(S)quark Masses
and Non-Abelian Horizontal Symmetries

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We present a model of quark and squark masses which is based on a non-Abelian horizontal symmetry. It leads to order of magnitude relations between quark mass ratios and mixing angles and to the successful exact relation \( \sin \theta_C = \sqrt{\frac{m_d}{m_s}} \) to better than 20\% accuracy. The non-Abelian symmetry also ensures the necessary squark degeneracy to suppress FCNC mediated by loops with squarks and gluinos, in the neutral meson systems.
One of the most puzzling open questions in particle physics is the origin of the small numbers in the Standard Model Lagrangian. These numbers include the mixing angles and the ratios of quark masses. Typically in physics, small numbers are associated with an explicitly broken symmetry which becomes exact when these numbers are set to zero [1]. Several authors have suggested to use a horizontal symmetry $\mathcal{H}$, which acts on the quarks and the leptons, as the underlying symmetry controlling the small numbers (for a recent discussion, see e.g. references [2]). Such an explicitly broken horizontal symmetry can arise naturally [3] when an exact symmetry $\mathcal{H}$ is spontaneously broken by an expectation value of some scalar field $\langle S \rangle$. The small numbers in the Lagrangian then appear as powers of the ratio $\lambda = \frac{\langle S \rangle}{M}$, where $M$ is the scale of higher energy physics which communicates the information about $\mathcal{H}$ breaking to the quarks. Explicit mechanisms based on massive fermions with mass $M$ were suggested in references [4] [5] [6]. For simplicity, we rescale $S$ and set $M = 1$.

The simplest $\mathcal{H}$ is Abelian and it easily leads to order of magnitude relations between quark mass ratios and angles [6] [3]. These explain the Wolfenstein parametrization of the CKM matrix

$$V_{\text{CKM}} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & \lambda^3 A (\rho + i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & \lambda^2 A \\
\lambda^3 A (1 - \rho + i\eta) & -\lambda^2 A & 1
\end{pmatrix}$$

with $A \sim 1$, $\lambda \sim 0.2$ and the mass ratios and phases

$$
m_u/m_c \sim \lambda^3 - \lambda^4 \quad m_c/m_t \sim \lambda^3 - \lambda^4 \\
m_d/m_s \sim \lambda^2 \quad m_s/m_b \sim \lambda^2 \\
m_b/m_t \sim \lambda^2 - \lambda^3 (\lambda^3) \quad \rho, \eta \sim 1 - \lambda 
$$

These estimates are valid in the TeV range (when the expressions at high energy, such as $10^{15}$ GeV, are different, we have indicated them in parentheses). Here and throughout this work we take $\tan \beta = \frac{\langle H_u \rangle}{\langle H_d \rangle} \sim 1$. Note that order of magnitude estimates as those in equation (2) are ambiguous and sensitive to the exact value of $\lambda$.

A better theory should be more predictive. One would like to find some exact or at least approximate relations and not only order of magnitude ones. Such relations can arise from a more detailed knowledge of the high energy theory. For example in grand unified
theories there are some exact relations among quark mass ratios and mixing angles [7]. The approach we will take here is based on a non-Abelian $\mathcal{H}$. The group theory of $\mathcal{H}$ can fix some of the couplings of order one as Clebsch-Gordan coefficients and lead to exact relations. A successful exact relation which we will try to explain is

$$\sin\theta_C = \sqrt{\frac{m_d}{m_s}}$$  \hspace{1cm} (3)

which is satisfied at about $5 - 10\%$, namely to $O(\lambda - \lambda^2)$. Recent CLEO results for $|V_{ub}/V_{cb}|$ [8] may favor $\rho, \eta \sim \lambda$, which would lead to another close-to-exact relation

$$|V_{td}| = |V_{cb}V_{us}|[1 + O(\lambda)] \ .$$ \hspace{1cm} (4)

A related issue is the squark spectrum. It is constrained by FCNC [9]. Usually these constraints are satisfied by postulating that all squarks are degenerate. In standard supergravity or string theory this degeneracy is not natural. However, under special circumstances in string theory the squarks might be naturally degenerate [10]. Also, if supersymmetry breaking is fed to the squarks by QCD interactions, the squarks are degenerate [11]. In reference [12] it was suggested that FCNC constraints can be satisfied with non-degenerate squarks by arranging that the quark and squark mass matrices be approximately diagonal in the same basis. This can be achieved with an Abelian $\mathcal{H}$. An alternative, which we will follow here is to use a non-Abelian $\mathcal{H}$ to guarantee the degeneracy of some of the squarks [13], along with alignment.

