Recent Progress in Understanding Quark and Gluon Distribution Functions for Large Nuclei at Small x*

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Abstract

I discuss the problem of computing the structure functions for very heavy nuclei at small Bjorken x. The approximations used in this description are physically motivated, and recent computations reviewed.

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1 INTRODUCTION

The theoretical question which will be addressed in this talk is the computation of quark and gluon distribution functions at small Bjorken $x$. The reason why such computations may be possible is because at small values of Bjorken $x$, which correspond to central values of rapidity, the density of partons per unit area satisfies

$$\frac{1}{\pi R^2} \frac{dN}{dy} \gg \Lambda_{QCD}^2$$

(1)

This density of partons per unit area is the only local parameter with dimensions of an energy squared which we can construct, and the coupling therefore should be evaluated at this scale. Defining

$$\Lambda^2 = \frac{1}{\pi R^2} \frac{dN}{dy}$$

(2)

the strong coupling constant at this scale must be

$$\alpha_s(\Lambda) << 1$$

(3)

so that it may be possible to formulate the problems in weak coupling.

There are several problems which might be solved if one can compute the quark and gluon distribution functions at small Bjorken $x$. At small $x$, it is expected that the gluon distribution function for a single proton behaves as [1]

$$\frac{dN}{dx} \sim \frac{1}{x^{1+\alpha_s}}$$

(4)

This implies that the distribution of partons for a proton diverges as $x \to 0$. If this is the case, then this problem should be possible to analyze using weak coupling methods. The distribution functions at small $x$ have recently been measured at HERA, and do seem to be singular at small $x$.

In addition to understanding the Lipatov enhancement, it would also be useful to compute quantities such as the ratio of sea quark to gluon distributions. For
example, if the typical energy scale has been increased due to a high density of partons, we should expect that heavier quarks will become of increasing importance. Charm quark production might for example become substantial.

Another interesting physical problem is deep inelastic scattering and di-lepton production using nuclei with $A >> 1$. If the value of Bjorken $x < A^{-1/3}$, then the nucleus as seen by a parton moving with that value of $x$ is Lorentz contracted to a scale size which is much smaller than the wavelength of the parton in a frame comoving with its longitudinal momentum. It is expected that in this kinematic region, there should be non-trivial effects which might screen the effects of the valence nuclear matter distribution. On the other hand, we expect that the distribution functions for partons should become large for large nuclei, and if the effects due to screening can be ignored (as we will see they can be when we do the computation) then

$$\frac{1}{\pi R^2} \frac{dN}{dy} \sim A^{1/3} \gg \Lambda_{QCD}^2$$

as $A \to \infty$. In addition to the questions about the Lipatov enhancement and the ratio of sea quarks to glue, one can also ask about the $A$ dependence of the distribution functions.

Finally, there is the problem of determining the initial conditions for quarks and gluons in heavy ion collisions. If one can understand the distribution functions, then these may provide information about the boundary conditions for the evolution of the matter into a quark-gluon plasma. Recall that since the density of partons per unit area is the only scale in the problem, and it goes like $A^{1/3}$, the energy density will have to scale as $E/V \sim A^{2/3}$, where this is the energy density scale at a time which corresponds to the dimensional scale constructed from the density of partons per unit area. In order for these considerations to be valid, the Lorentz contracted size of the nucleus must be smaller than the size scale constructed from the density of partons per unit area. This is $E_{CM}/A \gg \sqrt{A}$ which for large nuclei requires
that $E_{CM}/A >> 50$ GeV which is within the range accessible at RHIC.

In what follows, I will describe how to compute the quark and gluon distribution functions for very large nuclei. We have not yet succeeded in being able to compute for a single hadron. It may also be true that our weak coupling analysis may be marginal at best for realistic values of $A$, and therefore our results may only be useful as a theoretical laboratory which will give us some insight into the structure which we might expect for realistic nuclei. If it is true that weak coupling techniques cannot be used for the earliest stages of a heavy ion collision, then it also will imply that such techniques are probably never applicable in the subsequent evolution of the matter produced in such a collision.

