Scaling Violation in e^+e^- Fragmentation Functions: QCD Evolution, Hadronization and Heavy Quark Mass Effects

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Abstract

Tests of QCD in the process $e^+e^- \rightarrow hX$, where h is any hadron species, or any charged hadron, are proposed, concentrating on the following topics: decomposition of the total inclusive cross section into transverse, longitudinal and asymmetric parts; extraction of the gluon fragmentation function from the longitudinal part; flavour sum rules for the asymmetric part; power corrections due to quark masses and hadronization; measurement of α_s from a next-to-leading order analysis of scaling violation. We conclude that a number of novel tests can be performed using the very high statistics available at LEP, and that a comparison between LEP and lower-energy data, together with information on heavy quark fragmentation and some model-dependent assumptions about hadronization, could provide a useful measurement of α_s .

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1. Introduction

The inclusive single-particle production process

$$e^+e^- \rightarrow \gamma, Z^0 \rightarrow h X \quad (1.1)$$

where h represents either a given species of outgoing hadron, or a sum over all charged hadron species, has been studied over a wide range of energies at many different e^+e^- colliders. Up to now, however, analyses of the data on this process have mainly concentrated on comparing particle spectra and yields with model predictions, rather than on testing quantitative predictions of QCD. This is surprising since inclusive annihilation is closely related to the deep inelastic scattering process $e h \rightarrow e X$, which has been one of the most important testing-grounds for QCD. Now that very large amounts of precise data are available from the LEP collider, this imbalance should soon be rectified. Accordingly, the aim of this paper is to review critically the kinds of tests that can be performed, and to present some relevant new theoretical and phenomenological results.

For unpolarized beams and outgoing particles, the cross section for the process (1.1) at a given centre-of-mass energy $Q \equiv \sqrt{s}$ can depend only on the energy E of particle h and its angle θ with respect to the electron beam direction in the c.m. frame. It is convenient to introduce the energy fraction

$$x \equiv x_E = 2E/Q = 2pq/q^2, \quad (1.2)$$

where p and q are the four-momenta of the produced particle and virtual boson.[†] A standard tensor analysis (see for example refs. [1,2]) shows that, since the photon and Z^0 have spin one, the most general form of the differential cross section for the unpolarized process is

$$\frac{d^2\sigma^h}{dx d\cos\theta} = \frac{3}{8}(1 + \cos^2\theta)\frac{d\sigma_T^h}{dx} + \frac{3}{4}\sin^2\theta\frac{d\sigma_L^h}{dx} + \frac{3}{4}\cos\theta\frac{d\sigma_A^h}{dx}. \quad (1.3)$$

The three terms on the right-hand side will be referred to as the *transverse*, *longitudinal* and *asymmetric* contributions, respectively. The first two are associated with

[†]Experimental data are normally presented in terms of the scaled momentum $x_p = 2|\mathbf{p}|/Q$, which differs from the quantity (1.2) by mass terms. The consequences of this difference are discussed in sect. 3.

the corresponding polarization states of the virtual boson with respect to the direction of the observed hadron. The asymmetric contribution is due to parity-violating interference terms and is not present in purely electromagnetic annihilation.

In sect. 2 of the present paper, we start by discussing methods for the extraction of these three contributions from experimental data and their general structure according to QCD. Apart from subasymptotic corrections, which are expected to decrease like inverse powers of the centre-of-mass energy, they are expressible in terms of universal *fragmentation functions* which specify the x distributions of hadrons of type h coming from the fragmentation of each type of QCD parton at a given energy scale. The fragmentation functions incorporate the long-distance, non-perturbative physics of the *hadronization* process, in which the observed hadrons h are formed from the partons. The scale dependence of the fragmentation functions is governed by an evolution equation, similar to the evolution equation for parton densities. The initial condition for the evolution equation, (as in the case for parton densities), is not calculable in perturbation theory, and must at present be taken from experiment.

Multiplying the fragmentation functions are calculable coefficient functions (sometimes also called short distance cross sections) which are the cross sections for the inclusive production of each parton type in the given physical process. This yields predictions concerning the angular distributions of hadrons and the scaling violations in their energy spectra. The relevant coefficient functions and the formalism for predicting angular distributions and scaling violation to next-to-leading order are presented in sect. 2.

In addition to asymptotic scaling violation, there are various sources of subasymptotic terms, suppressed by inverse powers of the energy scale Q , which we discuss in sect. 3. The best-understood corrections of this type are heavy quark effects, which give rise to powers of m/Q where m is the relevant quark mass. We study dynamical mass corrections to the inclusive cross section up to first order in α_s (complete results to this order are given in the Appendix), as well as kinematical effects of heavy quark decays.

Power corrections are also expected to arise from non-perturbative effects such as the hadronization process. We find that in this respect inclusive annihilation is at some disadvantage compared with deep inelastic scattering, even though the accessible energies have generally been much higher. The reason is that in deep

inelastic scattering we can apply the techniques of the operator product expansion to show that power corrections are at most of order $1/Q^2$ relative to the leading terms. This result cannot be proven in the timelike kinematic region relevant for inclusive annihilation, and so we have to rely on models to estimate power corrections. We show that simple models as well as more sophisticated Monte Carlo studies of hadronization suggest that the power corrections to inclusive annihilation are in fact of order $1/Q$. Thus, as far as power corrections are concerned, annihilation at $Q^2 \sim m_Z^2 \sim 10^4 \text{ GeV}^2$ is only equivalent to deep inelastic scattering at $Q^2 \sim 100 \text{ GeV}^2$, assuming a hadronization scale of the order of 1 GeV. We propose a simple parametrization of hadronization power corrections, based on Monte Carlo studies, which amounts to a Q -dependent rescaling of x .

In sect. 3 we also perform Monte Carlo studies of the effects of hadronization on sum rules for the longitudinal and asymmetric contributions to the cross section. The former provides a possible new method for measuring the strong coupling α_s , while the latter could provide further information on the hadronization process.

In sect. 4 we apply the next-to-leading QCD formalism and hadronization studies of the previous sections to some phenomenological analyses of real and Monte Carlo data. We first discuss the gluon fragmentation function and use data on gluon jet fragmentation to predict the longitudinal component of the inclusive cross section. From this exercise we find that an experimental measurement of the longitudinal component would provide useful information on gluon fragmentation, especially at small x .

Next we perform an illustrative scaling violation analysis of the total inclusive cross section. Information on heavy (charm and bottom) quark fragmentation into light hadrons is an essential ingredient of the analysis. The flavour composition of the final state varies greatly between lower energies and the Z^0 region, leading to apparent scaling violation if the fragmentation functions of light and heavy quarks are significantly different. Since there are no experimental data available yet on the total fragmentation functions of heavy quarks, we use Monte Carlo data as input.

Using our simple parametrization of power corrections, we obtain acceptable fits to experimental data over the energy range $22 < Q < 92 \text{ GeV}$, with correlated values of the QCD scale $\Lambda_{\overline{\text{MS}}}$ and the coefficient B of the $1/Q$ power corrections. The required value for B is however unreasonably large. We then show that with a reasonable

modification of the heavy quark input we can obtain acceptable values for the B parameter. This means that experimental information on heavy quark fragmentation is essential, together with further study of hadronization, to limit the uncertainty in the power corrections, in order to make a useful determination of α_s . Progress on the experimental investigation of scaling violation is already being made^[3], which, when combined with the full next-to-leading formalism presented here, should enable a definitive result to be obtained in the near future.