The goal of this work is to use a non-Abelian horizontal symmetry $\mathcal{H}$ which

1. explains the order of magnitude relations (1) and (2) as in references [6][3],
2. leads to the approximate relations (3) and (4),
3. guarantees the degeneracy of the squarks $\tilde{d}$ and $\tilde{s}$ and the degeneracy of $\tilde{d}$ and $\tilde{s}$ to be compatible with the data on $K - \bar{K}$ mixing as in reference [13] and
4. aligns the quark and squark mass matrices as in references [12] [3] to be consistent with all other FCNC data.

Below we will present an example satisfying these requirements. Our example is not simple and is certainly not unique. It should be thought of as an existence proof that such a program is possible.
A few general arguments guide us in the search for a model.

We will use the following notation for the superfields. $Q_i$ denote the left-handed doublets and $\bar{d}_i (\bar{u}_i)$ denote the left-handed anti-down (anti-up) $SU(2)$ singlets. The mass matrices are $M^d$ and $M^u$.

Since the top mass is of the same order of magnitude as the Higgs VEVs, $M^u_{3,3}$ should not be suppressed by any power of the small parameter $\lambda$. Therefore, using the baryon number symmetry the fields $Q_3$ and $\bar{u}_3$ can be taken to transform trivially under $\mathcal{H}$.

To obtain equation (3) as a prediction we should use $\mathcal{H}$ to ensure the equality of $M^d_{1,2}$ and $M^d_{2,1}$, and make $M^d_{1,1} \ll (M^d_{1,2})^2 / M^d_{2,2}$. Also, we should make sure that the contribution to $V_{us}$ from the up sector is sufficiently small. This can be achieved when $Q = (Q_1, Q_2)$ transforms as a doublet of $\mathcal{H}$ and either $\bar{d} = (\bar{d}_1, \bar{d}_2)$ is a doublet and $\bar{d}_3$ a singlet or $\bar{d} = (\bar{d}_1, \bar{d}_2, \bar{d}_3)$ is a triplet. We will focus on the first of these possibilities. This choice of representations also guarantees the squark degeneracy mentioned above.

The $\bar{u}$ quarks could all be (perhaps non-trivial) singlets or $\bar{u} = (\bar{u}_1, \bar{u}_2)$ could be a doublet of $\mathcal{H}$. We will focus on the latter possibility, which also leads to the degeneracy of the squarks $\bar{u}$ and $\bar{c}$.

As explained in reference [6], at least one of the standard model singlet fields $S$ has to be in a multi-dimensional irreducible representation of $\mathcal{H}$; we take it to be a doublet. We also found it helpful to add an extra Standard Model singlet field $T$ to explain $M^d_{3,3} \ll M^u_{3,3}$ when $\tan \beta \sim 1$.

We now consider our explicit example. We start with the group $U(1) \times O(2)$ and label the representations in terms of the charges under the two factors. We assign the quarks and scalars to

$$
\begin{align*}
Q & \quad \bar{d} & \quad \bar{d}_3 & \quad \bar{u} & \quad S & \quad T \\
(8, 2) & \quad (16, 2) & \quad (12, 0) & \quad (11, 1) & \quad (-5, 1) & \quad (-4, 0)
\end{align*}
$$

where $\bar{d}_3$ and $T$ transform trivially under $O(2)$. Since both charges have the same parity,

\footnote{\(O(2) \cong Z_2 \ltimes U(1)\) has two-dimensional representations labeled by a positive integer charge under the $U(1)$ subgroup and two different one-dimensional representations with vanishing charge and different $Z_2$ assignments.}
only \( G = (U(1) \times O(2))/Z_2 \) is represented\(^2\). (The \( U(1) \) charges in equation (5) are smaller and look more natural, if the \( U(1) \) group is replaced by its \( Z_9 \) subgroup. This does not affect our results.)

