Scalar Field Quantization
on the 2+1 Dimensional Black Hole Background

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Abstract

The quantization of a massless conformally coupled scalar field on the 2+1 dimensional Anti de Sitter black hole background is presented. The Green’s function is calculated, using the fact that the black hole is Anti de Sitter space with points identified, and taking into account the fact that the black hole spacetime is not globally hyperbolic. It is shown that the Green’s function calculated in this way is the Hartle-Hawking Green’s function. The Green’s function is used to compute \( \langle T^\mu_\nu \rangle \), which is regular on the black hole horizon, and diverges at the singularity. A particle detector response function outside the horizon is also calculated and shown to be a fermi type distribution. The back-reaction from \( \langle T^\mu_\nu \rangle \) is calculated exactly for the vacuum spacetime (with zero mass) and is shown to give rise to a curvature singularity at \( r = 0 \) shielded by a horizon, indicating that the endpoint of evaporation should be regarded as a spacetime with a horizon.

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INTRODUCTION

The study of black hole physics is complicated by the many technical and conceptual problems associated with quantum field theory in curved spacetime. One serious difficulty is that exact calculations are almost impossible in 3+1 dimensions. In this paper we shall instead study some aspects of quantum field theory on a 2+1 dimensional black hole background. This enables us to obtain an exact expression for the Green’s function of a massless, conformally coupled scalar field in the Hartle-Hawking vacuum [1]. We use this Green’s function to study particle creation by the black hole, back-reaction and the endpoint of evaporation.

We shall work with the 2+1 dimensional black hole solution found by Bañados, Teitelboim and Zanelli (BTZ) [2]. It had long been thought that black holes cannot exist in 2+1 dimensions for the simple reason that there is no gravitational attraction, and therefore no mechanism for confining large densities of matter. This difficulty has been circumvented in the BTZ spacetime\(^1\), but not surprisingly, their solution has some features that we do not normally associate with black holes in other dimensions, such as the absence of a curvature singularity. It is interesting to ask whether this spacetime behaves quantum mechanically in a way consistent with more familiar black holes.

The spinless BTZ spacetime has a metric [2]

\[ ds^2 = -N^2 dt^2 + \frac{N^2}{r^2} dr^2 + r^2 d\phi^2 \]

where

\[ N^2 = \frac{r^2 - r_+^2}{\ell^2}, \quad r_+ = M \ell. \]

Here \( M \) is the mass of the black hole. The metric is a solution to Einstein’s equations with a negative cosmological constant, \( \Lambda = -\ell^{-2} \), and the curvature of the black hole solution is constant everywhere. As a result there is no curvature singularity as \( r \to 0 \). A Penrose diagram of the spacetime is given in Fig. 1.

An important feature of the BTZ solutions is that the solution with \( M = 0 \) (which we refer to as the vacuum solution), is not AdS\(_3\). Rather, it is a solution that is not globally Anti de Sitter invariant. It has no horizon, but does have an infinitely long throat for small \( r > 0 \), which is reminiscent of the extreme Reissner-Nordström solution in 3+1 dimensions. It is worth noting that there are other similarities between the spinless BTZ black holes, \( M \geq 0 \), and the Reissner-Nordström solutions for \( M \geq Q \). In particular, the temperature associated with the Euclidean continuation of the BTZ black holes has been computed in [2], and it was found to increase with the mass, and to decrease to zero as \( M \to 0 \). Thus, if we carry over the usual notions from four dimensional black holes, the \( M = 0 \) solution appears to be a stable endpoint of evaporation.

A feature of the BTZ solution that we shall make use of, is that the solution arises from identifying points in AdS\(_3\), using the orbits of a spacelike Killing vector field. It is

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\(^1\)A charged black hole solution in 2+1 dimensions had previously been found in Ref. [3]. For further discussions on the BTZ black hole, see Refs. [4].
this property that is the starting point of our derivation of a Green’s function on the black hole spacetime. We construct a Green’s function on AdS$_3$, and this translates to a Green’s function on the black hole via the method of images.

A Green’s function constructed in this way is only interesting if we can identify the vacuum with respect to which it is defined. We prove that our construction gives the Hartle-Hawking Green’s function. It is worth noting that for the BTZ black hole, there is a limited choice of vacua. Quantisation on AdS$_3$ is hampered by the fact that AdS$_3$ is not globally hyperbolic, and this necessitates the use of boundary conditions at spatial infinity [5] (see Fig. 2), as discussed in Appendix A. This problem carries over to the black hole solution, and as a result, the value of the field at spatial infinity is governed by either Dirichlet or Neumann type boundary conditions. Thus a Cauchy surface for the region R of the BTZ black hole is either the past or the future horizon only. With this knowledge, the natural definition of the Hartle-Hawking vacuum is with respect to Kruskal modes on either horizon, whereas there does not appear to be a natural definition of an Unruh vacuum (see [6,7] for a discussion of the various eternal black hole vacua). The definition of an Unruh vacuum might be possible given a description of the formation of a BTZ black hole from the vacuum via some sort of infalling matter, but as far as we are aware, no such construction has been found.

Having an explicit expression for the Hartle-Hawking Green’s function, we are able to obtain a number of exact results. As a check, we show that it satisfies the KMS thermality condition [8]. We then compute the expectation value of the energy-momentum tensor and the response of a particle detector for both nonzero $M$ and for the vacuum solution. For nonzero $M$ we address the issue of whether the response of the particle detector can be interpreted as radiation emitted from the black hole, although a clear picture does not emerge.

For the $M = 0$ solution, we find a non-zero energy-momentum tensor, although the corresponding Green’s function is at zero temperature, and there is no particle detector response. We interpret this as a sort of Casimir energy. Classically, the vacuum solution appears to be similar to the extremal Reissner-Nordström solution, in the sense that we expect that if any matter is thrown in, a horizon develops. Quantum mechanically, the $M = 0$ solution appears to be unstable to the formation of a horizon, when the back-reaction caused by the Casimir energy is taken into account. This suggests that the end-point of evaporation should be thought of as a solution with a horizon.

The paper is organized as follows. In section I we study the 1+1 dimensional solution which arises from a dimensional reduction of the BTZ black hole [9], and show that the vacuum defined by the Anti de Sitter (AdS) modes is the same as that defined by the Kruskal modes; with this encouraging result we tackle the 2+1 case. Section II contains a review of the essential features of the geometry of the BTZ black hole. In section III we construct the Wightman Green’s functions on the black hole spacetime from the AdS$_3$ Wightman function, using the method of images. We then show that the Green’s function coincides with the Hartle-Hawking Green’s function [1], by showing that it is analytic and bounded in the lower half of the complex $\bar{V}$ plane on the past horizon ($\bar{U} = 0$), where $\bar{V}, \bar{U}$ are the Kruskal null coordinates. We also compute the Wightman function for the $M = 0$ solution, and compare this to the $M \rightarrow 0$ limit of the results for $M \neq 0$. Section IV contains a calculation of $\langle T_{\mu\nu} \rangle$ for all $M \geq 0$. For the black hole solutions, it is regular on
the horizon, and for all \( M \) it is singular as \( r \to 0 \). In Section V the response function of a stationary particle detector outside the horizon is calculated and shown to be of a fermi type distribution. A discussion is given of how this response might be interpreted. In Section VI we calculate the back reaction on the \( M = 0 \) solution induced by \( \langle T_{\mu\nu} \rangle \), and show that the spacetime develops a curvature singularity and a horizon. Throughout this paper we use metric signature \((-+++)\), and natural units in which \( \hbar = c = 1 \).

I. 2-D BLACK HOLE

Let us begin by looking at quantum field theory on the region of Anti de Sitter spacetime in 1+1 dimensions described by the metric

\[
  ds^2 = -\left( \frac{r^2 - M\ell^2}{\ell^2} \right) dt^2 + \left( \frac{r^2 - M\ell^2}{\ell^2} \right)^{-1} dr^2 \quad 0 < r < \infty \quad -\infty < t < \infty,
\]

where \( M \) is the mass of the solution. This metric was discussed in [9] as the dimensional reduction of the spinless BTZ black hole, and can be thought of as being a region of AdS\(_2\) in Rindler-type co-ordinates. Under the change of co-ordinates

\[
  r = \sqrt{M\ell^2} \sec \rho \cos \lambda, \quad \tanh \left( \frac{\sqrt{M\ell}}{\ell} \right) = \frac{\sin \rho}{\sin \lambda},
\]

where we shall call \((\lambda, \rho)\) AdS co-ordinates, the metric becomes

\[
  ds^2 = \ell^2 \sec^2 \rho (-d\lambda^2 + d\rho^2)
\]

which for \(-\frac{\pi}{2} \leq \rho \leq \frac{\pi}{2}\) and \(-\infty < \lambda < \infty\) is just AdS\(_2\) [10].

It is possible to define Kruskal-like co-ordinates for this black hole, which do not coincide with the usual AdS co-ordinates. For \( r > M\ell^2 \), they are:

\[
  U = \left( \frac{r - \sqrt{M\ell}}{r + \sqrt{M\ell}} \right)^{\frac{1}{2}} \cosh \frac{\sqrt{M\ell}}{\ell} \quad t
\]

\[
  V = \left( \frac{r - \sqrt{M\ell}}{r + \sqrt{M\ell}} \right)^{\frac{1}{2}} \sinh \frac{\sqrt{M\ell}}{\ell} t. \quad (1.1)
\]

The metric then takes the form

\[
  ds^2 = \frac{-2\ell^2}{1 + UV} dU dV
\]

where \( \bar{U} = V + U, \bar{V} = V - U \), and the transformation between Kruskal and AdS co-ordinates is

\[
  \bar{U} = \tan \left( \frac{\rho + \lambda}{2} \right) \quad \bar{V} = \tan \left( \frac{\rho - \lambda}{2} \right)
\]

which is valid over all the Kruskal manifold. The Kruskal co-ordinates cover only the part of AdS\(_2\) with
\[-\frac{\pi}{2} \leq \rho \leq \frac{\pi}{2} \quad -\frac{\pi}{2} < \lambda < \frac{\pi}{2}\].