Finally in sect. 5 we summarize all our results and present our conclusions.

2. QCD evolution

2.1. Structure of inclusive cross sections

We consider first the general properties of the inclusive cross section (1.1) and its decomposition into transverse, longitudinal and asymmetric parts according to eq. (1.3). Note that in this paper the transverse and longitudinal parts are normalized in such a way that

$$\frac{d\sigma^h}{dx} = \int_{-1}^{+1} d\cos\theta \frac{d^2\sigma^h}{dx d\cos\theta} = \frac{d\sigma_T^h}{dx} + \frac{d\sigma_L^h}{dx} . \quad (2.1)$$

The three contributions can be extracted from the data, either by fitting the angular dependence for each value of x , or by weighting each particle in the final state with an appropriate angular factor. We have

$$\frac{d\sigma_P^h}{dx} = \int_{-1}^{+1} d\cos\theta W_P(\cos\theta) \frac{d^2\sigma^h}{dx d\cos\theta} \quad (2.2)$$

where

$$W_T(u) = 5u^2 - 1 , \quad W_L(u) = 2 - 5u^2 , \quad W_A(u) = 2u . \quad (2.3)$$

If particles of type h can be detected with equal efficiency at all angles, the weighting factors are just the expressions (2.3). If only a restricted angular region is accessible, then modified weights can be defined to extract the cross section contributions from data in that region. Using only $-v < \cos\theta < v$, for example, we have

$$\frac{d\sigma_P^h}{dx} = \int_{-v}^{+v} d\cos\theta W_P(\cos\theta; v) \frac{d^2\sigma^h}{dx d\cos\theta} \quad (2.4)$$

where the new weights are

$$\begin{aligned} W_T(u; v) &= [5u^2(3 - v^2) - v^2(5 - 3v^2)]/2v^5 \\ W_L(u; v) &= [v^2(5 + 3v^2) - 5u^2(3 + v^2)]/4v^5 \\ W_A(u; v) &= 2u/v^3. \end{aligned} \tag{2.5}$$

Neglecting power corrections, which we discuss in sect. 3, the QCD prediction for each part of the inclusive cross section has the general form

$$\frac{d\sigma_P^h}{dx} = \sum_i \int_x^1 \frac{dz}{z} C_{P,i}(z, \alpha_s(\mu), Q, \mu) D_i^h(x/z, \mu), \tag{2.6}$$

where the sum on i runs over all types of partons (quarks f , antiquarks $\bar{f}$ and gluons g), $C_{P,i}$ is a coefficient function calculable in perturbation theory, and D_i^h is the fragmentation function for particles of type h from partons of type i . Both the coefficient and fragmentation functions depend on an arbitrary factorization scale μ , in such a way that the dependence of the physical cross sections on μ would cancel if the calculation could be carried out to all orders in perturbation theory. In a finite-order calculation, however, some dependence on μ will remain in orders higher than those calculated. This is the scale dependence problem, common to all perturbative predictions. We study this problem quantitatively in sect. 4.2. Generally speaking, a scale value of $\mu \sim Q$ is expected to be most appropriate in eq. (2.6); if μ and Q are very different, then large logarithms of $R = \mu/Q$ appear in the coefficient functions and spoil the convergence of their perturbative expansions. In principle, the renormalization scale used for α_s in eq. (2.6) could be chosen differently from the factorization scale μ ; for simplicity, we have taken the two scales to be the same.

The parton fragmentation functions D_i^h cannot be calculated perturbatively. However, their dependence on the factorization scale μ can be deduced from that of the coefficient functions using the cancellation condition described above. Thus the fragmentation functions at a given scale μ_0 can be fitted to experimental data using eq. (2.6) and then evolved to a different scale using the perturbative *evolution equations* described in sect. 2.2. If we use input data at c.m. energy-squared Q_0 and set $\mu_0 = Q_0$, then evolve to $\mu = Q$, the resulting changes in the fragmentation functions, together with the running of α_s in the coefficient functions, gives rise to *scaling violation* in the x -dependence of the inclusive cross section. This effect is analogous to

the scaling violation observed in deep inelastic structure functions, and can be used to measure the QCD scale $\Lambda_{\overline{\text{MS}}}$, as we shall illustrate in sect. 4.2.

As a consequence of energy conservation, the inclusive cross section summed over all types of outgoing particles satisfies the sum rule

$$\frac{1}{2} \sum_h \int dx d\cos\theta x \frac{d^2\sigma^h}{dx d\cos\theta} = \sigma_{tot} = \sigma_T + \sigma_L \quad (2.7)$$

where

$$\sigma_P \equiv \frac{1}{2} \sum_h \int_0^1 dx x \frac{d\sigma_P^h}{dx} \quad (P = T, L, A) . \quad (2.8)$$

For the type of factorization scheme that we shall adopt, the fragmentation functions D_i^h satisfy the analogous sum rule

$$\sum_h \int_0^1 dz z D_i^h(z, \mu) = 1 , \quad (2.9)$$

which states that the total energy carried off by all fragments is equal to that of the original parton. From Eqs. (2.6), this implies that the integrated cross section contributions (2.8) are given in terms of the coefficient functions as

$$\sigma_P = \frac{1}{2} \sum_i \int_0^1 dz z C_{P,i}(z, \alpha_s(\mu), Q, \mu) \equiv \sum_i \sigma_{P,i} . \quad (2.10)$$

To order α_s^0 (the quark-parton model) the coefficient functions $C_{P,i}$ vanish for gluons and are given by the electroweak couplings for quarks and antiquarks. We have explicitly

$$\begin{aligned} C_{T,f} = C_{T,\bar{f}} &= [\delta(1-z) + \mathcal{O}(\alpha_s)] \sigma_{0,f}(s) \\ C_{A,f} = -C_{A,\bar{f}} &= [\delta(1-z) + \mathcal{O}(\alpha_s)] A_f(s) \end{aligned} \quad (2.11)$$

with

$$\begin{aligned} \sigma_{0,f}(s) &= \sigma_{0,f}^{(v)}(s) + \sigma_{0,f}^{(a)}(s) \\ \sigma_{0,f}^{(v)}(s) &= \frac{4\pi\alpha^2}{s} [e_f^2 + 2e_f v_e v_f \rho_1(s) + (v_e^2 + a_e^2) v_f^2 \rho_2(s)] \\ \sigma_{0,f}^{(a)}(s) &= \frac{4\pi\alpha^2}{s} (v_e^2 + a_e^2) a_f^2 \rho_2(s) \end{aligned}$$

$$A_f(s) = \frac{8\pi\alpha^2}{s} a_e a_f [e_f \rho_1(s) + 2v_e v_f \rho_2(s)] , \quad (2.12)$$

where e_i is the charge of i in units of the positron charge, $v_i = T_{3i} - 2e_i \sin^2 \theta_W$ and $a_i = T_{3i}$ are the vector and axial electroweak couplings, and[§]

$$\begin{aligned} \rho_1(s) &= \frac{1}{4 \sin^2 \theta_W \cos^2 \theta_W} \cdot \frac{s(m_Z^2 - s)}{(m_Z^2 - s)^2 + m_Z^2 \Gamma_Z^2} \\ \rho_2(s) &= \left(\frac{1}{4 \sin^2 \theta_W \cos^2 \theta_W} \right)^2 \frac{s^2}{(m_Z^2 - s)^2 + m_Z^2 \Gamma_Z^2} . \end{aligned} \quad (2.13)$$

For future reference, in connection with heavy quark mass corrections, we have separated the vector (v) and axial (a) contributions to the Born cross section in eq. (2.12). Neglecting quark masses, we thus have to order α_s^0 for n_F active flavours

$$\sigma_T = \sum_{f=1}^{n_F} \sigma_{0,f}(s) \equiv \sigma_0(s) , \quad (2.14)$$

and $\sigma_L = 0$.

