 , as a solution of the corresponding pair of equations with the numbers $2n + 3$ and $2n + 4$:

$$y_1 \sin \alpha_1 z_1 + (\cos \alpha_1 - \lambda)z_{2N+3} + \sin \alpha_1 z_{2N+4} = 0$$

$$y_1 \cos \alpha_1 z_1 - \sin \alpha_1 z_{2N+3} + (\cos \alpha_1 - \lambda)z_{2N+4} = 0 ,$$

giving

$$z_{2N+3} = \frac{\lambda y_1 \sin \alpha_1 z_1}{F(\lambda, \alpha_1)} .$$

The same is possible for z_{2n+5} by using equations number $2n + 5$ and $2n + 6$, etc., and the first equation (30) is rewritten as:

$$(a_{11} - \lambda + a_{12}\Delta + a_{12} \sum_{i=1}^N \frac{\lambda y_i \sin \alpha_i z_i}{F(\lambda, \alpha_i)})z_1 + a_{12}z_2 + \dots = 0$$

The similar procedure transforms the second equation of (29):

$$(a_{21} + a_{22}\Delta + a_{22} \sum_{i=1}^N \frac{\lambda y_i \sin \alpha_i z_i}{F(\lambda, \alpha_i)})z_1 + (a_{22} - \lambda)z_2 + \dots = 0 ,$$

and so forth, resulting in an addition of

$$\Psi = \Delta + \sum_{i=1}^N \frac{\lambda y_i \sin \alpha_i z_i}{F(\lambda, \alpha_i)}$$

to all the elements of the first column of (28).

Thus the system of the order $2(2N + 1)$ is reduced to the $2(N + 1)$ th order by eliminating the feedback oscillators' amplitudes. Consequently the order of its determinant which generates the characteristic equation is reduced. Similarly to the transformations in Section 7.1, we separate the determinant into two parts:

$$\Psi \cdot R_1(\lambda) + R_2$$

and equate it to

$$H(\lambda) = \frac{\prod_{i=1}^{2N+1} F(\lambda, \mu_i)}{\prod_{i=1}^N F(\lambda, \alpha_i)},$$

where μ_i are the assigned phase advances and α_i are the unknown ones. As a result, we have:

$$\Psi = \frac{H(\lambda) - R_2(\lambda)}{R_1(\lambda)}. \quad (31)$$

To proceed similarly to Section 7.1, we need first to demonstrate an essential *reciprocal* property of the polynomials $R_1(\lambda), R_2(\lambda)$, *i.e.* that they both have their roots in reciprocal pairs. The demonstration is based on the fact that for any set of oscillators the equation $\det(B - \lambda I) = 0$ has all its roots in reciprocal pairs. Then we set $\Psi = 0$, and see that

$$R_2(\lambda) = \det(B|_{\Psi=0} - \lambda I)$$

is a reciprocal polynomial (See Section 6).

At the next step we take $\Psi = \Delta$, corresponding to the single turn feedback alone, while the oscillators are all set to zero. Then the polynomial

$$\Delta \cdot R_1(\lambda) + R_2(\lambda)$$

is reciprocal. Let us show, that the $2N$ th power polynomial $R_1(\lambda)$ has its last coefficient $R_1(0)$ equal to zero. Indeed, $R_1(0) = \det B = 1$, while $R_2(0) = 1$, being equal to the determinant of B without a feedback. Hence,

$$\Delta \cdot R_1(0) + R_2(0) = \Delta \cdot R_1(0) + 1 = 1,$$

or $R_1(0) = 0$, and we conclude, that $R_1(\lambda)$ is also reciprocal, *Q.E.D.*

Everywhere above we used the fact that for all interactions at one point with feedback oscillators (or without them) by means of action on angle by coordinate we always have the system with reciprocal eigenvalues (See Section 6).