We shall now show that the notion of positive frequency in \((\lambda, \rho)\) (AdS) modes coincides with that defined in \((\tilde{U}, \tilde{V})\) (Kruskal) modes.

The AdS modes for a conformally coupled scalar field are normalized solutions of \(\Box \psi = 0\), subject to the boundary conditions

\[
\phi \left( \rho = \frac{\pi}{2} \right) = \phi \left( \rho = -\frac{\pi}{2} \right) = 0.
\]

The positive frequency modes are then

\[
\phi_m = \frac{1}{\sqrt{m \pi}} e^{-im \lambda} \sin m \rho \quad m \text{ even } \geq 0
\]

\[
\phi_m = \frac{1}{\sqrt{m \pi}} e^{-im \lambda} \cos m \rho \quad m \text{ odd } \geq 0
\]

and these define a vacuum state \(|0\rangle_A\) in the usual way.

The Kruskal modes are solutions of \(\Box \psi = 0\) with the boundary condition \(\psi(\tilde{U} \tilde{V} = -1) = 0\). Positive frequency solutions are given by

\[
\psi_\omega = N_\omega \left( e^{-i \omega \tilde{U}} - e^{i \omega \tilde{V}} \right) \quad \omega > 0
\]

where \(N_\omega = (8\pi \omega)^{-1/2}\), and these define \(|0\rangle_K\). These modes are analytic and bounded in the lower half of the complex \(\tilde{U}, \tilde{V}\) plane. In order to show equivalence of the two vacua \(|0\rangle_A\) and \(|0\rangle_K\), it is enough to show that the positive frequency AdS modes can be written as a sum of only positive frequency Kruskal modes. Because of the analyticity properties of the Kruskal modes, it is enough to show that the AdS modes are analytic and bounded in the lower half of the complex \(\tilde{U}, \tilde{V}\) plane \([6,11]\). Changing co-ordinates, we have

\[
\phi_m = \frac{1}{\sqrt{m 2i}} \left( e^{-2im \arctan \tilde{V}} - e^{-2im \arctan \tilde{U}} \right) \quad m \text{ even} \quad (1.3)
\]

\[
\phi_m = \frac{1}{\sqrt{m 2i}} \left( e^{-2im \arctan \tilde{V}} + e^{-2im \arctan \tilde{U}} \right) \quad m \text{ odd} \quad (1.4)
\]

Using the definition \(\arctan z = \frac{1}{2i} \ln \frac{1+iz}{1-iz} [15]\), (1.3) and (1.4) become

\[
\phi_m = \frac{1}{2 \sqrt{m i}} \left[ \left( \frac{1-i \tilde{U}}{1+i \tilde{V}} \right)^m \mp \left( \frac{1-i \tilde{U}}{1+i \tilde{U}} \right)^m \right]
\]

where \(\pm\) is for \(m\) odd or even. These modes can easily be seen to be bounded and analytic in the lower half of the complex \(\tilde{U}, \tilde{V}\) plane for all \(m\). This establishes that the vacuum defined by the AdS modes is the same as that defined by the Kruskal modes. Thus a Green’s function defined on this spacetime using AdS co-ordinates \((\lambda, \rho)\) corresponds to a Hartle-Hawking Green’s function, in the sense discussed in the Introduction.
II. THE GEOMETRY OF THE 2+1 DIMENSIONAL BLACK HOLE

In this paper, we shall be working only with the spinless black hole solution in 2+1 dimensions

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\phi^2 \]  

(2.1)

where

\[ N^2 = \frac{r^2 - r_+^2}{\ell^2}, \quad r_+ = M \ell. \]

As was shown in [2], this metric has constant curvature, and is a portion of three dimensional Anti de Sitter space with points identified. The identification is made using a particular killing vector \( \xi \), by identifying all points \( x_n = e^{2\pi n} x \). In order to see this most clearly, it is useful to introduce different sets of co-ordinates on AdS3.

AdS3 can be defined as the surface \( -v^2 - u^2 + x^2 + y^2 = -\ell^2 \) embedded in \( R^4 \) with metric \( ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \). A co-ordinate system \( (\lambda, \rho, \theta) \) which covers this space, and which we shall refer to as AdS co-ordinates, is defined by [5]

\[
\begin{align*}
  u &= \ell \cos \lambda \sec \rho \\
  v &= \ell \sin \lambda \sec \rho \\
  x &= \ell \tan \rho \cos \theta \\
  y &= \ell \tan \rho \sin \theta
\end{align*}
\]

where \( 0 \leq \rho \leq \frac{\pi}{2}, \quad 0 < \theta \leq 2\pi, \quad \text{and} \quad 0 < \lambda < 2\pi \). In these co-ordinates, the AdS3 metric becomes

\[ ds^2 = \ell^2 \sec^2 \rho (-d\lambda^2 + d\rho^2 + \sin^2 \rho \, d\theta^2). \]

AdS3 has topology \( S^1 \) (time) \( \times R^2 \) (space) and hence contains closed timelike curves. The angle \( \lambda \) can be unwrapped to form the covering space of AdS3, with \( -\infty < \lambda < \infty \), which does not contain any closed timelike curves. Throughout this paper we work with this covering space, and this is what we henceforth refer to as AdS3. As mentioned in the Introduction, even this covering space presents difficulties since it is not globally hyperbolic (see the discussion in Appendix A).

The identification taking AdS3 into the black hole (2.1) is most easily expressed in terms of co-ordinates \( (t, r, \phi) \), related in an obvious way to those used above, and defined on AdS3 by

\[
\begin{align*}
  u &= \sqrt{A(r)} \cosh \left( \frac{r - r_+}{\ell} \right) \\
  x &= \sqrt{A(r)} \sinh \left( \frac{r - r_+}{\ell} \right) \\
  y &= \sqrt{B(r)} \cosh \left( \frac{r + r_+}{\ell} \right) \\
  v &= \sqrt{B(r)} \cosh \left( \frac{r + r_+}{\ell} \right)
\end{align*}
\]

\( r > r_+ \)

\[
\begin{align*}
  u &= \sqrt{A(r)} \cosh \left( \frac{r + r_+}{\ell} \right) \\
  x &= \sqrt{A(r)} \sinh \left( \frac{r + r_+}{\ell} \right) \\
  y &= -\sqrt{-B(r)} \cosh \left( \frac{r - r_+}{\ell} \right) \\
  v &= -\sqrt{-B(r)} \cosh \left( \frac{r - r_+}{\ell} \right)
\end{align*}
\]

\( 0 > r > r_+ \).
Note that \(-\infty < \phi < \infty\). Under the identification \(\phi \rightarrow \phi + 2\pi n\), where \(n\) is an integer, these regions of AdS$_3$ become regions R and F of the black hole. Regions P and L are defined in an analogous way [2] (see Fig. 1 for a definition of regions F (future), P (past), R (right) and L (left)). The \(r = 0\) line is a line of fixed points under this identification, and hence there is a singularity there in the black hole spacetime of the Taub-NUT type [2,10].

Finally, it is possible to define Kruskal co-ordinates on the black hole. The relation between the Kruskal co-ordinates \(V\) and \(U\) and the black hole co-ordinates \(t\) and \(r\) is precisely as in (1.1) and (1.2). \(U, V\) and an unbounded \(\phi\) cover the region of AdS$_3$ which becomes the black hole after the identification.

**III. GREEN’S FUNCTIONS ON THE 2+1 DIMENSIONAL BLACK HOLE**

In this section we derive a Green’s function on the black hole spacetime, by using the method of images on a Green’s function on AdS$_3$. We then show that the resulting Green’s function is thermal, in that it obeys a KMS condition [8]. Using the analyticity properties discussed in the Introduction, the Green’s function is also shown to be defined with respect to a vacuum state corresponding to Kruskal co-ordinates on both the past and future horizons of the black hole. We therefore interpret it as a Hartle-Hawking Green’s function. Finally we derive the Green’s function for the \(M = 0\) solution directly from a mode sum, and compare it with the \(M \rightarrow 0\) limit of the black hole Green’s function.

**A. Deriving the Green’s Functions**

Since the black hole spacetime is given by identifying points on AdS$_3$ using a spacelike Killing vector field, we can use the method of images to derive the two point function on the black hole spacetime. Given the two point function \(G_{\Lambda}^+(x, x')\) on AdS$_3$,

\[
G_{\text{BH}}^+(x, x'; \delta) = \sum_n e^{-i\delta_n} G_{\Lambda}^+(x, x_n')
\]

Here \(x_n'\) are the images of \(x'\) and \(0 < \delta < \pi\) can be chosen arbitrarily. For a general \(\delta\) the modes of the scalar field on the black hole background will satisfy \(\phi_m(\tilde{e}^{2\pi n\xi} r) = e^{-i\delta_n} \phi_m(x)\). \(\delta = 0\) for normal scalar fields and \(\delta = \pi\) for twisted fields. From now on we will restrict ourselves to \(\delta = 0\).

This definition of the Green’s function on the black hole spacetime means that when summing over paths to compute the Feynman Green’s function \(G_F(x, x')\), we sum over all paths in AdS$_3$. Hence paths that cross and recross the singularities must be taken into account (compare this with the results of Hartle and Hawking [1]).