The transverse and longitudinal coefficient functions for light quarks and gluons have been computed to order α_s in Refs. [5], and the asymmetry coefficient function was computed in the present work. They take the form

$$\begin{aligned} C_{T,f}(z, \alpha_s, Q, \mu) &= \left[\delta(1-z) + \frac{\alpha_s}{2\pi} C_F c_{T,q}(z, \mu/Q) \right] \sigma_{0,f}(s) \\ C_{T,g}(z, \alpha_s, Q, \mu) &= \frac{\alpha_s}{2\pi} C_F c_{T,g}(z, \mu/Q) \sigma_0(s) \\ C_{L,f}(z, \alpha_s, Q, \mu) &= \frac{\alpha_s}{2\pi} C_F c_{L,q}(z, \mu/Q) \sigma_{0,f}(s) \\ C_{L,g}(z, \alpha_s, Q, \mu) &= \frac{\alpha_s}{2\pi} C_F c_{L,g}(z, \mu/Q) \sigma_0(s) \\ C_{A,f}(z, \alpha_s, Q, \mu) &= \left[\delta(1-z) + \frac{\alpha_s}{2\pi} C_F c_{A,q}(z, \mu/Q) \right] A_f(s) \end{aligned} \quad (2.15)$$

with $C_{T,\bar{f}} = C_{T,f}$, $C_{L,\bar{f}} = C_{L,f}$, $C_{A,\bar{f}} = -C_{A,f}$ and $C_{A,g} = 0$. The colour factor

[§]We note that an improvement to eq. (2.13) can be made by replacing the Z^0 width Γ_Z by an energy-dependent width $s\Gamma_Z/m_Z^2$ (see, for example, ref. [4]). The resulting changes are small compared with the strong-interaction effects discussed in this paper.

$C_F = 4/3$. In the $\overline{\text{MS}}$ factorization scheme

$$\begin{aligned}
c_{T,q}(z, \mu/Q) &= (1+z^2) \left(\frac{\ln(1-z)}{1-z} \right)_+ - \frac{3}{2} \left(\frac{1}{1-z} \right)_+ + 2 \frac{1+z^2}{1-z} \ln z + \frac{3}{2} (1-z) \\
&\quad + \left(\frac{2}{3} \pi^2 - \frac{9}{2} \right) \delta(1-z) - 2 \ln(\mu/Q) \left(\frac{1+z^2}{1-z} \right)_+ \\
c_{T,g}(z, \mu/Q) &= 2 \frac{1+(1-z)^2}{z} [\ln(1-z) + 2 \ln z - 2 \ln(\mu/Q)] - 4 \frac{1-z}{z} \\
c_{L,q}(z, \mu/Q) &= 1 \\
c_{L,g}(z, \mu/Q) &= 4 \frac{1-z}{z} \\
c_{A,q}(z, \mu/Q) &= c_{T,q}(z, \mu/Q) - (1-z) .
\end{aligned} \tag{2.16}$$

The ‘plus-prescription’ used in the definition of $c_{T,q}$ is such that

$$\int_0^1 dz f(z) [g(z)]_+ = \int_0^1 dz [f(z) - f(1)] g(z) . \tag{2.17}$$

We also define for future reference

$$\begin{aligned}
c_q(z, \mu/Q) &= c_{T,q}(z, \mu/Q) + c_{L,q}(z, \mu/Q) \\
c_g(z, \mu/Q) &= c_{T,g}(z, \mu/Q) + c_{L,g}(z, \mu/Q) .
\end{aligned} \tag{2.18}$$

Performing the integrations in eq. (2.10), we find that to order α_S

$$\begin{aligned}
\sigma_{T,f} &= \frac{1}{2} \left\{ 1 + \frac{8}{3} \frac{\alpha_S}{2\pi} C_F [\ln(\mu/Q) + 2] \right\} \sigma_{0,f}(s) \\
\sigma_{T,g} &= -\frac{8}{3} \frac{\alpha_S}{2\pi} C_F [\ln(\mu/Q) + 2] \sigma_0(s) \\
\sigma_{L,f} &= \frac{1}{4} \frac{\alpha_S}{2\pi} C_F \sigma_{0,f}(s) \\
\sigma_{L,g} &= \frac{\alpha_S}{2\pi} C_F \sigma_0(s)
\end{aligned} \tag{2.19}$$

so that

$$\sigma_T = \sigma_0 , \quad \sigma_L = \frac{3}{2} \frac{\alpha_S}{2\pi} C_F \sigma_0 = \frac{\alpha_S}{\pi} \sigma_0 , \quad \sigma_{tot} = \left(1 + \frac{\alpha_S}{\pi} \right) \sigma_0 . \tag{2.20}$$

Thus to first order the entire QCD correction to σ_{tot} comes from the longitudinal part σ_L . It is not known whether this remains true in higher orders; the $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ corrections to σ_{tot} are known but not the separate contributions of σ_T and σ_L . If these separate corrections were calculated, as has been done for deep inelastic lepton scattering^[6], they would provide a new independent determination of α_s . However, the non-perturbative power corrections to σ_T and σ_L separately are expected to be larger than those for σ_{tot} , and so the determination would probably not be so precise. We discuss this in more detail in sect. 3, where we also compute the power corrections due to heavy quark masses.

Observe that in the $\overline{\text{MS}}$ factorization scheme the integrals of the quark coefficient functions have the scale-independent values

$$\int_0^1 c_{T,q}(z, \mu/Q) dz = \frac{1}{2}, \quad \int_0^1 c_{L,q}(z, \mu/Q) dz = 1, \quad \int_0^1 c_{A,q}(z, \mu/Q) dz = 0, \quad (2.21)$$

which determine the order- α_s corrections to flavour sum rules for quantities of the form

$$\Sigma_P^Q = \sum_h \int_0^1 dx Q_h \frac{d\sigma_P^h}{dx}, \quad (2.22)$$

where Q is a conserved additive quantum number. In particular the flavour asymmetry sum rule,

$$\Sigma_A^Q = \sum_{h,f} \int_0^1 dz C_{A,f}(z, \alpha_s(\mu), Q, \mu) \int_0^1 dx Q_h [D_f^h(x, \mu) - D_{\bar{f}}^h(x, \mu)], \quad (2.23)$$

has no correction of this order. The integral of the fragmentation functions appearing in eq. (2.23) is also scale-independent in the $\overline{\text{MS}}$ scheme. Thus, defining

$$\hat{Q}_f = \sum_h Q_h^{(f)} \int_0^1 dx D_f^h(x, \mu) \quad (2.24)$$

we find simply

$$\Sigma_A^Q = 2 \sum_f \hat{Q}_f A_f(s), \quad (2.25)$$

where the constants $\hat{Q}_f$, which would be equal to the corresponding quark quantum numbers Q_f in perturbation theory, are sensitive to the non-perturbative hadronization process, as we discuss in more detail in sect. 3.4.