2 Summary of Results

Before going into a discussion of how to analyze the problem of computing distribution functions for very large nuclei, I will first summarize our results to date.[2] First I define

$$\mu^2 = \frac{4 N_{\text{valence}}}{3} \pi R^2 \sim 1.1 A^{1/3} \, \text{Fm}^{-2}$$

(6)

which is the the density of valence quark color charge squared per unit area. I will show that there is a many body theory which describes the quark and gluon distribution functions so long as we restrict our attention to parton transverse momenta which satisfy

$$q_T^2 << \mu^2$$

(7)

and we require that we are at small values of $x$

$$x << A^{-1/3}$$

(8)

There are two expansion parameters for this many body theory $\alpha_s(\mu)$ and $\alpha_s(\mu) \mu / q_T$
In the range $\alpha_s(\mu) \mu \ll q_T \ll \mu$ the gluon distribution function may be evaluated to lowest order in $\alpha_s(\mu)$ as

$$\frac{1}{\pi R^2} \frac{dN}{dx d^2 q_T} = \frac{\alpha_s(N_c^2 - 1) \mu^2}{\pi^2} \frac{1}{x q_T^2}$$  \hspace{1cm} (9)$$

This is just the Weizsacker-Williams distribution function for a Lorentz boosted distribution of Coulombic gluons, scaled by the factor $\mu^2$ which is the average value of the charge squared per unit area.

The result above is however only a formal result if it is true that there is a Lipatov enhancement. The Lipatov enhancement is of the form $x^{-c_{\alpha_s}}$. If we formally expand this as a series in $\alpha_s$, we find the expansion parameter is $\alpha_s(\mu) ln(1/x)$. Since we have assumed that $x \ll A^{-1/3}$ and $\mu^2 \sim A^{1/3}$, this expansion is not well behaved. To correct for this behavior if it occurs, it will be necessary to go beyond the naive weak coupling expansion. It should however still be true that weak coupling methods are still applicable.

If we sum to all orders in $\alpha_s, \mu$ but to first order in $\alpha_s$, we have shown that the gluon distribution function is of the form

$$\frac{1}{\pi R^2} \frac{dN}{dx d^2 q_T} = \frac{N_c^2 - 1}{\pi^2} \frac{1}{x} \frac{1}{\alpha_s} H_{\alpha_s(q_T^2 / \alpha_s^2 \mu^2)}$$  \hspace{1cm} (10)$$

where the function $lim_{y \to \infty} H(y) \to 1/y$. The function $H$ is a correlation function for an ultraviolet finite two dimensional Euclidean quantum field theory. The strong coupling limit of this field theory is the small $q_T$ limit. In this limit, we expect disorder since the theory is strongly coupled. This implies exponentially falling correlations in coordinate space which implies that $H(0)$ should be finite.

3 Setting the Problem Up

To begin computing the distribution functions, we will assume we are working in a frame where the nucleus is moving close to the speed of light. In this frame the
nucleus appears as a Lorentz contracted pancake. The value of Bjorken $x$ in this frame is essentially the ratio of longitudinal momentum of the parton to that of the projectile nucleus per nucleon.

Since the variation in the nuclear valence quark distribution function with respect to transverse coordinate is

$$
\frac{1}{N} \frac{dN}{dr} \sim \frac{1}{R_{nuc}}
$$

and since the average scale of transverse momenta is $q_T > 1/R_{nuc}$, we see that locally, the variation in the transverse nuclear matter distribution can be ignored. Therefore to compute the local properties of the gluon distributions function as a function of transverse coordinate, we need only consider a problem where the transverse matter distribution is assumed to be uniform and of infinite extent in the transverse direction. The problem is therefore of a sheet of valence quarks uniformly distributed on a thin sheet of infinite extent in the transverse direction. We will take the density of valence quarks to be $N_{quark}/\pi R^2 \sim A^{1/3}$.

The natural variable with which to analyze the dynamics are light cone variables,

$$
a^\pm = a^0 \pm a^z \sqrt{2}
$$

If we let $k^+$ be a parton light cone momentum and $p^+$ that of a projectile nucleon, then $x = k^+ / p^+$ In these variables, instead of constructing eigenstates of the Hamiltonian, it is simpler to construct the eigenstates of the generator of $x^+$ transformations,

$$
P^- = \frac{1}{\sqrt{2}} (H - P^z)
$$

Finally, when constructing the light cone Hamiltonian and action for the theory in terms of these variables, it is simplest to work in light cone gauge,

$$
A_- = A^+ = 0
$$
In this gauge, the light cone Hamiltonian has the form

\[ P^- = \int d^3x \frac{1}{4} F_{\tau}^2 + \frac{1}{2} (\rho_F + D_T \cdot E_T) \frac{1}{P^+ + 2}(\rho_F + D_T \cdot E_T) \]

\[ + \frac{1}{2} \psi \bar{\psi} (M - \gamma \cdot P_T) \frac{1}{P^+ + 2} (M + \gamma \cdot P_T) \psi \]  