As explained in Appendix A, boundary conditions at infinity must be imposed on any Green’s function on AdS$_3$ in order to deal with the fact that AdS$_3$ is not globally hyperbolic. From Appendix A, we have

\[
G_{\Lambda}^+ = G_{A1}^+ \pm G_{A2}^+(3.1)
\]

where \(+(-)\) corresponds to Neumann (Dirichlet) boundary conditions (from now on, it should be assumed that the upper (lower) sign is always for Neumann (Dirichlet) boundary conditions unless otherwise stated). The individual terms in (3.1) are given by
\[ G^+_{A_1}(x, x') = \frac{1}{4\sqrt{2\pi\ell}} (\cos(\Delta \lambda - i\epsilon) \sec \rho' \sec \rho - 1 - \tan \rho \tan \rho' \cos \Delta \theta)^{-\frac{1}{2}} \]
\[ G^+_{A_2}(x, x') = \frac{1}{4\sqrt{2\pi\ell}} (\cos(\Delta \lambda - i\epsilon) \sec \rho' \sec \rho + 1 - \tan \rho \tan \rho' \cos \Delta \theta)^{-\frac{1}{2}} \].

\(\Delta \lambda\) is defined as \(\lambda - \lambda'\), and similarly for all other co-ordinates.

The sign of the \(i\epsilon\) is proportional to \(\text{sign}(\sin \Delta \lambda)\). It is only important for timelike separated points, for which the argument of the square root is negative. In the three dimensional Kruskal co-ordinates on \(\text{AdS}_3\), the identification is only in the angular direction. For timelike separated points, \(\text{sign}\Delta \lambda = \text{sign}\Delta V\), where \(V\) is the Kruskal time. It follows that for all identified points the sign of \(i\epsilon\) in \(G(x, x')\) is the same.

We now work in the black hole co-ordinates \((t, r, \phi)\), so that the identification taking \(\text{AdS}_3\) into the black hole spacetime is given by \(\phi \rightarrow \phi + 2\pi n\). Under this identification, the two point function on the black hole background becomes

\[ G^+(x, x') = \frac{1}{4\sqrt{2\pi\ell}} \left[ G^1_1(x, x') \pm G^2_2(x, x') \right] \]

where for \(x, x' \in \text{region R}\)

\[ G^+_1(x, x') = \sum_{n=0}^{\infty} \left[ \frac{r_{r'}}{r_{r'}^2} \cosh \frac{r_+ (\Delta \phi + 2\pi n)}{\ell} - 1 - (r_+^2 - r_{r'}^2) \frac{1}{r_+^2} \frac{r_+ (\Delta \lambda - i\epsilon)}{\ell^2} \right]^{-\frac{1}{2}} \]  \hspace{1cm} (3.2)
\[ G^+_2(x, x') = \sum_{n=0}^{\infty} \left[ \frac{r_{r'}}{r_{r'}^2} \cosh \frac{r_+ (\Delta \phi + 2\pi n)}{\ell} + 1 - (r_+^2 - r_{r'}^2) \frac{1}{r_+^2} \frac{r_+ (\Delta \lambda - i\epsilon)}{\ell^2} \right]^{-\frac{1}{2}} \].  \hspace{1cm} (3.3)

For \(x, x' \in \text{region F}\),

\[ G^+_1(x, x') = \sum_{n=0}^{\infty} \left[ \frac{r_{r'}}{r_{r'}^2} \cosh \frac{r_+ (\Delta \phi + 2\pi n)}{\ell} - 1 + (r_+^2 - r_{r'}^2) \frac{1}{r_+^2} \frac{r_+ (\Delta \lambda - i\epsilon)}{\ell^2} + i\epsilon \text{ sign } \Delta V \right]^{-\frac{1}{2}} \]
\[ G^+_2(x, x') = \sum_{n=0}^{\infty} \left[ \frac{r_{r'}}{r_{r'}^2} \cosh \frac{r_+ (\Delta \phi + 2\pi n)}{\ell} + 1 + (r_+^2 - r_{r'}^2) \frac{1}{r_+^2} \frac{r_+ (\Delta \lambda - i\epsilon)}{\ell^2} + i\epsilon \text{ sign } \Delta V \right]^{-\frac{1}{2}} \].

Of course in this region \(\text{sign } \Delta V \neq \text{sign } \Delta t\). For \(x \in \text{region R and } x' \in \text{region F}\), we have

\[ G^+_1(x, x') = \sum_{n=0}^{\infty} \left[ \frac{r_{r'}}{r_{r'}^2} \cosh \frac{r_+ (\Delta \phi + 2\pi n)}{\ell} - 1 - (r_+^2 - r_{r'}^2) \frac{1}{r_+^2} \frac{r_+ (\Delta \lambda - i\epsilon)}{\ell^2} + i\epsilon \text{ sign } \Delta V \right]^{-\frac{1}{2}} \]
\[ G^+_2(x, x') = \sum_{n=0}^{\infty} \left[ \frac{r_{r'}}{r_{r'}^2} \cosh \frac{r_+ (\Delta \phi + 2\pi n)}{\ell} + 1 - (r_+^2 - r_{r'}^2) \frac{1}{r_+^2} \frac{r_+ (\Delta \lambda - i\epsilon)}{\ell^2} + i\epsilon \text{ sign } \Delta V \right]^{-\frac{1}{2}} \].

In all of these expressions, \(G^-(x, x') = \langle 0 | \phi(x') \phi(x) | 0 \rangle\) is obtained by reversing the sign of \(i\epsilon\). All of these expressions are uniformly convergent for \(x, x'\) real, and \(r, r' > \delta, \delta > 0\). Notice that as \(M \to 0\) \((r_+ \to 0)\), \(G(x, x')\) will diverge like \(\sum \frac{1}{n}\) unless we take the Dirichlet boundary conditions.
From these expressions, the Feynman Green’s function can easily be constructed and in fact has exactly the same form, but with the sign of $i\epsilon$ being strictly positive. It should be noted that none of these Green’s functions are invariant under Anti de Sitter transformations, as the Killing vector field defining the identification does not commute with all the generators of the AdS group.

**B. KMS condition**

A thermal noise satisfies a skew periodicity in imaginary time called Kubo-Martin-Schwinger (KMS) condition [8]

$$g_{\beta}(\Delta \lambda - \frac{i}{T}) = g_{\beta}(-\Delta \lambda)$$

where $g_{\beta}(\Delta \lambda) = G^{\pm}_{\beta}(x(\lambda), x(\lambda'))$ and $G^{\pm}_{\beta} = \langle \theta_{\beta} \phi(x) \phi(x') \theta_{\beta} \rangle_{\text{limit}}$ with the world line $x(\lambda)$ taken to be the one at rest with respect to the heat bath (for a more extensive discussion of the KMS condition, see [12]). We will show that $g(\Delta \lambda) = G^{+}_{A}(x(\lambda), x(\lambda'))$ with $x(\lambda) = (\frac{\lambda}{T}, r, \theta)$ and $b = (r^{2} - r_{+}^{2})^{1/2}/\ell$, satisfies this condition outside the horizon, with a local temperature $T = \frac{\ell}{2\pi(r^{2}_{+} - r_{+}^{2})^{1/2}}$, which agrees with the Tolman relation [13] $T = (g_{00})^{-1/2}T_{0}$, with $T_{0} = r_{+}/2\pi \ell^{2}$ the temperature of the black hole.

$g(\Delta \lambda)$ is defined as

$$g(\Delta \lambda) = \frac{1}{4\sqrt{2\pi \ell}}(g_{1}(\Delta \lambda) \pm g_{2}(\Delta \lambda))$$

where

$$g_{1}(\Delta \lambda) = \sum_{n=-\infty}^{\infty} \left[ \frac{r^{2}}{r_{+}^{2}} \cosh \frac{2\pi n r_{+}}{\ell} - 1 - \frac{(r^{2} - r_{+}^{2})}{r_{+}^{2}} \cosh \left( \frac{r_{+}}{\ell^{2}} \frac{\Delta \lambda}{b} - i\epsilon \right) \right]^{-\frac{1}{2}}$$

$$g_{2}(\Delta \lambda) = \sum_{n=-\infty}^{\infty} \left[ \frac{r^{2}}{r_{+}^{2}} \cosh \frac{2\pi n r_{+}}{\ell} + 1 - \frac{(r^{2} - r_{+}^{2})}{r_{+}^{2}} \cosh \left( \frac{r_{+}}{\ell^{2}} \frac{\Delta \lambda}{b} - i\epsilon \right) \right]^{-\frac{1}{2}}.$$

We will demonstrate the KMS property for each term in these sums.

Take a typical term in the sum. It has singularities at

$$\Delta \lambda_{n} = \pm \Delta \lambda_{n}^{0} + \frac{i}{T} p + i\epsilon$$

where $p$ is an integer. These singularities are square root branch points and the branch cuts go from $\left( \Delta \lambda_{n}^{0} + 2\pi \frac{i}{T} p + i\epsilon \to \infty + \frac{i}{T} p + i\epsilon \right)$ and $\left( -\Delta \lambda_{n}^{0} + 2\pi \frac{i}{T} p + i\epsilon \to -\infty - 2\pi \frac{i}{T} p + i\epsilon \right)$.

In any region without the branch cuts, $g_{1}$ and $g_{2}$ are analytic. Going around a branch point gives an additional minus sign. Now for a given $n$, if $\Delta \lambda$ is such that the expression inside the square root is positive, then $|\text{Real } \Delta \lambda| < \Delta \lambda_{n}$. In this region, $g_{1}^{n}$ and $g_{2}^{n}$ are analytic and periodic in $\frac{i}{T}$. What’s more $g^{n}(\Delta \lambda) = g^{n}(\Delta \lambda)$ as $\epsilon \to 0$. If on the other hand the expression inside the square root is negative, then because of the branch cuts, $g^{n}(\Delta \lambda - \frac{i}{T}) = -g^{n}(\Delta \lambda)$.