2.2. Evolution of fragmentation functions

The fragmentation functions obey Altarelli-Parisi evolution equations^[7] which are very similar to the evolution equations for the parton densities in the hadrons. The equations have the form

$$\frac{dD_j^h(x, \mu)}{d \log \mu^2} = \sum_i \int_x^1 P_{ij}(x/z, \mu) D_i^h(z, \mu) \frac{dz}{z} \quad (2.26)$$

where

$$P_{ij}(x, \mu) = \frac{\alpha_s(\mu)}{2\pi} P_{ij}^{(0)}(x) + \left(\frac{\alpha_s(\mu)}{2\pi} \right)^2 P_{ij}^{(1)}(x) + \mathcal{O}(\alpha_s^3). \quad (2.27)$$

The indices i and j run over all active quarks, antiquarks, and the gluon. At leading order we have

$$\begin{aligned} \frac{\partial D_g^h(x, \mu)}{\partial \log \mu^2} &= \frac{\alpha_s(\mu)}{2\pi} \int \frac{dy}{y} \left[P_{gg}^{(0)}(y) D_g^h(x/y, \mu) + \right. \\ &\quad \left. P_{qg}^{(0)}(y) \sum_{f=1}^{n_F} \left(D_f^h(x/y, \mu) + D_{\bar{f}}^h(x/y, \mu) \right) \right] \\ \frac{\partial D_f^h(x, \mu)}{\partial \log \mu^2} &= \frac{\alpha_s(\mu)}{2\pi} \int \frac{dy}{y} \left[P_{qq}^{(0)}(y) D_f^h(x/y, \mu) + P_{gq}^{(0)}(y) D_g^h(x/y, \mu) \right] \end{aligned} \quad (2.28)$$

with a similar equation for the antiquark $\bar{f}$, and

$$\begin{aligned} P_{qq}^{(0)}(z) &= C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right] \\ P_{gg}^{(0)}(z) &= 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \left(\frac{11}{12} - \frac{n_F T_F}{3C_A} \right) \delta(1-z) \right] \\ P_{gq}^{(0)}(z) &= C_F \frac{1+(1-z)^2}{z} \\ P_{qg}^{(0)}(z) &= T_F \left(z^2 + (1-z)^2 \right) \end{aligned} \quad (2.29)$$

where for SU(3) the colour factors are

$$C_A = 3, \quad C_F = \frac{4}{3}, \quad T_F = \frac{1}{2}. \quad (2.30)$$

Observe that eq. (2.28) is the transpose of the usual Altarelli-Parisi equation for structure functions. This relation does not persist at the next-to-leading order.

Introducing the Mellin transform of the fragmentation function

$$\tilde{D}^h(N, \mu) = \int_0^1 D^h(x, \mu) x^N \frac{dx}{x} , \quad (2.31)$$

the Altarelli-Parisi equations in N space simplify:

$$\frac{d\tilde{D}_j^h(N, \mu)}{d \log \mu^2} = \sum_i \tilde{P}_{ij}(N, \mu) \tilde{D}_i^h(N, \mu) \quad (2.32)$$

where

$$\tilde{P}_{ij}(N, \mu) = \int_0^1 P_{ij}(x, \mu) x^N \frac{dx}{x} . \quad (2.33)$$

We now introduce the total fragmentation function, obtained by summing over all hadron species

$$D_i(x, \mu) = \sum_h D_i^h(x, \mu) . \quad (2.34)$$

It satisfies the energy sum rule (2.9)

$$\int D_i(x, \mu) x dx = \tilde{D}_i(2, \mu) = 1 . \quad (2.35)$$

This equation holds in the leading logarithmic approximation. Its validity at next-to-leading order is a scheme-dependent issue (it is valid, for example, in the $\overline{\text{MS}}$ scheme). Together with eq. (2.32) it implies the relations

$$\sum_i \tilde{P}_{ij}(2, \alpha_s) = 1. \quad (2.36)$$

In leading order eqs. (2.36) give the easily verifiable relations

$$\tilde{P}_{gg}^{(0)}(2) + 2n_F \tilde{P}_{qg}^{(0)}(2) = 0 , \quad \tilde{P}_{qq}^{(0)}(2) + \tilde{P}_{gq}^{(0)}(2) = 0. \quad (2.37)$$

In higher order other splitting processes arise: besides the gg , qg , gq , and qq splitting functions, one has to introduce also those for $q'q$ (a quark splitting into a quark of different flavour), $\bar{q}q$ (a quark splitting into its own antiquark), and $\bar{q}'q$ (a quark splitting into an antiquark of different flavour). This last subprocess is equal to the qq' subprocess at order α_s^2 , but differences may arise in higher order. The general form of the evolution equations, restricted only by charge conjugation invariance, is

given by

$$\begin{aligned}
\frac{\partial D_g^h(x, \mu)}{\partial \log \mu^2} &= \int \frac{dy}{y} \left\{ P_{gg}(y, \mu) D_g^h(x/y, \mu) + P_{qg}(y, \mu) \sum_{f=1}^{n_F} [D_f^h(x/y, \mu) + D_{\bar{f}}^h(x/y, \mu)] \right\} \\
\frac{\partial D_f^h(x, \mu)}{\partial \log \mu^2} &= \int \frac{dy}{y} \left\{ P_{qq}(y, \mu) D_f^h(x/y, \mu) + P_{\bar{q}q}(y, \mu) D_{\bar{f}}^h(x/y, \mu) + \right. \\
&\quad \sum_{f'=1}^{n_F} (1 - \delta_{ff'}) [P_{q'q}(y, \mu) D_{f'}^h(x/y, \mu) + P_{\bar{q}'q}(y, \mu) D_{\bar{f}'}^h(x/y, \mu)] \\
&\quad \left. + P_{gq}(y, \mu) D_g^h(x/y, \mu) \right\} \\
\frac{\partial D_{\bar{f}}^h(x, \mu)}{\partial \log \mu^2} &= \int \frac{dy}{y} \left\{ P_{q\bar{q}}(y, \mu) D_{\bar{f}}^h(x/y, \mu) + P_{\bar{q}\bar{q}}(y, \mu) D_{\bar{f}}^h(x/y, \mu) + \right. \\
&\quad \sum_{f'=1}^{n_F} (1 - \delta_{ff'}) [P_{\bar{q}'q}(y, \mu) D_{f'}^h(x/y, \mu) + P_{q'q}(y, \mu) D_{\bar{f}'}^h(x/y, \mu)] \\
&\quad \left. + P_{g\bar{q}}(y, \mu) D_g^h(x/y, \mu) \right\}. \tag{2.38}
\end{aligned}$$

It is convenient to introduce the components of the fragmentation function

$$\begin{aligned}
D_S^h(x, \mu) &= \frac{1}{2n_F} \sum_{f=1}^{n_F} (D_f^h(x, \mu) + D_{\bar{f}}^h(x, \mu)) \\
D_A^h(x, \mu) &= \frac{1}{2n_F} \sum_{f=1}^{n_F} (D_f^h(x, \mu) - D_{\bar{f}}^h(x, \mu)) \\
D_{f-}^h &= D_f^h(x, \mu) - D_{\bar{f}}^h(x, \mu) - 2D_A^h(x, \mu) \\
D_{f+}^h &= D_f^h(x, \mu) + D_{\bar{f}}^h(x, \mu) - 2D_S^h(x, \mu). \tag{2.39}
\end{aligned}$$