The Yukawa terms in the Lagrangian lead to the mass matrices

\[
M^d = y_d \langle H_d \rangle \begin{pmatrix}
T S^4_2 & T^6 + \eta TS^2_1 S^2_2 & \gamma S^3_1 S^3_2 \\
T^6 + \eta TS^2_1 S^2_2 & T S^4_1 & \gamma S^3_1 S^3_2 \\
0 & 0 & T^3
\end{pmatrix}
\]

(6)

\[
M^u = y_u \langle H_u \rangle \begin{pmatrix}
\alpha TS^3_2 & \beta TS^3_1 S^2_2 & 0 \\
\beta TS^3_1 S^2_2 & \alpha TS^3_1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

(7)

here we have used a basis for the two-dimensional representations in which the \( U(1) \) sub-group of \( O(2) \) is diagonal. In writing these equations two of the Yukawa coefficients have been absorbed into the definitions of the fields \( S \) and \( T \). (We remind the reader that we have also divided the fields \( S \) and \( T \) by the scale \( M \) at which symmetry breaking is communicated to the quarks.)

A scalar potential which we do not discuss leads to expectation values to the two scalars in \( S \) and to the scalar \( T \). The three expectation values can be different and lead to three stages of symmetry breaking. Instead, we assume that first \( S_1 \) and \( T \) acquire expectation values of order \( \lambda \sim 0.2 \) and break \( G \) to \( Z_2 \) which is broken at a lower scale by \( \langle S_2 \rangle \sim \lambda^2 \sim 0.04 \). (In the version of the model based on \( Z_9 \subset U(1) \) this pattern of symmetry breaking is natural (i.e. \( \langle S_2 \rangle \ll \langle S_1 \rangle, \langle T \rangle \) is protected by a symmetry) only when another \( Z_4 \) group under which \( S \) and \( u \) have charge one is added.)

To leading order in \( \lambda \), the mass matrices of equations (6) and (7) lead to the mass

\(^2\) Note that \( G \cong Z_2 \ltimes (U(1) \times U(1)) \) which suggests a connection to string inspired models.

Non-Abelian discrete groups of the form \( S_m \ltimes (Z_n \times Z_n \cdots Z_n) \) with \( m \) factors of \( Z_n \) are common in Calabi-Yau compactifications. For \( m = 2 \) this is a discrete subgroup of our \( G \).
ratios\(^3\)

\[
\begin{align*}
\frac{m_s}{m_b} & \approx \frac{S_1^4}{T^2} \sim \lambda^2 \\
\frac{m_c}{m_t} & \approx \alpha T S_1^3 \sim \lambda^4 \\
\frac{m_d}{m_s} & \approx \frac{T^{10}}{S_1^8} \sim \lambda^2 \\
\frac{m_u}{m_c} & \approx \left(1 - \frac{\beta^2}{\alpha^2}\right) \frac{S_2^3}{S_1^2} \sim \lambda^3 \\
\frac{m_b}{m_t} & \approx \frac{y_d T^3}{y_u \tan \beta} \sim \lambda^3
\end{align*}
\]

and the mixing angles

\[
\begin{align*}
|V_{us}| & = \frac{T^5}{S_1^4} + \mathcal{O}\left(\frac{S_2^2}{S_1^2}\right) = \sqrt{\frac{m_d}{m_s}} [1 + \mathcal{O}(\lambda)] \sim \lambda \\
|V_{cb}| & \approx \frac{\gamma S_1 S_2}{T^3} \sim \lambda^2 \\
|V_{ub}| & \approx \frac{\gamma S_1 S_2}{T^3} \left(1 - \frac{\beta}{\alpha}\right) \sim \lambda^4 \\
|V_{td}| & = |V_{us} V_{cb}| [1 + \mathcal{O}(\lambda)] \sim \lambda^3
\end{align*}
\]