As $g^{n}(\Delta \lambda) = (-A + i\epsilon \text{ sign } \Delta \lambda)^{-\frac{1}{2}}$ and because our definition of the square root is with a branch cut along the negative real axis, we see that $g^{n}(\Delta \lambda) = -g^{n}(-\Delta \lambda)$ ($A$ is only a function of $|\Delta \lambda|$). This shows that the KMS condition is satisfied, and hence that $G^{+}$ is a thermal Green’s function.
C. Identifying the Vacuum State

In the region R where \( r > r_+ \), the Kruskal co-ordinates are defined as

\[
U = \left( \frac{r - r_+}{r + r_+} \right)^{\frac{1}{2}} \cosh \frac{r_+ \ell}{2},
\]

\[
V = \left( \frac{r - r_+}{r + r_+} \right)^{\frac{1}{2}} \sinh \frac{r_+ \ell}{2}.
\]

Defining \( \tilde{V} = V - U \) and \( \tilde{U} = V + U \), \( r \) is given by

\[
\frac{r}{r_+} = \frac{1 - \tilde{U} \tilde{V}}{1 + \tilde{U} \tilde{V}}.
\]

In these co-ordinates the two point function becomes

\[
G^J_{\tilde{U}, \tilde{V}, \phi; \tilde{U}', \tilde{V}', \phi'} = \frac{1}{\sqrt{24 \pi \ell}} \sum_n \frac{1}{1 + \tilde{U} \tilde{V}} \left[ (1 - \tilde{U} \tilde{V})(1 - \tilde{U}' \tilde{V}') \cosh \left( \frac{r_+}{\ell} (\Delta \phi + 2 \pi n) \right) \mp (1 + \tilde{U} \tilde{V})(1 + \tilde{U}' \tilde{V}') + 2(\tilde{V} \tilde{U}' + \tilde{U} \tilde{V}') + 2i \varepsilon \sign \Delta V \right]^{-\frac{1}{2}}
\]

where \( \mp \) is for \( J = 1, 2 \). Here the sign of \( i \varepsilon \) is the same as \( \sign \Delta V \) which is the same as \( \sign \Delta \lambda \) for timelike separated points. For \( x, x' \in \mathbb{R} \), this is just \( \sign \Delta t = \sign (\tilde{V} \tilde{U}' - \tilde{U} \tilde{V}') \).

This expression is valid all over the Kruskal manifold.

As discussed in the Introduction, the Hartle-Hawking Green’s function is defined to be analytic and bounded in the lower half complex plane of \( \tilde{V} \) on the past horizon \( (\tilde{U} = 0) \), when \( \tilde{U}', \tilde{V}', \phi, \phi' \) are real, or in the lower half plane of \( \tilde{U} \) on the future horizon \( (\tilde{V} = 0) \).

On the past horizon \( \tilde{U} = 0 \) we have

\[
G^J_{\tilde{V}'} = \frac{1}{\sqrt{24 \pi \ell}} \sum_n \frac{1}{1 + \tilde{U} \tilde{V}'} \left[ (1 - \tilde{U}' \tilde{V}') \cosh \left( \frac{r_+}{\ell} (\Delta \phi + 2 \pi n) \right) \mp (1 + \tilde{U}' \tilde{V}') + 2\tilde{V} \tilde{U}' + 2i \varepsilon \sign \Delta V \right]^{-\frac{1}{2}}.
\]

In order to prove analyticity and boundedness we will show that the singularities occur in the upper half plane of \( \tilde{V} \). Hence every term in the sum is a holomorphic function in the lower half plane. We will then use Weierstrass’s Theorem on sums of holomorphic functions in order to prove that the \( G^J \) are analytic in the lower half of the complex \( \tilde{V} \) plane.

\( G^J_{\tilde{V}} \) has singularities when

\[
\tilde{V} = \frac{\pm (1 + \tilde{U}' \tilde{V}') - (1 - \tilde{U}' \tilde{V}') \cosh \left( \frac{r_+}{\ell} (\Delta \phi + 2 \pi n) \right)}{2(1 + i \varepsilon) \tilde{U}'},
\]

Now suppose that \( x' \in \mathbb{R} \), then \(-1 \leq \tilde{U}' \tilde{V}' < 0 \) and \( \tilde{U}' > 0 \). Defining \( \tilde{U} \tilde{V}' = -a, (1 > a > 0) \), the singularity occurs at

\[
\tilde{V} = \frac{\pm (1 - a) - (1 + a) \cosh \left( \frac{r_+}{\ell} (\Delta \phi + 2 \pi n) \right)}{2(1 + i \varepsilon) \tilde{U}'}.
\]
We see that when \( \epsilon \to 0 \), \( \tilde{V} \) is real and negative. Hence 
\[
\tilde{V} = \frac{-A}{1+i\epsilon} \approx -A + i\epsilon \quad \text{with} \quad A > 0,
\]
so that the singularities are in the upper half plane. Similarly, for the future horizon \( \tilde{V} = 0 \),
there are singularities when
\[
\tilde{U} = \frac{\pm(1 + \tilde{U}'\tilde{V}') - (1 - \tilde{U}'\tilde{V}') \cosh(\Delta \phi + 2\pi n)}{(1 - i\epsilon)\tilde{V}'}. 
\]
For \( x' \in \mathbb{R} \), then \(-1 < \tilde{U}'\tilde{V}' < 0\), and \( \tilde{V}' < 0 \), so that 
\[
\tilde{U} = \frac{A}{1+i\epsilon} = A + i\epsilon, \quad \text{with} \quad A > 0, 
\]
so the singularities are in the upper half plane of \( \tilde{U} \). At this point it should be noted that for \( G \)
we get singularities in the lower half plane of \( \tilde{U} \) on the surface \( \tilde{V} = 0 \), and singularities in
the lower half plane of \( \tilde{V} \) on the surface \( \tilde{U} = 0 \).

For \( x' \in F \), if \( \tilde{U} = 0 \) and \( x \) and \( x' \) connected by a null geodesic, then \( \Delta V < 0 \). This
is the case because for timelike and null separations, sign \( \Delta V = -\text{sign} \Delta r \) (\( r \) is a timelike
co-ordinate in \( F \)) and \( \Delta r \) is always positive if \( x \) is on the horizon. Then it can be checked
that the singularities are again in the upper half plane of either \( \tilde{U} \) or \( \tilde{V} \).

Now that we have established that each term in the infinite sum is holomorphic in the
lower half plane of \( \tilde{V} \) on the past horizon (and in \( \tilde{U} \) on the future horizon) we will use
Weierstrass’s Theorem. This states [14] that if a series with analytic terms
\[
f(z) = f_1(z) + f_2(z) + \cdots
\]
converges uniformly on every compact subset of a region \( \Omega \), then the sum \( f(z) \) is analytic in
\( \Omega \), and the series can be differentiated term by term. It is easily seen that unless \( \tilde{U}'\tilde{V}' = 1 \),
i.e., \( x' \) is at the singularity, the sum converges uniformly on every compact subset of the
lower half plane. For \( \tilde{U}'\tilde{V}' = 1 \) the sum diverges and the Green’s function becomes singular
at \( r = 0 \). This is because \( r = 0 \) is a fixed point of the identification.

To conclude we have shown that our Green’s function is analytic on the past horizon
in the lower half \( \tilde{V} \) complex plane, and similarly on the future horizon in the lower half \( \tilde{U} \)
complex plane. Its singularities occur when \( x, x' \) can be connected by a null geodesics either
directly or after reflection at infinity (see Appendix A and Ref. [7]). We conclude that the
Green’s function we have constructed is the Hartle-Hawking Green’s function as defined in
the Introduction, for both Neumann and Dirichlet boundary conditions.

D. The \( M = 0 \) Green’s function

The black hole solution as \( M \to 0 \) is the spacetime with metric [2]
\[
ds^2 = -\left(\frac{r}{\ell}\right)^2 dt^2 + \left(\frac{\ell}{r}\right)^2 dr^2 + r^2 d\phi^2
\]
with \( r > 0 \), and \( t \) and \( \phi \) as in (2.1). Defining \( z = \frac{\ell}{r} \) and \( \gamma = \frac{\ell}{r} \) the metric becomes
\[
ds^2 = \frac{\ell^2}{z^2}(-d\gamma^2 + dz^2 + d\phi^2). \tag{3.4}
\]

The modes for a massless conformally coupled scalar field are solutions of the equation
\[ \Box \phi - \frac{1}{8} R \phi = 0 \]

where again \( R = -6 \ell^{-2} \), and are given by

\[ \phi_{km} = N_\omega \sqrt{\frac{\tau}{\ell}} e^{-i\omega t} e^{im\phi} e^{ikz} \]

where \( \omega^2 = k^2 + m^2 \), \( m \) is an integer, and \( N_\omega = (8\pi^2 \omega)^{\frac{1}{2}} \) is a normalization constant such that \( (\psi_{mk}, \psi_{m'k'}) = \delta_{mm'} \delta(k-k') \).

As in quantization on \( \text{AdS}_3 \), care must be taken at the boundary \( z = 0 \), which is at spatial infinity. The metric (3.4) is conformal to Minkowski spacetime with one spatial coordinate periodic and the other restricted to be greater than zero. As in the case of \( \text{AdS}_3 \), we impose the boundary conditions

\[ \frac{1}{\sqrt{z}} \psi = 0 \quad \text{or} \quad \frac{\partial}{\partial z} \left( \frac{1}{\sqrt{z}} \psi(z) \right) = 0 \]

at \( z = 0 \), corresponding to Dirichlet or Neumann boundary conditions in the conformal Minkowski metric. Our approach will be to first calculate the Green’s function without boundary conditions and then use the method of images to impose them.

Summing modes, we obtain the two point function

\[
\hat{G}(x, x') = \frac{1}{8\pi^2 \omega} \frac{\sqrt{zz'}}{\ell} \sum_m \int_k e^{-i\omega k} e^{im\phi} e^{ikz} dk
\]

\[
= \frac{1}{2\pi} \frac{\sqrt{zz'}}{\ell} \sum_m e^{im\phi} G_2(y, y', m)
\]

where \( G_2(y, y', m) \) is the massive 1+1 dimensional Green’s function and \( y = (\gamma, z) \).