Then we take all the $2N$ roots of $R_1(\lambda)$:

$$R_1(\lambda_i) = 0, \quad (i = 1, \dots, 2N),$$

and the numerator $H(\lambda) - R_2(\lambda)$ in (31), being divisible by the $2N$ th power polynomial $R_1(\lambda)$, yields a set of $2N$ equations:

$$H(\lambda_i) - R_2(\lambda_i) = 0, \quad (i = 1, \dots, 2N)$$

for N unknown oscillator frequencies, which are involved in the denominator of $H(\lambda)$:

$$H(\lambda) = \frac{\prod_{i=1}^{2N+1} F(\lambda, \mu_i)}{\prod_{i=1}^N F(\lambda, \alpha_i)}$$

Figure 5: An idealistic single-turn feedback compensates for the mode $m = 0$ tune slope.

Such a set of equations can in general be inconsistent, however in our case half of the equations are redundant, due to reciprocal property of $R_1(\lambda)$ and $R_2(\lambda)$. So, changing λ to $1/\lambda$, we see that equations

$$H(\lambda_i) - R_2(\lambda_i) = 0$$

and

$$H\left(\frac{1}{\lambda_i}\right) - R_2\left(\frac{1}{\lambda_i}\right) = 0$$

are reducible to each other.

Thus, similarly to Section 7.1, we have N equations for N unknown oscillator frequencies, and the $\cos \alpha_i$ are to be found as a complete set of N roots of a certain equation of the N th order. The coupling coefficients for all these oscillators can be found from a set of linear equations, provided that the above mentioned polynomials are reciprocal.

In Figures 5-8 one can see different cases with an idealistic single turn feedback and two feedback oscillators (Fig. 7 shows the solution with total compensation of current dependence of the eigenfrequencies).

7.4 Extension of the Algorithm to Radial Modes

An important feature of the hollow beam model is that it contains only one unperturbed mode $m = 0$ with nonzero dipole moment, a feature that was essential for evaluating the determinant and eigenvalues in the above algorithm. In this manner we could avoid directly solving a high-order nonlinear set of equations.

Given a general distribution over the synchrotron amplitudes, an infinite number of unperturbed radial modes with nonzero dipole moment appears. According to the numerical

Figure 6: Mode frequencies Ω vs. the beam intensity, two feedback oscillators are included, 50% compensation of the TMC.

Figure 7: The same with full compensation.

Figure 8: The same with 5-fold overcompensation. The synchrotron resonance can be seen.

procedure in Section 5, the bunch now is divided into several rings, each of which is specified by an amplitude variable (*i.e.* the Hamiltonian) H . Hence, we obtain as many unperturbed modes with $m = 0$, as we have rings.

However, we can always form a single linear combination of these modes to give a nonzero total dipole moment, and the complementary combinations, to give zero total dipole moment. The proof comes from linear algebra: a linear space can always be split into a direct sum of two subspaces:

- i) a one-dimensional line $a\mathbf{v}$ with arbitrary a and $\mathbf{v} = (1, \dots, 1)$;
- ii) a hyperplane, spanned by all the vectors $\mathbf{x} = (\dots, x_i, \dots)$, which are orthogonal to $\mathbf{v}$, so that $(\mathbf{x} \cdot \mathbf{v}) = 0 \Rightarrow \sum x_i = 0$.

If the x_i here denote the dipole moment of the rings, then all the combinations but one (namely, vector $\mathbf{v}$) have zero total dipole moment.

Such change of the basic set (now consisting of linear combinations, which themselves are not eigenvectors of the system) recovers the former situation. The previous algorithm for the hollow beam is again valid, the role of the single mode carrying the dipole moment and, therefore, coupled to the feedback system, is taken over by the “total dipole” combination.

7.5 Restrictions of the Algorithm

Here we give a few cases where the algorithm fails either partly, or completely.

Multiple Roots, which can occur at the stage when we find the feedback oscillator frequencies from a high-order polynomial equation, seriously complicate the problem. We can either avoid the multiplicity by tuning the assigned frequencies, or we must deal with the enlarged set of equations, including equations for zero derivative(s) of the polynomials

at the multiple roots.¹⁰

The Vanishing Coupling of certain modes to the zeroth mode will apparently exclude them from a set of modes curable by feedback, because the only means to affect them with a feedback, which does not resolve intra-bunch motion, was to use the mode $m = 0$ as a mediator of the interaction. This problem has no solution.