The (hermitian) squark mass matrices are 6 × 6 matrices. The off-diagonal 3 × 3 blocks are proportional to \(\bar{m} \langle H_u, d\rangle\) where \(\bar{m} \sim 1\) TeV is a typical squark mass and are similar to the quark mass matrices (in order of magnitude but not necessarily the same coefficients). If \(\bar{m}^2\) is much larger than the entries in the off-diagonal blocks, we can focus on the diagonal 3 × 3 blocks. The leading order contributions to the mass square matrices are

\[
\begin{align*}
\tilde{M}_Q^2 & = \bar{m}^2 \delta^Q = \bar{m}^2 \left(\begin{array}{ccc}
a_Q & c_Q S_1^2 S_1^{*2} & d_Q S_2 S_1^{*2} T^2 \\
a_Q & d_Q S_2 S_1^{*2} T^2 & b_Q \end{array}\right) \sim \bar{m}^2 \left(\begin{array}{ccc}
1 & \lambda^6 & \lambda^5 \\
\lambda^5 & 1 & \lambda^5 \\
\lambda^5 & \lambda^5 & 1
\end{array}\right) \\
\tilde{M}_d^2 & = \bar{m}^2 \delta^d = \bar{m}^2 \left(\begin{array}{ccc}
a_d & c_d S_2 S_1^{*2} & d_d S_2 S_1^{*2} T \\
a_d & d_d S_2 S_1^{*2} T & b_d \\
\end{array}\right) \sim \bar{m}^2 \left(\begin{array}{ccc}
1 & \lambda^6 & \lambda^4 \\
\lambda^4 & 1 & \lambda^4 \\
\lambda^4 & \lambda^4 & 1
\end{array}\right) \\
\tilde{M}_u^2 & = \bar{m}^2 \delta^u = \bar{m}^2 \left(\begin{array}{ccc}
a_u & c_u S_2 S_1^{*} & d_u S_1 T^4 + d_u S_1 S_2 T^4 \\
a_u & d_u S_1 T^4 + d_u S_1 S_2 T^4 & b_u \\
\end{array}\right) \sim \bar{m}^2 \left(\begin{array}{ccc}
1 & \lambda^3 & \lambda^5 \\
\lambda^5 & 1 & \lambda^5 \\
\lambda^5 & \lambda^5 & 1
\end{array}\right)
\end{align*}
\]

where the coefficients \(a_i, b_i, c_i, d_i, d'_u\) are constants of order one. Note that the first two diagonal entries are equal up to corrections proportional to \((S_1 S_1^{*} - S_2 S_2^{*})\) which are of order \(\lambda^2\). The third diagonal entry can be different.

\(^3\) Off-diagonal wave function renormalization which depends on \(\lambda\) should also be taken into account [14] [3]. However, because of the quantum numbers of the fields in our model this effect is negligible.
The FCNC constraints on the squark mass matrices are best presented in the basis related by supersymmetry to the basis where the quark mass matrices are diagonal. We transform the matrices $\tilde{\delta}$ in equations (10), (11) and (12) to this basis and find four new matrices $\delta^u$, $\delta^d$, $\delta^u$ and $\delta^d$. The values of the experimental upper bounds and the order of magnitude predictions in our model are

\[
\begin{array}{cccccccc}
(\delta^d_{ds} \delta^d_{ds})^{1/2} & \delta^d_{ds} & \delta^d_{ds} & (\delta^d_{db} \delta^d_{db})^{1/2} & \delta^d_{db} & \delta^d_{db} & (\delta^u_{uc} \delta^u_{uc})^{1/2} & \delta^u_{uc} & \delta^u_{uc} \\
0.006 & 0.05 & 0.05 & 0.04 & 0.1 & 0.1 & 0.04 & 0.1 & 0.1 \\
\lambda^3 & \lambda^3 & \lambda^3 & \lambda^{3.5} & \lambda^3 & \lambda^4 & \lambda^{3.5} & \lambda^4 & \lambda^3 \\
\end{array}
\] (13)

The experimental upper bounds suffer from multiplicative ambiguities of order $3 - 4$. Similarly, there is also a factor of order one ambiguity in these predictions of our model which arises from the coefficients we do not know.