Now [11],

\[ G_2(y, y', m) = \frac{1}{2\pi} K_0(|m|d) \quad m \neq 0 \]

\[ = -\frac{1}{2\pi} \log d + \lim_{n \to 1} \frac{\Gamma(\frac{n}{2}) - 1}{n\pi} \quad m = 0 \]

where \( d = \epsilon + i\Delta \text{ sign } \Delta t \), with \( \Delta = ((\Delta \gamma)^2 - (\Delta z)^2)^{\frac{1}{2}} \) for timelike separation, and \( d = ((\Delta z)^2 - (\Delta \gamma - i\epsilon)^2)^{\frac{1}{2}} \) for spacelike separation. Here \( K_0 \) is a modified Bessel function. It follows that

\[ \hat{G}(x, x') = \frac{1}{4\pi^2} \frac{\sqrt{zz'}}{\ell} \left[ 2 \sum_{m>0} \cos(m\Delta \beta) K_0(md) - \log d \right] \]

where the infinite constant in the \( m = 0 \) expression was dropped to regularize the infrared divergences of the 1+1 dimensional Green’s function. Using [15]

\[
\sum_{m=1}^\infty K_0(mx) \cos(mxI) = \frac{1}{2} \left( c + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2\sqrt{x^2 + (xt)^2}} + \frac{\pi}{2} \sum_{l \neq 0} \left[ \left( x^2 + (2\ell \pi - xt)^2 \right)^{-\frac{1}{2}} - \frac{1}{2\ell \pi} \right].
\]
Here $c$ is the Euler constant. This expression is valid for $x > 0$ and real $t$, and gives the following expression for spacelike separated $x$ and $x'$:

\[
\hat{G}(x, x') = \frac{1}{4\pi} \frac{\sqrt{z \bar{z}^\prime}}{\ell} \left[ \sum_n \left[ (\Delta z)^2 + (\Delta \phi + 2\pi n)^2 - (\Delta \gamma - i\epsilon)^2 \right]^{-\frac{1}{2}} - \sum_{n \neq 0} \frac{1}{2\pi n} + c_1 \right]
\]

\[= \frac{1}{4\pi} \frac{\sqrt{z \bar{z}^\prime}}{\ell} F(x, x') \]

Here $c_1 = c - \ln 4\pi$. Although the above formula was only true for $x > 0$ and real $t$, the result is analytic for every real $\Delta z$, $\Delta \gamma$ and $\Delta \phi$. Hence it is also correct for timelike separated points.

This is just what is expected from the conformality to the Minkowski space, other than the $\sum 1 = c_1$ which regularizes $\hat{G}$. Now the boundary condition can be easily put in by writing

\[G^+(x, x') = \frac{1}{4\pi} \frac{\sqrt{z \bar{z}^\prime}}{\ell} (F(x, x') + F(x, \bar{x}')) \]

where $\bar{x}' = (\gamma', -z', \phi)$. Notice that for Dirichlet boundary conditions, this agrees with the $M \to 0$ limit of (3.2) and (3.3).

Going to the $(t, r, \phi)$ co-ordinates we have

\[G^+_{M=0} = \frac{1}{4\pi} (rr')^{-\frac{1}{2}} (G^+_1 \pm G^+_2) \]

with

\[G^+_1 = \sum_n \left[ \ell^2 \left( \frac{r'}{rr'} - r \right)^2 + (\Delta \phi + 2\pi n)^2 - \left( \frac{\Delta t - i\epsilon}{\ell} \right)^2 \right]^{-\frac{1}{2}} - \sum_{n \neq 0} \frac{1}{2\pi n} + c_1 \]

\[G^+_2 = \sum_n \left[ \ell^2 \left( \frac{r + r'}{rr'} \right)^2 + (\Delta \phi + 2\pi n)^2 - \left( \frac{\Delta t - i\epsilon}{\ell} \right)^2 \right]^{-\frac{1}{2}} - \sum_{n \neq 0} \frac{1}{2\pi n} + c_1 \]

**E. Computation of $\langle \phi^2 \rangle$**

$\langle \phi^2 \rangle$ is defined as $\langle \phi^2 \rangle = \lim_{x = x'} \frac{1}{2} \hat{G}_{\text{Reg}}(x, x')$ where $G = G^+ + G^-$ is the symmetric Green's function. In order to compute $\langle \phi^2 \rangle$, we need to regularize $\hat{G}$. Now only the $n = 0$ term in $G^+_1$ is infinite and is just a Green's function on AdS$_3$. Hence, we can use the Hadamard development in AdS$_3$ to regularize $G$ [11]:

\[G_{\text{Had}} = \frac{-i \Delta t^\frac{1}{2}}{2\sqrt{2\pi} \sigma^2} \]

where
\[
\sigma = \frac{\ell^2}{2} [ar \cos Z]^2, \quad \Delta^{-\frac{1}{2}} = \frac{\sin \left(\frac{2\sigma}{\ell^2}\right)^{\frac{1}{2}}}{\left(\frac{2\sigma}{\ell^2}\right)^{\frac{1}{2}}}, \quad \text{and} \quad Z = \frac{\cos \Delta \lambda - \sin \rho \sin \rho' \cos \Delta \theta}{\cos \rho \cos \rho'}
\]
(here \(\Delta\) is the Van Vleck determinant). Defining
\[
G_{\text{Reg}}(x, x') = G_{\text{BH}}(x, x') - G_{\text{Had}}(x, x')
\]
we get
\[
\langle \phi^2 \rangle = \frac{1}{4\sqrt{2\pi} \ell^2} \sum_{n \neq 0} \left( \cosh \left( \frac{r_+}{\ell} + 2\pi n \right) - 1 \right)^{-\frac{1}{2}} \pm \sum_{n} \left( \cosh \left( \frac{r_+}{\ell} + 2\pi n \right) - 1 + 2 \left( \frac{r_+}{r} \right)^2 \right)^{-\frac{1}{2}}
\]
which, for Dirichlet boundary conditions, can be seen to be regular as \(M \to 0\) (that is \(r_+ \to 0\)), and to coincide in this limit with the \(M = 0\) result for Dirichlet boundary conditions.

**IV. THE ENERGY-MOMENTUM TENSOR**

The energy-momentum tensor for a massless conformally coupled scalar field in AdS\(3\) is given by the expression
\[
T_{\mu \nu}(x) = 3 \frac{\partial_\mu \phi(x) \partial_\nu \phi(x)}{4} - \frac{1}{4} g_{\mu \nu} g_{\rho \sigma} \partial_\rho \phi(x) \partial_\sigma \phi(x) - \frac{1}{4} \nabla_\mu \partial_\nu \phi(x) \phi(x) + \frac{1}{96} g_{\mu \nu} R \phi^2(x)
\]
where \(R = -6\ell^{-2}\). In order to compute \(T_{\mu \nu}\) one differentiates the symmetric two-point function \(G = \langle 0 | \phi(x) \phi(x') + \phi(x') \phi(x) | 0 \rangle\) [11], and then takes the coincident point limit. This makes \(T_{\mu \nu}\) divergent and regularization is needed. A look at our Green’s function reveals that only the \(n = 0\) term in \(G_1\) diverges as \(x \to x'\), so only the \(T_{\mu \nu}\) derived from it should be regularized.

The \(n = 0\) term is just the Green’s function in AdS\(3\) in accelerating co-ordinates. The vacuum in which this Green’s function is derived is symmetric under the Anti de Sitter group and AdS\(3\) is a maximally symmetric space. Hence [16] \(T_{\mu \nu} = \frac{1}{3} g_{\mu \nu} T\) where \(T = g^{\mu \nu} T_{\mu \nu}\). For a conformally coupled massless scalar field \(T = 0\) (there is no conformal anomaly in 2+1 dimensions) so \(T_{\mu \nu}^{\text{AdS}} = 0\).

Having shown that we may drop the \(n = 0\) term in \(G_1\), after a somewhat lengthy calculation we arrive at the result for \(M \neq 0\),
\[
T^\nu_\mu(x) = \frac{1}{4\sqrt{2\pi} \ell^2} \left( \frac{r_+}{r} \right)^3 \left\{ (a_1 \pm a_2 \pm \left( \frac{r_+}{r} \right)^2 a_3 ) \text{ diag}(1, 1, -2) \right. \\
\left. \pm \frac{r^2 - r_+^2}{r^2} \left( -b_1 + b_2 \left( \frac{r_+}{r} \right)^2 \right) \text{ diag}(-1, 0, 1) \right\}
\]
(4.1)
where
\[ a_1 = \frac{1}{8} \sum_{n \neq 0} f^{-\frac{1}{2}}(n) + \frac{1}{2} \sum_{n \neq 0} f^{-\frac{1}{2}}(n) \]
\[ a_2 = \frac{1}{8} \sum_{n \neq 0} g(n, r)^{-\frac{1}{2}} \]
\[ a_3 = \frac{1}{4} \sum_{n \neq 0} g(n, r)^{-\frac{1}{2}} \]
\[ b_1 = \frac{3}{4} \sum_{n \neq 0} g(n, r)^{-\frac{1}{2}} \]
\[ b_2 = \frac{3}{2} \sum_{n \neq 0} g(n, r)^{-\frac{1}{2}} \]

and \( f(n) = \cosh(\frac{r}{L} - 2\pi n) - 1, \)
\( g(n, r) = \cosh(\frac{r}{L} - 2\pi n) - 1 + 2 \left( \frac{r}{L} \right)^2. \)
\( \text{diag}(a, b, c) \) is in \((t, r, \theta)\) co-ordinates. As expected the \( n = 0 \) term from \( G_2 \) did not contribute.

