Abstract
We investigate axially symmetric asymptotically flat vacuum self-gravitating system. A class of initial data with apparent horizon was numerically constructed. The examined solutions satisfy the Penrose inequality. The prior analysis of a massive system and the present results suggest that either massive or sourcefree configurations fulfill the Penrose inequality.
1. Introduction.

The problem of the Cosmic Censorship Hypothesis is one of the most significant issues of the classical gravity. The feasible technique of testing it is the examination of the validity of the Penrose inequality [1]. Breaking of the Penrose inequality points to the violation of the Cosmic Censorship.

The geometry created by massive nonspherical (spheroidal) bodies has been numerically analyzed in [2]. The Penrose inequality in this case has been satisfied. We would like to stress out that those calculations was made under the assumption of conformal flatness. Conformal flatness may be intuitively understood as a restriction to the case without gravitational waves - a conformally flat system without matter is flat. Therefore the subject of our interest in this paper is in some sense an orthogonal situation: a geometry of the collapsing pure gravitational wave. After a few numerical tests [3] it has been recently analytically established [4] that vacuum solutions with apparent horizon exist.

In this paper we are numerically examining the following formulation of the inequality:

Let \( \Sigma \) be an asymptotically flat Cauchy surface with a non-negative energy density, \( m \) - the Arnowitt-Deser-Misner mass and \( S \) the area of an external apparent horizon. Then

\[
m \geq \sqrt{\frac{S}{16\pi}}.
\]  

(1)

Penrose argues that if the above is not true then the Cosmic Censorship will be broken. A more refined version of (1) has been proposed in [5]. The inequality (1) is still proven only for a restricted class of geometries [6,7] (see also a short review in [8]).

2. Main equations.

We examine the formation of apparent horizons on the initial hypersurface \( \Sigma \) which is required to be \textbf{momentarily static} and \textbf{axially symmetric} (Brill waves). In the case of vacuum constraints on \( \Sigma \) reduces to

\[
(3) \ R[g_{ij}] = 0.
\]

\( R[g_{ij}] \) is 3-dimensional scalar curvature of \( \Sigma \). The axially symmetric line element may be written as follows [9]:

\[
g_{ij}dx^idx^j = \Phi^4(r, \theta) \left( e^{-q(r, \theta)} (dr^2 + r^2d\theta^2) + r^2 \sin^2 \theta d\varphi^2 \right).
\]

Due to vanishing of the scalar curvature, \( \Phi \) and \( q \) are constrained by [9]

\[
\Delta \Phi + f \Phi = 0,
\]

(2)

where \( \Delta \) denotes 3-dimensional flat laplacian and

\[
f = -\frac{1}{8} \left( \frac{\partial^2 q}{\partial r^2} + \frac{1}{r} \frac{\partial q}{\partial r} + \frac{1}{r^2} \frac{\partial^2 q}{\partial \theta^2} \right).
\]

(3)
Regularity of metric on the axis implies the boundary conditions for $\Phi$ and $q$.

$$\frac{\partial \Phi}{\partial \theta} (\theta = 0) = 0, \quad \frac{\partial q}{\partial \theta} (\theta = 0) = 0, \quad q(\theta = 0) = 0. \quad (4)$$

The equivalent group of conditions must hold for $\theta = \pi$. Moreover asymptotically solution is flat: $\Phi \to 1 + \frac{m}{2r}, \; q \to 0$. Where $m$ is the ADM mass (see equation (1)). The above requirements may be satisfied by following choices of function $q$ [3]:

$$q_1(r, \theta) = \frac{Ar^2 \sin^2 \theta}{1 + r^5}. \quad (5)$$

Inserting $q$ in (3) we obtain

$$f_1 = -\frac{A}{8} \left( 2U - 45U^2 r^5 \sin^2 \theta + 50U^3 r^{10} \sin^2 \theta \right), \quad (6)$$

where

$$U = \frac{1}{1 + r^5}. \quad (7)$$

Similarly we analyze another $q$:

$$q_2(r, \theta) = Ae^{-\alpha^2} r^2 \sin^2 \theta. \quad (8)$$

Now $f$ has the following form:

$$f_2 = -\frac{A}{4} e^{-\alpha^2} (1 - 6\alpha r^2 \sin^2 \theta + 2\alpha^2 r^4 \sin^2 \theta). \quad (9)$$

The above formulas we use in the next analysis for calculation of function $\Phi$. Because of the regularity the determinant of the metric must have no zeros. Therefore we consider only solutions which are everywhere positive; $\Phi > 0$.

A surface $S$ is called an apparent horizon if the expansion of outgoing future directed null geodesics which are orthogonal to $S$ vanishes everywhere on $S$. In the momentarily static case expansion is the divergence of normal unit vector $n^i$ to $S$.

$$D_i n^i = 0. \quad (10)$$

Hence on time symmetric $\Sigma$ notion of apparent horizon coincides with definition of minimal surface. We investigate the external apparent horizon which is the outermost apparent horizon that surrounds the region of concentration of gravitational energy. Equation (10) for our metric takes the form

$$r_{\theta \theta} + \frac{r^3}{r^2} \left( \frac{4\Phi}{\Phi} - \frac{q_\theta}{2} + \text{cot} \theta \right) - r_\theta^2 \left( \frac{4\Phi}{\Phi} - \frac{q_r}{2} + \frac{3}{r^2} \right) +$$
\[ +r_\theta \left( \frac{4\Phi_\theta}{\Phi} - \frac{q_0}{2} \csc(\theta) \right) - r^2 \left( \frac{4\Phi_r}{\Phi} - \frac{q_0}{2} + \frac{q}{r} \right) = 0 \quad (11) \]

The area of \( S \) is given by

\[ S = 2\pi \int_0^\pi \Phi^4 e^{-q/2} \sqrt{r^2 + r^2 \sin^2(\theta)} \, d\theta \quad (12) \]

3. Numerical techniques and results of calculations.

Originally we intended to solve the eq. (2) by the use of standard method on lattice. The numerical error was too big and therefore we solve the partial differential equation (2) by expanding the functions \( \Phi \) and \( f \) in a series of the Legendre polynomials [3]. We obtained an infinite system of ordinary differential equations for \( r \) dependent functions. We approximately solved this system by retaining only leading terms (up to \( l=5 \)). Numerical results shows that the contribution from the higher terms is very small and safely can be neglected. Our numerical analysis of the system just described and the equation for external apparent horizon \( (11) \) was based of the fourth order Runge-Kutta method. For \( q_2 \) we examined only one value of \( \alpha = 0.1 \). The solutions for other values of \( \alpha \) may be obtained from the previous ones by simple scaling transformations.

We analyzed only such configurations for which the shape of the external apparent horizon significantly diverges from spherical symmetry. In the case of spherical symmetry the inequality \( (1) \) reduces to an identity. In the below tables we collect the most important results of our calculations. The first table corresponds to \( q \) given by formula \( (5) \) and the second one to \( (8) \) with \( \alpha = 0.1 \).

<table>
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<th>( m )</th>
<th>( r_0 )</th>
<th>( r_{\pi/2} )</th>
<th>( S )</th>
<th>( m - \sqrt{S/16\pi} )</th>
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Table 1.

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<th>( A )</th>
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Table 2

Summarizing, the obtained result confirm the validity of the Penrose inequality for the analyzed class of Brill waves. Although the proof of the inequality \( (1) \) is still uncompleted it seems that our belief in this conjecture is rather reasonable.

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References