(15)

In this equation,

\[ P^\mu = \frac{1}{i} \partial^\mu - g \tau \cdot A^a \]  

(16)

and \( E_k = -\partial_\tau A_k \). The Lagrangean is generated in the usual way by adding in

\(-i \psi \bar{\psi} + E_k \partial_\tau A_k \) The fermion charge density is

\[ \rho_{\psi}^a = \overline{\psi} \gamma^\tau T^a \psi \]  

(17)

The quantized fields are

\[ \psi_\alpha(x) = \int_{k+>0} \frac{d^3k}{(2\pi)^3} (b_\alpha(k)e^{ikx} + d_\alpha^\dagger(k)e^{-ikx}) \]

\[ A^a_\tau(x) = \int_{k+>0} \frac{d^3k}{(2\pi)^3 \sqrt{2k^+}} (a^a_\tau(k)e^{ikx} + a^a_\tau^\dagger(k)e^{-ikx}) \]  

(18)

The last ingredient we need to set the problem up is how to describe the valence quarks. These quarks are traveling close to the speed of light. For small values of \( \alpha_s \), these quarks occasionally emit a small \( x \) gluon. This emission does not change the path of the valence quark. It should therefore be a good approximation to treat the trajectories for the valence quarks as straight line propagation at light velocity, that is

\[ J_\alpha^+ = \rho_{\alpha}(x^+, x_T) \delta(x^-) \]  

(19)

Unlike the case for QED, in general in QCD the charge density will have to depend on the time \( x^+ \) since the extended current conservation condition requires that

\[ \partial_\tau Q^a + f^{abc} A_b Q_c = 0 \]  

(20)

This forces the charge to rotate as

\[ \tau \cdot Q(x^+) = U(x^+) \tau \cdot Q(0) U^\dagger(x^+) \]  

(21)
where

\[ U(x^+) = T \exp \left( \int_0^{x^+} dy^+ \tau \cdot A_+(y^+) \right) \]  

(22)

To summarize, we have shown the problem which must be solved is to compute the ground state expectation values for a system with the valence quarks traveling with light velocity localized along an infinite sheet in the transverse space. The density of the valence quarks is uniform. This problem is well posed since the constraint on the valence quarks is equivalent to specifying the space-time coordinates of the electromagnetic charge and baryon number. These operators commute with the QCD Hamiltonian.

4 The Example of QED

In QED, the problem we want to solve is the photon structure function generated by a fast moving electron. We will ignore pair production of electron-positron pairs. The source for the electron is \( x^+ \) independent and is

\[ \rho_e = e \delta(x^-) \delta^2(x_T) \]  

(23)

The light cone Hamiltonian is

\[ P^- = \int d^3x \left( \frac{1}{4} E_T^2 + \frac{1}{2} (\rho_e + \nabla_T \cdot E_T) \frac{1}{p_{+2}} (\rho_e + \nabla_T \cdot E_T) \right) \]  

(24)

For this Hamiltonian, the ground state is a coherent state

\[ | \Psi > = C \exp \left( i \int d^3x A^\mu(x) E^\mu(x) \right) | 0 > \]  

(25)

Letting \( P^- \) operate on this state, we see that the ground state has \( P^- = 0 \) so that the classical field is purely longitudinal and

\[ B_T = 0 \]

\[ \nabla_T \cdot E_T = -e \rho_e \]  

(26)
The solution for these equations are that

\[ A_T = e \frac{1}{k^+} \frac{\tilde{k}_T}{k_T^2} \]  

(27)

In space-time, the vector potential is \( \theta(x^-) \vec{V} \lambda \) which is for \( x^- < 0 \) vanishing and a pure gauge for \( x^- > 0 \).