For \( M = 0 \) we get from Sec. III D

\[ T^\mu_\nu(x) = \frac{1}{\sqrt{24\pi r^3}} \left\{ \frac{1}{2\sqrt{2}} \sum_{n>0} (n\pi)^{-3} \text{diag}(1, 1, -2) \right\} \]
\[ \pm \frac{3}{4\sqrt{2}} \left\{ \left( \frac{\pi}{r} \right)^2 + \left( \frac{\ell}{r} \right)^2 \right\}^{-\frac{1}{2}} \sum_{n>0} \left( (\pi n)^2 + \left( \frac{\ell}{r} \right)^2 \right)^{-\frac{1}{2}} \text{diag}(-1, 0, 1) \} \] \quad (4.2)

As we can see from (4.1), far away from the black hole, \( T^\mu_\nu \) obeys the strong energy condition [10] only for the Dirichlet boundary conditions, while for the Neumann boundary conditions, the energy density is negative in this limit. It is also worth noting that for Dirichlet boundary conditions, as \( M \) decreases, although the temperature decreases, the energy density increases; just the opposite occurs for Neumann boundary conditions. In the other limit, \( M \to \infty, T^\mu_\nu \to 0 \) for both sets of boundary conditions, which suggests the presence of a Casimir effect. On the horizon, \( T^\mu_\nu \) is regular, and hence in the semiclassical approximation, the horizon is stable to quantum fluctuations; on the other hand, at \( r = 0, T^\mu_\nu \) diverges. Our Green's function was thermal in \((t, r, \theta)\) co-ordinates, but although \( T^\mu_\nu \sim T^\mu_\nu_{\text{loc}} \) for large \( r \), it is not of a thermal type [13].

**V. THE RESPONSE OF A PARTICLE DETECTOR**

In this section we calculate the response of a particle detector which is stationary in the black hole co-ordinates \((t, r, \phi)\), and outside the black hole. The simplest particle detector can be described by an idealized point monopole coupled to the quantum field through an interaction described by \( L_{\text{int}} = cm(\lambda) \phi [x(\lambda)] \) where \( \lambda \) is the detector’s proper time, and \( c \ll 1 \). The probability for the detector to undergo a transition from energy \( E_1 \) to \( E_2 \) [11] is

\[ R(E_1/E_2) = e^2 \left| \langle E_2 | m(0) | E_1 \rangle \right|^2 F(E_2 - E_1) \]

to lowest order in perturbation theory, where

\[ F(\omega) = \lim_{s \to 0} \lim_{\lambda_0 \to \infty} \frac{1}{2\lambda_0} \int_{-\lambda_0}^{\lambda_0} d\lambda \int_{-\lambda_0}^{\lambda_0} d\lambda' e^{-i\omega(\lambda - \lambda')} e^{-S[\lambda - S[\lambda'] g(\lambda, \lambda')}. \]
\( g(\lambda, \lambda') = G^+(x(\lambda), x(\lambda')) \) and \( x(\lambda) \) is the detector trajectory.

\( F(\omega) \) is called the response function. It represents the bath of particles that the detector sees during its motion \([17]\). We take \( x(\lambda) = (\frac{\lambda}{b}, r, \theta) \) where \( b = \left(\frac{r^2 - r_+^2}{\ell^2}\right)^{\frac{1}{2}} \). Because \( g(\lambda, \lambda') = g(\Delta \lambda) \), then

\[
F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \Delta \lambda} g(\Delta \lambda) d(\Delta \lambda)
\]

(5.1)

where \( g(\Delta \lambda) = g_1(\Delta \lambda) \pm g_2(\Delta \lambda) \). Here

\[
g_i(\Delta \lambda) = \frac{r_+}{\sqrt{24\pi \ell}} (r^2 - r_+^2)^{-\frac{1}{2}} \sum_n \left[ \frac{r_+^2}{r^2 - r_+^2} \left( \frac{r^2}{r_+^2} \cosh \left( \frac{r_+^2 + 2\pi \ell n}{r_+^2} \right) - 1 \right) - \cosh \frac{r_+^2}{\ell^2} \left( \frac{\Delta \lambda}{b} - i\epsilon \right) \right]^{-\frac{1}{2}}.
\]

(5.2)

In this expression \( -(+) \) is for \( g_1(g_2) \).

Defining

\[
cosh \alpha_n = \frac{r_+^2}{r^2 - r_+^2} \left( \frac{r^2}{r_+^2} \cosh \left( \frac{r_+^2 + 2\pi \ell n}{r_+^2} \right) - 1 \right)
\]

and

\[
cosh \beta_n = \frac{r_+^2}{r^2 - r_+^2} \left( \frac{r^2}{r_+^2} \cosh \left( \frac{r_+^2 + 2\pi \ell n}{r_+^2} \right) + 1 \right)
\]

we have from Appendix B that

\[
F(\omega) = \frac{1}{2} \frac{1}{e^{\omega/T} + 1} \sum_n \left( P_{\frac{\omega}{T} - \frac{1}{2}}(\cosh \alpha_n) \pm P_{\frac{\omega}{T} + \frac{1}{2}}(\cosh \beta_n) \right)
\]

where \( T = \frac{r_+}{2\pi(r_+^2 - r_+^2)^{\frac{1}{2}}} \) is the local temperature. This looks like a fermion distribution with zero chemical potential and a density of states

\[
D(\omega) = \frac{\omega}{2\pi} \sum_n \left( P_{\frac{\omega}{T} - \frac{1}{2}}(\cosh \alpha_n) \pm P_{\frac{\omega}{T} + \frac{1}{2}}(\cosh \beta_n) \right).
\]

Notice that \( F(\omega) \) is finite on the horizon in contrast with black holes in two and four dimensions (see \([18]\)). This seems to be a consequence of the Fermi type distribution. Statistical inversion of the type found above was first noted in Ref. \([12]\).

If the mass of the black hole satisfies \( e^{2\pi \sqrt{M}} \gg 1 \) and \( \omega/T \gg 1 \), then far from the horizon, \( r \gg r_+ \), we can sum the series, and for Dirichlet boundary conditions we obtain

\[
F(\omega) \simeq 2\pi \ell^2 T^2 \frac{1}{e^{\omega/T} + 1} \left[ \left( \frac{\omega}{2\pi T} \right)^2 + \frac{8e^{-3\pi \sqrt{M}}}{\pi} \left( \frac{\omega}{T} \right)^{\frac{1}{2}} \left( \sin \frac{\omega \sqrt{M}}{T} - \cos \frac{\omega \sqrt{M}}{T} \right) \right].
\]

A similar result holds for Neumann boundary conditions at large \( r \),
\[ F(\omega) \approx \frac{1}{e^{\omega/T} + 1} \left[ 1 + 4 \left( \frac{\omega}{T} \right)^{\frac{1}{2}} e^{-\pi \sqrt{M} \left( \sin \frac{\omega \sqrt{M}}{T} + \cos \frac{\omega \sqrt{M}}{T} \right)} \right] \]

where the approximation improves for large \( M \) as before.

It seems clear that the particle detector response will consist of a Rindler-type effect \([11]\), and, if present, a response due to Hawking radiation (real particles). The former is due to the fact that a stationary particle detector is actually accelerating, even when \( r \to \infty \) (there is no asymptotically flat region). This is reflected in the fact that the high \( \omega \) behaviour of \( F(\omega) \) for \( r \gg r_+ \) and \( M \gg 1 \) is governed by the \( n = 0 \) term in \( G^+ \), which is AdS invariant. Hence all observers connected by an AdS transformation (the asymptotic symmetry group) register the same response, even though they might be in relative motion; this means that \( F(\omega) \) as a whole cannot be interpreted as real particles \((\text{see } [19,20])\) for a discussion of this point). Unfortunately, one cannot filter out these effects in a simple way, and further work is needed in order to find the spectrum of the Hawking radiation.

Finally, for \( M = 0 \), we may again define

\[ F_i(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \Delta \lambda} g_i(\Delta \lambda) d\Delta \lambda. \]

Now, however, \( g_1 \) and \( g_2 \) are analytic in the lower half complex plane of \( \Delta \lambda \). Hence for \( \omega > 0 \) we can close the integral in an infinite semicircle in the lower half plane and by Cauchy’s theorem \( F_i(\omega) = 0 \) for \( \omega > 0 \) so that no particles are detected by a stationary particle detector.

**VI. BACK-REACTION**

For the black hole solution with \( M \not= 0 \), we have shown in Sec. IV that the energy-momentum tensor is finite at the horizon and diverges as \( r \to 0 \). We therefore expect that the only qualitative difference in the solutions if we take into account quantum fluctuations is that a curvature singularity will appear at \( r = 0 \), although in this limit the semi-classical approximation breaks down.

For the \( M = 0 \) solution, we expect the backreaction to have a more striking effect. We compute it in the usual way by inserting the expectation value of the energy-momentum tensor \((4.2)\), into Einstein’s equations,

\[ G_{\mu\nu} = \Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}. \]

The first thing to note is that although the external solution is of constant curvature everywhere, the perturbed solution is not, and indeed the curvature scalar \( R_{\mu\nu} R^{\mu\nu} \) may be seen to diverge at the origin, \( r = 0 \). Inserting the expression \((4.2)\) into Einstein’s equation gives

\[ R_{\mu\nu} R^{\mu\nu} = (8\pi G T_{\nu}^{\nu} - 2\Lambda \delta^\nu_\nu)(8\pi G T^\mu_\mu - 2\Lambda \delta^\mu_\mu) \]

\[ = (8\pi G)^2 T^\mu_\nu T^{\nu}_\mu + 12\Lambda^2 \]

\[ > (8\pi G)^2 (T^\nu_\nu)^2 + 12\Lambda^2 = 12\Lambda^2 + \frac{G^2 \zeta(3)}{4\pi^2 \ell^3} \frac{1}{r^6}, \]
where $\zeta(x)$ is the Riemann Zeta function. This result shows that a singularity develops at $r = 0$, and is consistent with the viewpoint that the non-singular $M = 0$ solution is generally unstable as a vacuum solution.