The evolution equations in terms of these quantities become

$$\begin{aligned}
\frac{\partial D_g^h(x, \mu)}{\partial \log \mu^2} &= \int \frac{dy}{y} [P_{gg}(y, \mu) D_g^h(x/y, \mu) + 2n_F P_{qg}(y, \mu) D_S^h(x/y, \mu)] \\
\frac{\partial D_S^h(x, \mu)}{\partial \log \mu^2} &= \int \frac{dy}{y} [P_S(y, \mu) D_S^h(x/y, \mu) + P_{gq}(y, \mu) D_g^h(x/y, \mu)] \\
\frac{\partial D_A^h(x, \mu)}{\partial \log \mu^2} &= \int \frac{dy}{y} P_A(y, \mu) D_A^h(x/y, \mu)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial D_{f-}^h(x, \mu)}{\partial \log \mu^2} &= \int \frac{dy}{y} P_-(y, \mu) D_{f-}^h(x/y, \mu) \\
\frac{\partial D_{f+}^h(x, \mu)}{\partial \log \mu^2} &= \int \frac{dy}{y} P_+(y, \mu) D_{f+}^h(x/y, \mu)
\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}
P_S(x, \mu) &= P_{qq}(x, \mu) + P_{\bar{q}q}(x, \mu) + (n_F - 1) [P_{q'q}(x, \mu) + P_{\bar{q}'q}(x, \mu)] \\
P_A(x, \mu) &= P_{qq}(x, \mu) - P_{\bar{q}q}(x, \mu) + (n_F - 1) [P_{q'q}(x, \mu) - P_{\bar{q}'q}(x, \mu)] \\
P_-(x, \mu) &= P_{qq}(x, \mu) - P_{\bar{q}q}(x, \mu) - [P_{q'q}(x, \mu) - P_{\bar{q}'q}(x, \mu)] \\
P_+(x, \mu) &= P_{qq}(x, \mu) + P_{\bar{q}q}(x, \mu) - [P_{q'q}(x, \mu) + P_{\bar{q}'q}(x, \mu)] .
\end{aligned} \tag{2.41}$$

The evolution mixes the so-called singlet component D_S^h with the gluon component D_g^h . All the other components (usually called non-singlet) evolve independently. At the order α_s^2 , $P_{q'q}$ and $P_{\bar{q}'q}$ are equal, and we have the simplified expressions

$$\begin{aligned}
P_S(x, \mu) &= P_{qq}(x, \mu) + P_{\bar{q}q}(x, \mu) + 2(n_F - 1)P_{q'q}(x, \mu) \\
P_A(x, \mu) &= P_-(x, \mu) = P_{qq}(x, \mu) - P_{\bar{q}q}(x, \mu) \\
P_+(x, \mu) &= P_{qq}(x, \mu) + P_{\bar{q}q}(x, \mu) - 2P_{q'q}(x, \mu).
\end{aligned} \tag{2.42}$$

At this order all the $-$ type components of the fragmentation function obey the same evolution equation. There is therefore no separate evolution equation for the D_{f-}^h and D_A^h fragmentation functions. Furthermore the relevant splitting function $P_A = P_-$ has the property

$$\int_0^1 dx P_A(x, \mu) = \tilde{P}_A(1, \mu) = 0 , \tag{2.43}$$

which ensures that flavour asymmetries of the form given in eq. (2.23) are scale independent, as stated in eq. (2.25).

Expressions for the splitting functions at next-to-leading order are given in ref. [5]. They can be used to solve the evolution equations numerically. In this work, we have solved the equations using two techniques.[¶] The first is to discretize the evolution equations in x -space, and solve them step by step. This method has the advantage that any initial conditions can be specified. However, it is slow and not very precise,

[¶]Computer programs for both methods of solution are available from the authors.

since many numerical integrations have to be performed. The second method uses instead the Mellin transform of the fragmentation functions (see eq. (2.31)), and solves the evolution equations analytically in N space for complex N . The fragmentation functions in x space are then obtained by performing the inverse Mellin transform numerically. The advantage of the second method is that one need only perform a single integration in order to get the value of a fragmentation function at a given x point. It has however the disadvantage that the initial condition for the evolution in N space must be given as an analytic function of N . We have found that using a function of the form

$$D_i(x, \mu_0) = (r_0 + x c_1 + x^2 c_2 + x^3 c_3 + x^4 c_4) x^{a-1} (1-x)^b \quad (2.44)$$

which has the Mellin transform

$$\tilde{D}_i(N, \mu_0) = \sum_{i=0,4} c_i \frac{\Gamma(a + N + i - 1) \Gamma(b + 1)}{\Gamma(a + b + N + i)} \quad (2.45)$$

we can fit fragmentation function data starting from x values as small as 0.05. In practice, the lower limit on x is not a problem. The evolution at a point x always involves the fragmentation function at larger x values. This is the case whether we evolve forwards or backwards in μ . Therefore, given the fragmentation functions at a given scale μ_0 and for values of x larger than a given x_0 , we can compute the fragmentation function at any other scale μ for $x > x_0$.

The solution of the evolution equation for non-singlet fragmentation functions has a simple form. Following ref. [8], defining the variable

$$t = \frac{2}{b_0} \log \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \quad (2.46)$$

we have

$$\tilde{D}_{f\pm}^h(N, \mu) = \tilde{E}_{\pm}(N, \mu, \mu_0) \tilde{D}_{f\pm}^h(N, \mu_0) \quad (2.47)$$

with

$$\begin{aligned} \tilde{E}_{\pm}(N, \mu, \mu_0) = \exp & \left[\tilde{P}_{qq}^{(0)}(N) t + \right. \\ & \left. \frac{2}{b_0} \frac{\alpha_s(\mu_0) - \alpha_s(\mu)}{2\pi} \left(-\frac{b_1}{2b_0} \tilde{P}_{qq}^{(0)}(N) + \tilde{P}_{\pm}^{(1)}(N) \right) \right] \end{aligned} \quad (2.48)$$

where

$$\begin{aligned} b_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F n_F \\ b_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_G T_F n_F - 4C_F T_F n_F. \end{aligned} \quad (2.49)$$

In eq. (2.48) terms of order $\alpha_s^2(\mu_0)$ and $\alpha_s^2(\mu)$ have been consistently neglected.

In the singlet case it is still possible to give a closed analytic expression for the solution of the evolution equation. We introduce the singlet evolution matrices

$$\tilde{\mathcal{P}}^{(0)}(N) = \begin{vmatrix} \tilde{P}_{gg}^{(0)}(N) & 2n_F \tilde{P}_{gq}^{(0)}(N) \\ \tilde{P}_{qg}^{(0)}(N) & \tilde{P}_{qq}^{(0)}(N) \end{vmatrix} \quad (2.50)$$

$$\tilde{\mathcal{P}}^{(1)}(N) = \begin{vmatrix} \tilde{P}_{gg}^{(1)}(N) & 2n_F \tilde{P}_{gq}^{(1)}(N) \\ \tilde{P}_{qg}^{(1)}(N) & \tilde{P}_S^{(1)}(N) \end{vmatrix}. \quad (2.51)$$

We then define

$$\lambda_{\pm}(N) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2n_F \tilde{P}_{qg}^{(0)}(N) \tilde{P}_{gq}^{(0)}(N)}{(\tilde{P}_{gg}^{(0)}(N) + \tilde{P}_{qq}^{(0)}(N))^2}} \quad (2.52)$$

$$\tilde{\mathcal{M}}_{\pm}(N) = \frac{\tilde{\mathcal{P}}^{(0)}(N) - \lambda_{\pm}(N)}{\lambda_{\pm}(N) - \lambda_{\mp}(N)}. \quad (2.53)$$