The field above is precisely the Weizsacker-Williams field for the Lorentz boosted Coulomb field. The distribution function for the photons can be computed and is

\[ F(k^+, k_T) = \frac{1}{(2\pi)^3} a^1(k)a(k) > = \frac{2\epsilon^2}{(2\pi)^3} \frac{1}{k^+k_T^2} \]  

(28)

or

\[ F(x, k_i) = \frac{\alpha}{\pi^2} \frac{1}{xk_T^2} \]  

(29)

5 The Distribution Functions for QCD

For QCD, we have not been successful in constructing the ground state wavefunction in the presence of the external source corresponding to the valence quarks. We have however concentrated on computing ground state expectation values. To do this, we first note that

\[ Z = \lim_{T \to -\infty} \sum_N \langle N | e^{iTP^-} | N \rangle \]  

(30)

will project onto the ground state. The sum over \( N \) here includes a sum over the color labels of the external source of color charge generated by the valence quarks.

To do the sum over \( N \), we break our transverse space into a grid of squares with size \( d^2x \gg \pi R^2/N_{\text{quark}} \sim A^{-1/3}Fm^2 \). Our approximation will therefore only be good when we look at transverse momentum scales where \( q_T^2 \ll \mu^2 \). In these limits, the number of valence quarks in each square is much larger than 1. If this is the case, then typically the charge in each square will be much larger than 1. If \( Q \) is this charge, then \( Q^2 \gg Q \sim [Q, Q] \) so that the charge may be treated classically.
If the total charge of interest is also much less than the maximum possible charge in the square, then the density of states for charge $Q$ is $e^{-Q^2/2\mu^2}$. Summing over the states in the definition of $Z$ is therefore equivalent to inserting into the path integral the integration

$$\exp \left(-\frac{1}{2\mu^2} \int d^2 x_T \rho^2(x) \right) \quad (31)$$

where $\rho$ is the charge density per unit area.

For such a Gaussian charge distribution we have

$$< \rho^a(x) \rho^i(y) > = \mu^2 \delta^{ai} \delta^{(2)}(x_T - y_T) \quad (32)$$

where it is straightforward to estimate $\mu^2 = 1.1 A^{1/3} Fm^2$

The problem we must solve is therefore that described by the theory in the presence of an arbitrary external source of surface charge on the light cone and then integrating over all possible values of the charge. The fields generated correspond to a stochastic source of charge. The problem has therefore been reduced to a many body theory. It is possible to integrate out the sources entirely and get an action in terms of the quark and gluon fields with a term involving $\mu^2$ which is associated with the valence quark charge density. The value of $\mu^2$ sets the scale for the coupling constant. For large $\mu^2$ the coupling is small and weak coupling methods should be reliable.

We can now compute the gluon distribution function to lowest order in $\alpha_s$ and to lowest order in $\mu^2$. We first compute the change in the propagator induced by the sources. This is

$$\delta < A^a A^i > = \int [d \rho] g^2 \delta^{ai} \delta^{ij} < \left( \frac{\nabla_i}{\partial + \nabla_i^2} \right) \rho^a(x) \left( \frac{\nabla_i}{\partial + \nabla_i^2} \right) \rho^i(x) \quad (33)$$

or

$$\delta D^{\mu \nu}(k, q) = g^2 \mu^2 \delta^{ai} (2\pi)^4 \delta(k^-) \delta(q^-) \delta^{(2)}(k_T - q_T) \delta^{ai} \delta^{ij} \frac{k_T^i q_T^j}{k^+ q^+ k_T^2 q_T^2} \quad (34)$$
Using this form of the propagator, it is now easy to show that

\[
\frac{1}{\pi R^2} \frac{dN}{d^3k} = \frac{\alpha_s \mu^2 (N_2^2 - 1)}{\pi R^2 k^2 T^2} \frac{1}{k^2 T^2}
\]

This is just the Weizsacker-Williams distribution function weighted by the average charge squared per unit area. This result reflect the RMS fluctuations in the stochastic background field induced by the source associated with the valence quarks.

We will soon see that this result is valid only in the range of momentum where \( \alpha_s^2 \mu^2 \ll k_T^2 \ll \mu^2 \). The last term in the previous limit is just the region of validity of our derivation of the many body theory. The first term is the limit of validity of assuming that \( \mu^2 \) is small and that one can expand to first order in this quantity.

We can generalize our results to all orders in \( \alpha_s \mu \) and first order in \( \alpha_s \). To do this, we solve the classical problem of computing the fields in the background of an arbitrary source and then integrate over the source. This classical problem will be accurate to first order in \( \alpha_s \).