Having shown that the back-reacted metric becomes singular, it remains to check whether a horizon develops. In order to see this, it is useful to begin with a general static, spherically symmetric metric, which we take to be

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + e^{2A} d\theta^2$$

where $N$ and $A$ are functions of $r$ only. A linear combination of Einstein’s equations implies that

$$(N^2)'' = 2A + 16\pi G T^\phi.$$  

Integrating Eq. (6.1) once, we obtain the result

$$(N^2)' = 2Ar + \frac{G}{4\pi} \sum_{n>0} \left\{ \frac{4\ell^2}{n^2\pi^2 r^2} \pm \left[ 1 - \left( 1 + \frac{\ell^2}{n^2\pi^2 r^2} \right)^{-\frac{1}{2}} \right] \right\}$$  

where an integration constant has been included to make the result finite. A second integration gives

$$N^2 = Ar^2 - \frac{G}{4\pi^2} \sum_{n>0} \left\{ \frac{4\ell^2}{n^3\pi r} \pm \left[ \left( 1 + \frac{\ell^2}{n^2\pi^2 r^2} \right)^{-\frac{1}{2}} - 1 \right] \frac{n\pi r}{\ell} + \frac{\ell}{n\pi r} \left( 1 + \frac{\ell^2}{n^2\pi^2 r^2} \right)^{-\frac{1}{2}} \right\}$$  

where the second integration constant has been set to zero. The two integration constants ensure that $N^2 \to Ar^2 + o(1)$ as $r \to \infty$. Having obtained an expression for $N$, we note the following. First of all, $A$ is given in terms of $N$ by the equation

$$A' = \frac{2Ar^3n^3 + G\zeta(3)}{2r^3n^3(N^2)'}$$

which we shall not attempt to integrate. The important thing to notice is that $A'$ diverges where $(N^2)' = 0$, so if the singularity at $r = 0$ is to be taken seriously, it is important that $(N^2)'$ should not vanish for any finite, non-zero $r$. To see that this is indeed the case, note that, writing $x = \frac{\ell^2}{n^2\pi^2 r^2}$, the quantity inside the square brackets of Eq. (6.2),

$$x \pm \left( 1 - (1 + x)^{-\frac{1}{2}} \right)$$

is positive for all $x > 0$.

Finally, to confirm then the solution (6.3) has a horizon, note that it may be rewritten as

$$N^2 = Ar^2 - \frac{G\zeta(3)}{2\pi^3 r}$$

$$- \frac{G}{4\pi^2} \sum_{n>0} \left\{ \frac{1}{n^2\ell} \pm \left[ \left( 1 + \frac{\ell^2}{n^2\pi^2 r^2} \right)^{\frac{1}{2}} - 1 \right] \frac{n\pi r}{\ell} + \frac{\ell}{n\pi r} \left( 1 + \frac{\ell^2}{n^2\pi^2 r^2} \right)^{-\frac{1}{2}} \right\}$$
In this case the quantity in the curly braces is clearly positive for the upper sign. For the lower sign, writing \( y = \frac{r}{\pi r} \), the quantity is

\[
2y - \left( (1 + y^2)^{\frac{1}{2}} - 1 \right) \frac{1}{y} - y(1 + y^2)^{\frac{1}{2}} = \frac{(1 + 2y^2) \left[ (1 + y^2)^{\frac{1}{2}} - 1 \right]}{y(1 + y^2)^{\frac{3}{2}}} \geq 0.
\]

It follows that

\[
N^2 \leq \Delta r^2 - \frac{G\zeta(3)}{2\pi r}.
\]

Since \( N^2 \) is monotonic and \( N^2 \to \infty, r \to \infty, N^2 \to -\infty, r \to 0 \), it follows that the back-reacted metric has a single horizon.

We regard this result as being indicative of the fact that the \( M = 0 \) solution is unstable, in the sense that it will develop a horizon and a curvature singularity due to quantum fluctuations. Recall that \( T^\mu_{\nu} \) in this case appears to be just the Casimir energy of the spacetime as it is associated with a zero temperature Green’s function. Note that the horizon appears in a region sufficiently close to \( r = 0 \) that the semi-classical approximation may break down, i.e., fluctuations in \( T^\mu_{\nu} \) will be of the order of \( T^\mu_{\nu} \). However, if there are \( n \) independent scalar fields present, then the ratio of the fluctuations to \( T^\mu_{\nu} \) becomes negligible in the vicinity of the horizon, as \( n \) becomes large.

The appearance of a horizon may be contrasted in an obvious way with 4-dimensional Minkowski spacetime, regarded as the \( M = 0 \) limit of the Schwarzschild solution. Minkowski spacetime has no Casimir energy associated with it, and is stable in this sense.

VII. CONCLUSIONS

In this paper we presented some aspects of quantization on the 2+1 dimensional black hole geometry. We obtained an exact expression for the Green’s function in the Hartle-Hawking vacuum and for the expectation value of the energy-momentum tensor, but we found some difficulty in interpreting the particle detector response as Hawking radiation. We feel that further investigation on this question is required. If the black hole evaporates, then we see from section VI that due to quantum fluctuations the endpoint of evaporation will still have a horizon and a singularity.

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NOTE ADDED

After this work was completed, we received two preprints, K. Shiraishi and T. Maki, Akita Junior College preprint AJC-HEP-18, and A. R. Steif, Cambridge preprint DAMTP93/R20,
hep-th/9308032, in which the Green’s function on the BTZ black hole spacetime is computed. Neither of these papers takes into account the fact that the spacetime is not globally hyperbolic due to the timelike nature of spatial infinity, that is they work with the transparent boundary conditions defined at the end of Appendix A.

APPENDIX A: SCALAR FIELD QUANTIZATION ON ADS$_3$

The derivation of a scalar field propagator on AdS$_3$ is reviewed. This computation is complicated by the fact that AdS$_3$ is not globally hyperbolic. In the AdS co-ordinate system defined in Sec. II, spatial infinity is the $\rho = \frac{\ell}{4}$ surface which is seen to be timelike (see Fig. 2). Information can escape or leak in through this surface in a finite co-ordinate time, spoiling the composition law property of the propagator. In order to resolve this problem and define a good quantization scheme on AdS$_3$, we follow [5] and use the fact that AdS$_3$ is conformal to half of the Einstein Static Universe (ESU) $R \times S^2$.

The metric of ESU is

$$ds^2 = -d\lambda^2 + d\rho^2 + \sin^2 \rho \, d\theta^2$$

where $-\infty < \lambda < \infty$, $0 < \rho \leq \pi$, and $0 < \theta \leq 2\pi$. Positive frequency modes on ESU are solutions of

$$\Box \psi^E - \frac{1}{8}R \psi^E = 0$$

where $R = 2$, and are given by

$$\psi^E_{\ell m} = N_{\ell m} e^{-i\omega \lambda} Y^\ell_m(\rho, \theta) \quad \omega > 0$$

(A1)

where $Y^\ell_m$ are the spherical harmonics, $\omega = \ell + \frac{1}{2}$, $m$ and $\ell$ are integers with $\ell \geq 0$, $|m| \leq \ell$, and $N_{\ell m} = \frac{1}{\sqrt{2\ell + 1}}$. These modes are orthonormal in the inner product [11]

$$\langle \psi_1, \psi_2 \rangle = -i \int \psi_1 \partial_\mu \psi_2^* [g_{\Sigma} - g_{\Sigma}]^\mu_\nu d\Sigma$$

where $\Sigma$ is a spacelike Cauchy surface. i.e. $(\psi_{\ell m}, \psi_{\ell' m'}) = \delta_{\ell \ell'} \delta_{m m'}$, $(\psi_{\ell m}, \psi_{\ell m}^*) = 0$, and $(\psi_{\ell m}^*, \psi_{\ell' m'}^*) = -\delta_{\ell \ell'} \delta_{m m'}$. As usual the field operator is expanded in these modes $\phi = \sum_{\ell m} \psi_{\ell m} a_{\ell m} + \psi_{\ell m}^* a_{\ell m}^+$ so that $a, a^+$ destroy and create particles, and define the vacuum state $\langle 0 \rangle_E$.

The two point function is defined as

$$G^E_{\ell}(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle_E = \sum \psi^E_{\ell m}(x) \psi^E_{\ell m}^*(x').$$

Inserting (A1),

$$G^E_{\ell}(x, x') = \sum_{\ell} \frac{1}{2\ell + 1} e^{-i(\ell + \frac{1}{2})(\lambda - \lambda')} \int \frac{d^4Y}{4\pi^2} \left( Y^\ell_m(\rho, \theta) Y^\ell_{m'}(\rho', \theta') \right).$$

Using $(Y^\ell_m)^* = (-1)^m Y^\ell_{-m}$ and $\sum_{m = -\ell}^{\ell} (-1)^m Y^\ell_m(x) Y^\ell_{-m}(x') = \frac{2\ell + 1}{4\pi} P_{\ell}(\cos \alpha)$ where $\alpha$ is the angle between $(\rho, \theta)$ and $(\rho', \theta')$, we get

$$G^E_{\ell}(x, x') = \frac{1}{2\ell + 1} \left( \frac{2\ell + 1}{4\pi} \right) P_{\ell}(\cos \alpha).$$
\[ G_E^+(x, x') = \frac{1}{4\pi} e^{-\frac{1}{2}(\lambda - \lambda')} \sum_{n=0}^\infty e^{-i(n\lambda - \lambda')} P_n(\cos \alpha). \]