An Idealistic Single-Turn Feedback was implemented in the algorithm as a point-wise interaction with the single (thin lens) unknown parameter Δ , additional to the $2N$ oscillator parameters. In large machines with a substantial collective phase advance between the single-turn feedback pick-up and kicker, direct representation of a real feedback with such a model is no longer possible. To compensate for the $m = 0$ mode tune slope we can use one more feedback oscillator.

Let us consider a simple model of two oscillators with current-dependent coupling, the first oscillator corresponds to the synchrotron mode $m = 0$, and the second one is an artificial oscillator imitated by means of the transverse reactive feedback system.

At low current the coupling coefficients are negligible, so for the $m = 0$ mode oscillator we have a single-turn transfer matrix in terms of its dynamical variables d, d' (dipole moment of the bunch and its conjugate momentum):

$$\begin{bmatrix} \cos(\mu_\beta + kI) & \sin(\mu_\beta + kI) \\ -\sin(\mu_\beta + kI) & \cos(\mu_\beta + kI) \end{bmatrix},$$

where μ_β is the betatron phase advance, kI is the current-dependent tuneshift; $\beta' = 0$, $\beta = 1$ are taken to simplify the expressions.

Let us first assume that the feedback pickup and kicker stand at the same azimuth. We now introduce the second oscillator, which is realized by the feedback system, as a pair of dynamical variables x, x' with their single-turn transfer matrix (which simulates free oscillations of this oscillator), and μ is its phase advance:

$$\begin{bmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{bmatrix}$$

On passage through the kicker we will apply a kick to the mode oscillator, and its strength will be proportional to the “coordinate” x of the second (artificial) oscillator. We have also to add a kick to the momentum variable of the second oscillator on each passage through the pickup, and its strength should be proportional to the measured dipole moment of the bunch, *i.e.* to the coordinate variable d of the $m = 0$ mode:

$$\Delta x' = k_{fb} \cdot d$$

Thus the single-turn transfer matrix M for these two coupled oscillators, where a mutual kick is followed by a free oscillation, takes the form of the product:

$$M = \begin{bmatrix} \cos(\mu_\beta + kI) & \sin(\mu_\beta + kI) & 0 & 0 \\ -\sin(\mu_\beta + kI) & \cos(\mu_\beta + kI) & 0 & 0 \\ 0 & 0 & \cos \mu & \sin \mu \\ 0 & 0 & -\sin \mu & \cos \mu \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_x & 0 \\ 0 & 0 & 1 & 0 \\ k_{fb} & 0 & 0 & 1 \end{bmatrix}.$$

¹⁰We have in mind, that multiplicity of roots reduces the number of “useful” equations, however a corresponding number of equations is replenished from the condition, that the derivatives of the polynomial (up to a certain order) at such a point are zero, hence the number of equations will again suffice for evaluation of the unknown feedback oscillators’ frequencies, and then their coupling coefficients.

Here μ is the unknown tune of the feedback oscillator, while k_x and k_{fb} are the two coupling coefficients. Since they are always involved as a product in all the final formulas, we have only two unknown quantities (μ and $k_x \cdot k_{fb}$) to be found from the two *a priori* assigned eigenvalues $\lambda_{1,2} = e^{i\mu_{1,2}}$ of this 4-dimensional matrix (another pair of eigenvalues is just their inverse).

We can now look for the two unknowns so as to provide for the current-independent eigenfrequencies $\mu_{1,2}$ of the two normal modes of the system. After multiplication the single-turn transfer matrix becomes:

$$M = \begin{bmatrix} \cos(\mu_\beta + kI) & \sin(\mu_\beta + kI) & k_x \sin(\mu_\beta + kI) & 0 \\ -\sin(\mu_\beta + kI) & \cos(\mu_\beta + kI) & k_x \cos(\mu_\beta + kI) & 0 \\ k_{fb} \sin \mu & 0 & \cos \mu & \sin \mu \\ k_{fb} \cos \mu & 0 & -\sin \mu & \cos \mu \end{bmatrix}.$$