The matrices $\tilde{\mathcal{M}}$ satisfy the equations

$$\tilde{\mathcal{M}}_+(N) + \tilde{\mathcal{M}}_-(N) = 1, \quad \tilde{\mathcal{M}}_+(N)\tilde{\mathcal{M}}_-(N) = 0, \quad \tilde{\mathcal{M}}_{\pm}^2(N) = \tilde{\mathcal{M}}_{\pm}(N), \quad (2.54)$$

$$\lambda_+(N)\tilde{\mathcal{M}}_+(N) + \lambda_-(N)\tilde{\mathcal{M}}_-(N) = \tilde{\mathcal{P}}^{(0)}(N). \quad (2.55)$$

The solution of the singlet evolution equation is then given by

$$\begin{vmatrix} \tilde{D}_g^h(N, \mu) \\ \tilde{D}_S^h(N, \mu) \end{vmatrix} = \tilde{\mathcal{E}}(N, \mu, \mu_0) \begin{vmatrix} \tilde{D}_g^h(N, \mu_0) \\ \tilde{D}_S^h(N, \mu_0) \end{vmatrix} \quad (2.56)$$

with

$$\tilde{\mathcal{E}}(N, \mu, \mu_0) = \left[\tilde{\mathcal{M}}_+ e^{\lambda_+ t} + \tilde{\mathcal{M}}_- e^{\lambda_- t} \right]$$

$$\begin{aligned}
& + \frac{\alpha_s(\mu_0) - \alpha_s(\mu)}{\pi b_0} \left\{ \tilde{\mathcal{M}}_+ \tilde{\mathcal{P}}^{(1)} \tilde{\mathcal{M}}_+ e^{\lambda_+ t} + \tilde{\mathcal{M}}_- \tilde{\mathcal{P}}^{(1)} \tilde{\mathcal{M}}_- e^{\lambda_- t} \right\} \\
& - \frac{\alpha_s(\mu_0) - \alpha_s(\mu)}{\pi b_0} \left(\frac{b_1}{2b_0} \right) \left\{ \tilde{\mathcal{M}}_+ \lambda_+ e^{\lambda_+ t} + \tilde{\mathcal{M}}_- \lambda_- e^{\lambda_- t} \right\} \\
& + \tilde{\mathcal{M}}_+ \tilde{\mathcal{P}}^{(1)} \tilde{\mathcal{M}}_- \frac{\alpha_s(\mu) e^{\lambda_- t} - \alpha_s(\mu_0) e^{\lambda_+ t}}{2\pi \left(\lambda_- - \lambda_+ - \frac{b_0}{2} \right)} \\
& + \tilde{\mathcal{M}}_- \tilde{\mathcal{P}}^{(1)} \tilde{\mathcal{M}}_+ \frac{\alpha_s(\mu) e^{\lambda_+ t} - \alpha_s(\mu_0) e^{\lambda_- t}}{2\pi \left(\lambda_+ - \lambda_- - \frac{b_0}{2} \right)} \Big] \quad (2.57)
\end{aligned}$$

where the N dependence of $\lambda_{\pm}$, $\tilde{\mathcal{M}}_{\pm}$ and $\tilde{\mathcal{P}}^{(1)}$ has been omitted for ease of notation.

The splitting functions of ref. [5] are given in the $\overline{\text{MS}}$ scheme. In order to compute physical quantities they should be convoluted with the appropriate coefficient functions. For example, in order to compute the next-to-leading order transverse, longitudinal, or asymmetric part of the inclusive cross section one should use the coefficient functions given in eqs. (2.15) and (2.16). It is sometime convenient to use schemes which differ from the $\overline{\text{MS}}$ scheme. In analogy with the deep inelastic scattering case, it is particularly convenient here to introduce an *annihilation scheme*, in which the total fragmentation function in e^+e^- annihilation is given by the parton model formula. That is, defining

$$F^h(x) \equiv F_{tot}^h(x) = F_T^h(x) + F_L^h(x) \quad (2.58)$$

where

$$F_P^h(x) = \frac{1}{\sigma_{tot}} \frac{d\sigma_P^h}{dx}, \quad (2.59)$$

we require

$$F^h(x, Q) = \frac{1}{\sigma_{tot}} \sum_f \sigma_{0,f}(s) D_f^{h(AS)}(x, Q) \quad (2.60)$$

where the superscript (AS) stands for “annihilation scheme”. In moment space this implies that

$$\begin{aligned}
\tilde{D}_S^{h(AS)}(N, Q) &= \left(1 + \frac{\alpha_s}{2\pi} C_F \left[\tilde{c}_q(N, \mu/Q) - \frac{3}{2} \right] \right) \tilde{D}_S^h(N, \mu) \\
&\quad + \frac{\alpha_s}{2\pi} C_F \tilde{c}_g(N, \mu/Q) \tilde{D}_g^h(N, \mu), \\
\tilde{D}_{\pm}^{h(AS)}(N, Q) &= \left(1 + \frac{\alpha_s}{2\pi} C_F \left[\tilde{c}_q(N, \mu/Q) - \frac{3}{2} \right] \right) \tilde{D}_{\pm}^h(N, \mu). \quad (2.61)
\end{aligned}$$

The functions $\tilde{c}_g$ and $\tilde{c}_q$ are the Mellin transform of c_g and c_q , defined in eq. (2.18). In this scheme the gluon fragmentation function is unmodified. In general, the fragmentation functions in a new scheme obey modified evolution equations, identical in form to eqs. (2.48) and (2.57), but with modified next-to-leading kernels. Defining the matrix

$$\tilde{\mathcal{C}}(N, R) = \begin{vmatrix} 0 & 0 \\ \tilde{c}_g(N, R) & \tilde{c}_q(N, R) - \frac{3}{2} \end{vmatrix}, \quad (2.62)$$

where R stands for the ratio μ/Q , the change of scheme from $\overline{\text{MS}}$ to AS has the form

$$\begin{vmatrix} \tilde{D}_g^{h(A\mathcal{S})}(N, Q) \\ \tilde{D}_S^{h(A\mathcal{S})}(N, Q) \end{vmatrix} = \tilde{\mathcal{C}}(N, \mu/Q) \begin{vmatrix} \tilde{D}_g^h(N, \mu) \\ \tilde{D}_S^h(N, \mu) \end{vmatrix} \quad (2.63)$$

and the modified next-to-leading singlet kernel is

$$\tilde{\mathcal{P}}^{(1)(A\mathcal{S})}(N, R) = \tilde{\mathcal{P}}^{(1)}(N) + \tilde{\mathcal{C}}(N, R)\tilde{\mathcal{P}}^{(0)}(N) - \tilde{\mathcal{P}}^{(0)}(N)\tilde{\mathcal{C}}(N, R) - \frac{b_0}{2}\mathcal{C}(N, R). \quad (2.64)$$

The non-singlet kernels are instead modified as follows

$$\tilde{P}_{\pm}^{(1)(A\mathcal{S})}(N, R) = \tilde{P}_{\pm}^{(1)}(N) - \frac{b_0}{2}\tilde{c}_q(N, R), \quad (2.65)$$

and the evolution equations become

$$\begin{aligned} \begin{vmatrix} \tilde{D}_g^{h(A\mathcal{S})}(N, Q) \\ \tilde{D}_S^{h(A\mathcal{S})}(N, Q) \end{vmatrix} &= \tilde{\mathcal{E}}^{(A\mathcal{S})}(N, R, \mu, \mu_0) \begin{vmatrix} \tilde{D}_g^{h(A\mathcal{S})}(N, Q_0) \\ \tilde{D}_S^{h(A\mathcal{S})}(N, Q_0) \end{vmatrix} \\ \tilde{D}_{\pm}^{h(A\mathcal{S})}(N, Q) &= \tilde{P}_{\pm}^{(A\mathcal{S})}(N, R, \mu, \mu_0)\tilde{D}_{\pm}^{h(A\mathcal{S})}(N, Q_0) \end{aligned} \quad (2.66)$$

where $\mu/Q = \mu_0/Q_0 = R$. The matrix $\tilde{\mathcal{E}}^{(A\mathcal{S})}$ is given by the formula (2.57) with $\tilde{\mathcal{P}}^{(1)}$ replaced by $\tilde{\mathcal{P}}^{(1)(A\mathcal{S})}$. Observe that with this definition we still have some arbitrariness in fixing the ratio $R = \mu/Q$. Changing R affects the evolution equations only at even higher orders in the strong coupling. The most common choice is $R = 1$. We shall however study the effect of varying R in the range $\frac{1}{2}$ to 2, in order to assess the possible importance of unknown higher order corrections.