Further, using \( \sum_{n=0}^\infty P_n(x) z^n = (1 - 2xz + z^2)^{-\frac{1}{2}} \) for \(-1 < x < 1 \) and \(|z| < 1 \) and as usual giving \( \Delta \lambda \) a small negative imaginary part for convergence, we get

\[ G_E^+ = \frac{1}{4\sqrt{2\pi}} (\cos(\Delta \lambda - i\epsilon) - \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \Delta \theta)^{-\frac{1}{2}} \]

where the square root is defined with a branch cut along the negative real axis and the argument function is between \((-\pi, \pi) \) \( [21] \). From now we shall call this two point function \( G_{1,E}^+ \) and define \( G_{2,E}^+(x, x') = G_{1,E}^+(\hat{x}, x') \) where \( \hat{x} = (\lambda, \pi - \rho, \theta) \). Then,

\[ G_{2,E}^+ = \frac{1}{4\sqrt{2\pi}} (\cos(\Delta \lambda - i\epsilon) + \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \Delta \theta)^{-\frac{1}{2}} \]

and \( G_{2,E}^+ \) satisfies also the homogeneous equation \( (\Box - \frac{1}{8} R)G = 0 \). Conformally mapping these solutions to AdS3, where \( G_A^+ = \sqrt{\cos \rho \cos \rho'} G_E^+ \) we get

\[ G_{1,A}^+(x, x') = \frac{1}{4\sqrt{2\pi \ell}} (\cos(\Delta \lambda - i\epsilon) \sec \rho \sec \rho' - 1 - \tan \rho \tan \rho' \cos \Delta \theta)^{-\frac{1}{2}} \]

and

\[ G_{2,A}^+(x, x') = \frac{1}{4\sqrt{2\pi \ell}} (\cos(\Delta \lambda - i\epsilon) \sec \rho \sec \rho' + 1 - \tan \rho \tan \rho' \cos \Delta \theta)^{-\frac{1}{2}}. \]

It can be seen that \( G_{1,A}^+ \) and \( G_{2,A}^+ \) are functions of \( \sigma(x, x') = \frac{1}{2}[\frac{1}{2}(u - u')^2 + (v - v')^2 + (x - x')^2 + (y - y')^2] \), which is the distance between the spacetime points \( x, x' \) in the 4-dimensional embedding space.

In order to deal with the problem of global hyperbolicity, it was shown in [5] that imposing boundary conditions on the ESU modes gives a good quantization scheme on the half of ESU with \( \rho \leq \frac{\pi}{2} \), thus inducing a good quantization scheme on AdS4. It may be checked that this method also works in \( 2+1 \) dimensions. The boundary conditions on the ESU modes are either Dirichlet

\[ \psi^E_{\ell,m} (\rho = \frac{\pi}{2}) = 0 \quad \text{obeyed by } \psi_{\ell,m} \text{ with } \ell + m = \text{ odd} \]

or Neumann

\[ \frac{\partial}{\partial \rho} \psi^E_{\ell,m} (\rho = \frac{\pi}{2}) = 0 \quad \text{obeyed by } \psi_{\ell,m} \text{ with } \ell + m = \text{ even}. \]

It is easily verified that the combination \( G_E^+ = G_{1,E}^+ \pm G_{2,E}^+ \) has the right boundary condition where the \(+(-)\) signs are for Neumann (Dirichlet) boundary conditions.

Some remarks are in order: if \( x, x' \) are restricted such that \(-\pi < \lambda(x) - \lambda(x') < \pi \) then

1. \( G_{2,E}^+ \) is real for spacelike points, imaginary for timelike points and singular for \( x, x' \) which can be connected by a null geodesic.
(2) $G^+_2$ has the same property when $x \to \hat{x}$, and if $0 \leq \rho(x'), \rho(x) < \frac{\pi}{2}$ then $G_2$ has singularities when $x, x'$ can be connected by a null geodesic bouncing off $\rho = \frac{\pi}{2}$ boundary.

From this we see that if we take the modes in $\text{AdS}_3$ as

$$\psi^A_{\ell,m} = (\cos \rho)^{\ell + \frac{1}{2}} e^{-i(\ell + 1)\lambda} Y^m(\rho, \theta) \quad \ell + m = \text{odd} \quad \text{or} \quad \ell + m = \text{even}$$

then these modes give rise to a well-behaved propagator [5]. The two point function is then

$$G^+_A = \sqrt{\cos \rho \cos \rho' (G^+_1 A \pm G^+_2 A)}$$

where $+(-)$ are for Neumann (Dirichlet). The two point function has singularities whenever $x, x'$ can be connected by a null geodesic directly or by a null geodesic bouncing off infinity (null geodesics remain null geodesics by a conformal transformation). All other properties listed before also stay the same.

Note that it is possible to define a quantization scheme on $\text{AdS}_3$ without using boundary conditions (i.e., just using $G^+_1 A$), which is referred to as transparent boundary conditions in Ref. [5]. However this requires the use of a two-time Cauchy surface, and its physical interpretation is unclear.

**APPENDIX B: CALCULATING THE RESPONSE FUNCTION**

We are interested in an integral of the type

$$J(\omega) = \frac{\ell^2 b}{\pi r^2} \int_{-\infty}^{\infty} e^{-\frac{\pi r^2 t}{2\tau}} (\cosh \alpha_n - \cosh(t - \ell t))^{-\frac{1}{2}} dt$$

where $T = \frac{r^2}{2\pi (r^2 - r_+^2)}$ is the local temperature. $J(\omega) = I_1(\omega) + I_2(\omega) + I_3(\omega)$ where $I_1$ is the integral from $-\infty$ to $-\alpha_n$, $I_2$ is from $-\alpha_n$ to $\alpha_n$, and $I_3$ is from $\alpha_n$ to $\infty$. Recall that the square root is defined with the cut along the negative real axis. Then

$$I_1 = \frac{\ell^2 b}{ir_+} \int_{-\alpha_n}^{-\infty} e^{-\frac{\pi r^2 t}{4\tau}} (\cosh t - \cosh(\alpha_n))^{-\frac{1}{2}} dt$$

$$I_3 = \frac{\ell^2 b}{ir_+} \int_{\alpha_n}^{\infty} e^{-\frac{\pi r^2 t}{4\tau}} (\cosh t - \cosh(\alpha_n))^{-\frac{1}{2}} dt$$

$$I_2 = \frac{2\ell^2 b}{r_+} \int_{0}^{\alpha_n} \cos \frac{\omega t}{2\pi T} (\cosh \alpha_n - \cosh t)^{-\frac{1}{2}} dt.$$  

Using [15]

$$\int_{-\alpha_n}^{-\infty} \frac{e^{-\frac{\pi r^2 t}{4\tau}}}{(\cosh t - \cosh(\alpha_n))^{-\frac{1}{2}}} = \sqrt{\frac{\pi}{2}} Q_\nu(\cosh(\alpha_n)) \quad \text{Re} \nu > -1 \quad \alpha > 0$$

$$\int_{0}^{\alpha_n} \frac{\cosh(\nu + \frac{1}{2})t}{(\cosh \alpha_n - \cosh t)^{-\frac{1}{2}}} = \frac{\pi}{\sqrt{2}} P_\nu(\cosh(\alpha_n)) \quad \alpha > 0$$

where $P_\nu$ and $Q_\nu$ are associated Legendre functions of the first and second kind respectively, we get
\[ I_3 = -\frac{i\sqrt{2\ell^2 b}}{r_+} Q_{\frac{i\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \]
\[ I_2 = \frac{\sqrt{2\pi \ell^2 b}}{r_+} P_{\frac{i\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \]
\[ I_1 = \frac{i\sqrt{2\ell^2 b}}{r_+} Q_{\frac{i\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n). \]

Now using \( Q_\nu(z) - Q_{-\nu-1}(z) = \pi \cot(\nu \pi) P_\nu(z) \) [15]
\[ J(\omega) = \frac{2\sqrt{2\pi \ell^2 b}}{r_+} P_{\frac{i\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \frac{1}{e^{\omega/T} + 1}, \]
and \( F_{1,2}(\omega) \), defined by (5.1) and (5.2) in an obvious way, are given by
\[ F_1(\omega) = \frac{1}{2} \frac{1}{e^{\omega/T} + 1} \sum_{n \neq 0} P_{\frac{i\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \]
\[ F_2(\omega) = \frac{1}{2} \frac{1}{e^{\omega/T} + 1} \sum_{n} P_{\frac{i\omega}{2\pi T} - \frac{1}{2}}(\cosh \beta_n). \]

Notice that although the formulae that we used were not correct when \( \alpha = 0 \), nevertheless the \( \alpha_0 \) term came out correctly, since
\[ \int_{-\infty}^{\infty} e^{-i\omega t} (1 - \cosh(2\pi T t - i\epsilon))^{-\frac{1}{2}} \]
\[ = i \int_{-\infty}^{\infty} e^{-i\omega t} \left( \sqrt{2} \sinh(\pi T t - i\epsilon) \right)^{-\frac{1}{2}} - i \int_{0}^{\infty} e^{-i\omega t} \left( \sqrt{2} \sinh(\pi T t - i\epsilon) \right)^{-\frac{1}{2}} \]
\[ = \int_{-\infty}^{\infty} e^{-i\omega t} \left( \sqrt{2} \sinh(\pi T t + \epsilon) \right)^{-1}. \]

This gives [12]
\[ F_0^0(\omega) = \frac{T}{2} \int_{-\infty}^{\infty} e^{-i\omega t} (2i \sinh(\pi T t + \epsilon) + 1)^{-1} = \frac{1}{2} \frac{1}{e^{\omega/T} + 1} \]
which is exactly what we got before as \( P_\nu(1) = 1 \).

Combining the results for \( F_1 \) and \( F_2 \), we have
\[ F(\omega) = \frac{1}{2} \frac{1}{e^{\omega/T} + 1} \sum_{n} \left( P_{\frac{i\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \pm P_{\frac{i\omega}{2\pi T} - \frac{1}{2}}(\cosh \beta_n) \right). \]
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FIGURES

FIG. 1. A Penrose diagram of (a) the $M \neq 0$ black hole, and (b) the $M = 0$ solution. Information can leak through spatial infinity, unless we impose boundary conditions at $r = \infty$.

FIG. 2. A Penrose diagram of $AdS_3$. Information can leak in or out through spatial infinity, and thus $\Sigma$ is not a Cauchy surface unless we impose boundary conditions at $r = \infty$. 