For matrices, whose eigenvalues come in reciprocally-inverse pairs, we have a general formula:

$$\cos \mu_{1,2} = \frac{1}{4} \text{tr} M \pm \frac{1}{2} \sqrt{\frac{1}{4} \text{tr}^2(A - D) + \text{tr} BC + 2 - \det A - \det D} \quad (32)$$

that is expressed in terms of 2×2 blocks A, B, C, D , into which our matrix can be partitioned:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Current-independent (and equal) solutions $\mu_1 = \mu_2 = \mu_\beta$ appear at zero value of the square root in (32), hence we find a set of equations:

$$\begin{aligned} \text{tr} BC &= -\frac{1}{4} \text{tr}^2(A - D) \\ \frac{1}{4} \text{tr} M &= \cos \mu_\beta, \end{aligned}$$

and the desired solutions are

$$\begin{aligned} \cos \mu &= 2 \cos \mu_\beta - \cos(\mu_\beta + kI) \\ k_x \cdot k_{fb} &= -\frac{[\cos \mu - \cos(\mu_\beta + kI)]^2}{4 \sin \mu \sin(\mu_\beta + kI)} \\ &= -\frac{[\cos \mu_\beta - \cos(\mu_\beta + kI)]^2}{\sin \mu \sin(\mu_\beta + kI)}. \end{aligned} \quad (33)$$

This can be directly verified by substituting (33) into (32).

The influence of the neglected $m = -1$ and $m = +1$ modes on this solution can be evaluated numerically. In Fig. 9 the eigenvalues are plotted vs. the beam current for a small strength of the additional oscillator. In Fig. 10 the oscillator tune and coupling correspond to the solution (33). One can see a rather strong enhancement of the TMC threshold.

However this oscillator seriously upsets the general algorithm and we would defer the further details concerning this option.

The Separate Location of the Feedback Pick-up and the Kicker may still destroy the reciprocal property of the eigenvalues, which was proved only for point-wise interaction

Figure 9: Mode frequencies Ω of the system with one additional feedback oscillator. Feedback action is small.

Figure 10: Mode frequencies Ω of the system with one additional feedback oscillator. Interaction with the feedback oscillator eliminates the slope of the “zero” mode (See solution (33)).

Figure 11: A single turn reactive feedback with the kicker action delayed by one revolution compensates for the mode $m = 0$ tune slope. However, imaginary parts appear without a threshold, grow with the beam intensity and cause the instability in spite of separation of the real tunes.

of the beam with the feedback. Only the case where there are no large impedances (and no collective phase advance) in the machine section between the pick-up and kicker is apparently reducible to the point-wise interaction by recalculation of the measured dipole moment to the kicker azimuth from the known lattice functions. Preliminary results (see Fig. 11) of a test of the general case with a computer code is negative: the presence of transverse impedance components in the section between the feedback pick-up and kicker leads to the occurrence of imaginary parts in mode frequencies which start without a threshold and grow with the beam intensity. This results in the instability of at least one of the modes (one should have in mind that the sum of the imaginary mode frequencies is always zero for systems with undamped oscillators since $\det M = 1$).

Damping of the Feedback Oscillators is unlikely to obtain the desired sign of imaginary parts in all mode frequencies. This problem is of practical interest by itself even with no relevance to the stability. Of course, the kick matrices responsible for coupling the beam modes to the feedback oscillators have vacant elements for us to introduce more additional parameters. These possibilities include forces which are proportional to velocities, incrementation of the coordinate variable of the feedback oscillator (obviously feasible if this oscillator is represented with digital hardware), etc.

The general study of such systems is unfortunately impeded by the occurrence of an intractable set of non-linear equations. Even the existence of a solution is difficult to prove.

8 Conclusion

In conclusion we again emphasize the importance of the generalized symplecticity of the modes+oscillators system in the regular procedure for finding the feedback oscillators parameters aimed at elimination of the TMC instability thresholds. In fact this property allows further conclusions: we can be sure, that imaginary parts of the mode frequencies will only appear when their real parts merge. This means that the higher modes which we neglect when constructing the solution for a finite mode set, will be stable unless their coupling grows high enough (with the beam intensity) to merge them, provided that the feedback oscillators' frequencies stay well apart from these modes' frequencies. This should be taken into account when we assign the modes frequencies: the solutions for the feedback oscillators' frequencies should not fall in the domain where the neglected modes' frequencies dwell.

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