3. Power corrections

In order to test the predictions of QCD evolution as presented above, we need to be sure that other sources of scaling violation are negligible or at least under control.

Subasymptotic terms which decrease like inverse powers of Q belong to this category. In this section we consider various possible sources of such power corrections: mass effects in heavy quark production and decays, and the hadronization process.

Before turning to these more complex issues, we should note that kinematic power corrections arise from the common practice of using the momentum fraction $x_p = 2p/Q$ in place of the energy fraction $x = 2E/Q$. For a particle of mass m we have

$$x_p = x - \frac{2m^2}{xQ^2} + \mathcal{O}(1/Q^4) . \quad (3.1)$$

Thus such corrections are formally of order $1/Q^2$, albeit strongly enhanced at small x . At $x = 0.05$ and $Q = 22$ GeV, the lowest values considered in our analysis of scaling violation, the correction for pions is comparable with the experimental systematic errors, becoming rapidly smaller at higher values of x and/or Q .

3.1. Heavy quark effects

Heavy quarks are potential sources of large power corrections to the scaling behaviour of fragmentation functions. The mass of the b quark is about 5 GeV, and therefore naive estimate of power suppressed effects due to b production are not small. In most cases it is however possible to compute these effects and correct for them. In this section we will not attempt to actually compute the corrections, since our phenomenological analysis is only given for purpose of illustration. We will however show how these corrections may be computed.

We will focus, for ease of presentation, on b production, although all these considerations also apply to charm.

We can distinguish two sources of power suppressed effects due to heavy quarks. One is associated with production dynamics, in the sense that the evolution equations will receive dynamical corrections of the order m_b/Q . A second source is heavy quark decay, which gives rise to kinematic power corrections. We consider each of these sources in turn.

3.1.1. Heavy quark production

A systematic analysis of mass effects in the evolution of the fragmentation function has (to our knowledge) never been given. In the case of bottom production below LEP energies, it is however reasonable to rely on fixed order perturbation theory. We therefore computed the single inclusive cross section in the production of massive quarks up to the order α_s . Since at order α_s we may also produce a gluon, we need both the heavy quark and the gluon single inclusive cross section. The single inclusive heavy quark production cross section is given by

$$\begin{aligned}\frac{d\sigma_{T,L}^q}{dx} &= \sigma_{0,f}^{(v)}(s)F_{T,L}^{(v)}(x, \rho) + \sigma_{0,f}^{(a)}(s)F_{T,L}^{(a)}(x, \rho) \\ \frac{d\sigma_A^q}{dx} &= A_f(s)F_A(x, \rho)\end{aligned}\tag{3.2}$$

where $\rho = 4m^2/s$ and the functions $\sigma_{0,f}^{(v)}(s)$, $\sigma_{0,f}^{(a)}(s)$ and $A_f(s)$ are given in eqs. (2.12). In general we decompose

$$F_P^{(u)}(x, \rho) = \delta(1-x)B_P^{(u)}(\rho) + \frac{C_F\alpha_s}{2\pi}Q_P^{(u)}(x, \rho)\tag{3.3}$$

$$Q_P^{(u)}(x, \rho) = \delta(1-x)S_P^{(u)}(\rho) + \left[\frac{1}{1-x}\right]_+ R_P^{(u)}(x, \rho)\tag{3.4}$$

where P stands as usual for L (longitudinal), T (transverse) or A (asymmetric), and u stands for v (vector) or a (axial). The “plus” prescription is defined in eq. (2.17). For all quantities we will refer to the total cross section combination $L + T$ without any suffix. Thus for example

$$Q^{(u)}(x, \rho) = Q_T^{(u)}(x, \rho) + Q_L^{(u)}(x, \rho).\tag{3.5}$$

We have for the Born terms

$$B_T^{(v)}(\rho) = \beta, \quad B_L^{(v)}(\rho) = \frac{1}{2}\beta\rho, \quad B_T^{(a)}(\rho) = \beta^3, \quad B_L^{(a)}(\rho) = 0, \quad B_A(\rho) = \beta^2.\tag{3.6}$$

where $\beta = \sqrt{1-\rho}$. For the cross section for producing a gluon in a heavy quark initiated event, we have

$$\frac{d\sigma_P^g}{dx} = \frac{C_F\alpha_s}{2\pi} \left[\sigma_{0,f}^{(a)}(s)G_P^{(a)}(x, \rho) + \sigma_{0,f}^{(v)}(s)G_P^{(v)}(x, \rho) \right].\tag{3.7}$$

where now P stands for T or L (there is no asymmetric component in this case).

The analytical results for $S_P^{(u)}(\rho)$, $R_P^{(u)}(x, \rho)$ and $G_{T,L}^{(u)}(x, \rho)$ are given in Appendix A.

We begin by discussing the mass corrections to the energy sum rules for the longitudinal and total cross sections. From eq. (3.6) we immediately see by expanding in powers of ρ that power corrections start at the order ρ for all non-vanishing quantities, except for the total cross section combination, for which the term of order ρ cancels. In fig. 1 we show the corrections of order α_s to the second moment of the inclusive cross section, for the longitudinal and total cross section combinations, for both vector and axial couplings. We see that when $\mathcal{O}(\alpha_s)$ radiative corrections are

Figure 1: Corrections of order α_s to the second moments of the longitudinal and total inclusive cross sections, for both vector and axial couplings. The plot is in units of α_s/π (which corresponds to the quantity $C_F B_P^{(u)}$ in the text).

added the moments approach the massless values with corrections of order m^2 . The plot also shows that when $m > 0$ not all of the $\mathcal{O}(\alpha_s)$ contribution to the energy sum rule comes from the longitudinal term. As an example, for $m = 5 \text{ GeV}$ and $Q = 100 \text{ GeV}$ one can read off from the graph that the effect of the b mass correction to the longitudinal cross section is minus 10% in units of α_s/π , to be compared with the leading term α_s/π . The correction to the Born term is instead $\rho = 0.01$. Assuming that $\alpha_s/\pi = 0.04$, we would then conclude that the mass effects reduce the cross

section by an amount of 0.01 (Born) plus 0.004 ($\mathcal{O}(\alpha_s)$ term) out of a total of α_s/π , a 35% correction to the b quark contribution.