The bound on $K - \bar{K}$ mixing is satisfied because the squarks in the first two generations have sufficiently degenerate masses. $B - \bar{B}$ and $D - \bar{D}$ mixing bounds are satisfied because of alignment [12]; in the basis where the quark mass matrices are diagonal, the squark mass matrices are also approximately diagonal.

CP violation in the K system constrains $\text{Im} \ \delta^d_{ds}$ and $\text{Im} \ \delta^d_{ds}$ to be about an order of magnitude below the values quoted in equation (13). If all phases in the various mass matrices are of order one, the leading contribution to these imaginary parts is of order $\lambda^3$ which is too large. Therefore, to be consistent with the experimental value of $\epsilon_K$ some of the phases should be small. This can arise only from a better understanding of the origin of CP violation.

So far we have not discussed all the terms in the Lagrangian. Without the “$\mu$ term” $\mu H_u H_d$, the theory has another $U(1)_{PQ}$ symmetry. The horizontal symmetry can be taken as any subgroup $\mathcal{H} \subset G \times U(1)_{PQ}$. One should only make sure that the extra terms which are allowed by $\mathcal{H}$ do not ruin our results. The group $\mathcal{H}$ can be continuous or discrete. If it has discrete factors, one needs to make sure that there is no cosmological problem with domain walls when $\mathcal{H}$ is broken.

If squark masses are ever measured, a number of patterns can be found:

1. All twelve squarks are degenerate. The explanation of this fact is probably unrelated to the origin of quark masses.
2. The squark pairs \((\tilde{d}, \tilde{s}), (\tilde{d}, \tilde{s}), (\tilde{u}, \tilde{c}), (\tilde{u}, \tilde{c})\) are degenerate. The different pairs are not necessarily degenerate and neither are the other squarks. Such a situation arises in models with a horizontal symmetry where \(\tilde{u}\) is in a doublet of \(\mathcal{H}\) as in the model of reference [13] and in our model.

3. The degeneracy is as in the previous case except that the pair \((\tilde{u}, \tilde{c})\) is not degenerate. This can be a signal that \(\tilde{u}\) is in a reducible representation of \(\mathcal{H}\). For example, we can replace the doublet \(\tilde{u}\) in our model with two singlets \(\tilde{u}_1\) and \(\tilde{u}_2\) transforming under \(G\) like \((12, 0)\) and \((10, 0)\) respectively and find a model with a massless up quark. In this model, \(D - \bar{D}\) mixing is suppressed by one more power of \(\lambda\) relative to the aligned models of reference [12]. However, it is still much larger than predicted by the Standard Model or by supersymmetric models where all squarks are degenerate. This model is less constrained in the quark sector, allowing for smaller \(m_u/m_c\) and larger \(V_{ub}\) than the previous model.

4. No two squarks are degenerate as in the aligned models of references [12],[3].

To summarize, our model is natural. We include in the Lagrangian all terms consistent with the symmetry with coefficients of order one. \(\mathcal{H}\) is broken in two stages. At the first step all scalars invariant under the unbroken symmetry acquire VEVs and the other scalars are protected.

The only small parameters are in the VEVs of the singlets \(S\) and \(T\) which break \(\mathcal{H}\). By raising them to various powers which are determined by \(\mathcal{H}\), we generate all the small parameters of the squark sector. We find several order of magnitude relations as well as the approximate relation \(\sin \theta_C = \sqrt{\frac{m_u}{m_c}}\) at the 20\% level (order \(\lambda\)). With an appropriate choice of Yukawa couplings of order one, our results are consistent with all known phenomenology of masses and mixing angles. Also, as in reference [13], we explain some of the degeneracy among the squarks thus ensuring the suppression of FCNC.

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References