We must solve the equations of motion

\[
D_\mu F^{\mu \nu} = g \delta^{\mu \nu} \delta(x^-) \rho(x^+, x_T)
\]

There is a solution of these equations of motions with \( A_+ = A_- = 0 \) which also has \( F_T = 0 \). This solution is of the form

\[
A_j(x^+, x_T) = \theta(x^-) \alpha_j
\]

The condition that \( F_T = 0 \) is equivalent to the condition that the field \( \alpha_j \) is a gauge transformation of the vacuum configuration of a two dimensional gauge theory. The condition that \( D_T \cdot E_t = -g \rho \) is equivalent to the two dimensional gauge condition

\[
\nabla \cdot \alpha = -g \rho
\]

The above field configuration may be written as

\[
\tau \cdot \alpha = \frac{1}{ig} U(x_T) \nabla U^\dagger(x_T)
\]
and the gauge condition is
\[
\vec{\nabla}(U\vec{\nabla}U^\dagger) = ig^2 \rho
\]  
(40)

The integration over the sources may be written as
\[
\int [dU] \exp \left(-\frac{1}{\mu^2 g^4} \text{tr} \left( \vec{\nabla}_T \cdot U\frac{1}{i} \vec{\nabla}_T U^\dagger \right)^2 \right)
\]  
(41)

There is of course a Fadeev-Popov determinant for this measure which comes from restricting to Feynman gauge, but we will not be concerned with this measure here. (The determinant does not affect the arguments we present for the validity of perturbative expansions nor the scaling behavior in \( k_T \).)

Note that the coupling constant for this theory is \( g^2 \mu \) so that the expansion parameter is \( g^2 \mu / q_T \). The fluctuations over the external charge generate an ultraviolet finite theory. The correlation function of \( U\vec{\nabla}U^\dagger / g \) generates the modifications due to the sources of the gluon propagator. This correlation function should die exponentially at long distances corresponding to small coupling \( \alpha_s \), that is, in momentum space, the correlation function should be finite at zero momentum.

The step function \( \theta(x^-) \) in the solution of the classical field equations guarantees that the distribution of gluons is proportional to \( 1/k^+ \) when the theory is solved to first order in \( \alpha_s \). The gluon distribution function is therefore of the form
\[
\frac{1}{\pi R^2} \frac{dN}{dx d^2 k_T} = \frac{N_2^2 - 1}{\pi^2} \frac{1}{x \alpha_s} H\left(\frac{k_T^2}{\alpha_s^2 \mu^2}\right)
\]  
(42)

where \( \lim_{y \to \infty} H(y) = 1/y \)

6 Summary

There are of course many problems which must be solved before this approach can provide a realistic theory of the distribution functions at small \( x \). The first order corrections in \( \alpha_s \) must at least be understood. This will generate the induced contribution of sea quarks. It will also presumably lead to the Lipatov enhancement.
This must be nontrivial since for this theory where the valence quarks are localized to a delta function on the light cone, there is no scale of \( P^+ \), the momentum per nucleon of the nucleon in the nucleus. This can only arise from the cutoff dependence of a regularized delta function. Such a dependence will not affect the physics to leading order in \( \alpha_s \) nevertheless, since if we change the cutoff by a finite amount \((\Lambda/k^+)^{c\alpha_s}\), changes by only an amount proportional to \( \alpha_s \).

In addition to the above problems, the issue of how large a transverse momentum one can use to define a sensible theory remains open. Even though our approximations were only valid for \( q_T << \mu \) the validity of assuming straight line trajectories along the light cone should be valid in a much broader region. The extent to which the theory can be solved in the large momentum transfer region and the extent of the region of validity of the external source approximation remains open.

Finally, there is the issue of actually computing structure functions for deep inelastic scattering or Drell-Yan particle production. We have here only discussed computing the expectation values of quark distributions in the ground state wavefunction. When a probe of momentum \( q^2 \) is introduced, in general there will be two parameters of interest \( q^2 \) and \( \mu^2 \). For \( q^2 >> \mu^2 \), presumably one can analyze the distribution functions using Altarelli-Parisi equations. For \( q^2 \leq \mu^2 \), the situation must be more complicated since here the momentum of the quarks inside the hadron wavefunction are important and one is not in the scaling region.

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