A similar discussion for higher moments requires a separation of the logarithmic and power mass effects in the fixed order results. In ref. [2] it was shown that there is a relation between the $\overline{\text{MS}}$ coefficient functions for massless quarks and the small mass limit of the production cross section for massive quarks, given by

$$Q_P^{(u)}(x, \rho) = c_{P,q}(x, \mu/Q) + b_P \left[\frac{1+x^2}{1-x} \left(\log \frac{\mu^2}{m^2(1-x)^2} - 1 \right) \right]_+ + \mathcal{O}(m^2/Q^2), \quad (3.8)$$

where $b_P = B_P(0) = \delta_{PT} + \delta_{PA}$. In the present work we also find a similar relation for gluons, given by

$$G_P^{(u)}(x, \rho) = c_{P,g}(x, \mu/Q) + 2b_P \frac{1+(1-x)^2}{x} \left(\log \frac{\mu^2}{m^2 x^2} - 1 \right) + \mathcal{O}(m^2/Q^2). \quad (3.9)$$

We therefore define

$$\begin{aligned} M_{P,q}^{(u)}(x, \rho) &= Q_P^{(u)}(x, \rho) - c_{P,q}(x, \mu/Q) - b_P \left[\frac{1+x^2}{1-x} \left(\log \frac{\mu^2}{m^2(1-x)^2} - 1 \right) \right]_+ \\ M_{P,g}^{(u)}(x, \rho) &= G_P^{(u)}(x, \rho) - c_{P,g}(x, \mu/Q) - 2b_P \frac{1+(1-x)^2}{x} \left(\log \frac{\mu^2}{m^2 x^2} - 1 \right) \end{aligned} \quad (3.10)$$

The above expression can be used to correct for quark mass effects in the fragmentation function. Notice that there are two contributions, one due to the heavy quark inclusive fragmentation function, and the other coming from the radiated gluon. The first effect gives rise to the following correction to the heavy quark inclusive fragmentation function

$$\delta F_P^{(u)}(x, \rho) = \delta(1-x)[B_P^{(u)}(\rho) - B_P^{(u)}(0)] + \frac{C_F \alpha_s}{2\pi} M_{P,q}^{(u)}(x, \rho) \quad (3.11)$$

which should be used as input to the decay formalism of the next subsection. The second contribution directly affects the total fragmentation function. It is given by

$$\delta D_P(x, \rho) = \frac{C_F \alpha_s}{2\pi} \int_x^1 M_{P,g}^{(u)}(x/z, \rho) D_g(z, \rho) \frac{dz}{z}. \quad (3.12)$$

We will not attempt to actually compute the mass corrections numerically in the present work. We have merely provided here the theoretical tools one needs in order

to perform such a calculation. We will simply present the effect of the corrections, as an illustration, for the hypothetical case in which the heavy quark does not decay, and the gluon fragments into a single hadron only. In this case, the Born term is simply obtained from eq. (3.11). For the $\mathcal{O}(\alpha_s)$ terms we define for $P = T, L$

$$M_P^{(u)}(x, \rho) = 2M_{P,q}^{(u)}(x, \rho) + M_{P,g}^{(u)}(x, \rho), \quad (3.13)$$

which is the quantity relevant to the inclusive cross section. We plot various moments of M as a function of $\sqrt{\rho} = 2m/Q$. In fig. 2 and fig. 3 we plot the power corrections to moments of the total and longitudinal fragmentation function for vector currents.

As can be seen, mass effects vanish quadratically in the mass. In the total cross

Figure 2: Quark mass power effects in the total fragmentation function.

section case, they increase with increasing moments. This is easily understood, since high moments correspond to the large- x region.

In fig. 4 we show the mass effects in the asymmetry. In this case also the first moment (which is infinite for the transverse and longitudinal cross sections) is plotted. One immediately notices that the first moment behaves linearly in the mass. This implies a correction to the forward-backward asymmetry, and to sum rules of the form (2.25), of order m/Q ^[9,10,11]. We see therefore that corrections linear in the heavy quark mass may actually arise in moments of the fragmentation functions.

Figure 3: Quark mass power effects in the longitudinal fragmentation function.

Figure 4: Quark mass power effects in the asymmetric fragmentation function.

3.1.2. Heavy quark decays

A heavy quark will decay into light decay products. The decay system will appear more and more boosted as the energy of the hadron containing the heavy quark gets higher. It is a simple exercise to obtain the scaling violation effects induced by the decay. Let us assume that the heavy hadron B is produced unpolarized, with a

distribution $F^B(x, \cos \theta_B, Q)$, where θ_B is the angle between the heavy hadron velocity and the electron beam axis. We have as usual

$$F^B(x, \cos \theta_B, Q) = \frac{3}{8}(1 + \cos^2 \theta_B) F_T^B(x, Q) + \frac{3}{4} \sin^2 \theta_B F_L^B(x, Q) + \frac{3}{4} \cos \theta_B F_A^B(x, Q) \quad (3.14)$$

and

$$F^B(x, Q) = F_T^B(x, Q) + F_L^B(x, Q). \quad (3.15)$$

Let us define $H^B(E^*)$ to be the energy distribution of the decay products in the heavy hadron rest frame. We assume isotropic decay, which is appropriate for B production, and also that we can neglect the masses of the decay products. Let us call θ^* the angle of a decay product with respect to the flight direction of the heavy hadron. The energy of a decay product and its decay angle in the laboratory are

$$E = \frac{1}{2}yQ = \gamma_x E^* (1 + \beta_x \cos \theta^*), \quad \cos \tilde{\theta} = \frac{\cos \theta^* + \beta_x}{1 + \beta_x \cos \theta^*} \quad (3.16)$$

where $\gamma_x = 1/\sqrt{1 - \beta_x^2} = E_x/m = xQ/2m$, E_x being the energy of the heavy hadron and m its mass. Note that in this section we use $x = 2E_x/Q$ to represent the energy fraction of the heavy hadron and $y = 2E/Q$ for that of the observed decay product. The angle of the decay product with respect to the beam axis is given by

$$\cos \theta = \sin \tilde{\theta} \sin \theta_B \cos \phi + \cos \tilde{\theta} \cos \theta_B, \quad (3.17)$$

where ϕ is the azimuth of the decay product around the direction of the heavy hadron.

Consider first the total fragmentation function, which is the inclusive cross section without any θ -dependent weight. We have

$$\begin{aligned} F(y, Q) &= \int F^B(x, \cos \theta_B, Q) dx \int H^B(E^*) \\ &\quad \delta(y - 2\gamma_x E^* [1 + \beta_x \cos \theta^*]/Q) dE^* d\cos \theta^* d\cos \theta_B \\ &= \frac{Q}{2} \int \frac{F^B(x, Q)}{\gamma_x \beta_x} dx \int \Theta(E\alpha_x - E^*) \Theta(E^* - E/\alpha_x) H^B(E^*) \frac{dE^*}{E^*} \end{aligned} \quad (3.18)$$

where

$$\alpha_x = \sqrt{\frac{1 + \beta_x}{1 - \beta_x}} = \frac{2E_x}{m} \frac{1 + \beta_x}{2}. \quad (3.19)$$

Taking moments, we find

$$\begin{aligned}
\tilde{F}(N, Q) &= \int F(y, Q) y^{N-1} dy = \frac{Q}{2} \int \frac{F^B(x, Q)}{\gamma_x \beta_x} dx \\
&\cdot \int \left[\int y^{N-1} \theta(E \alpha_x - E^*) \theta(E^* - E / \alpha_x) dE \right] H^B(E^*) \frac{dE^*}{E^*} \\
&= \frac{Q}{2} \int \frac{F^B(x, Q)}{\gamma_x \beta_x} dx \int \left(\frac{2E^*}{Q} \right)^N \frac{1}{N} \left(\alpha_x^N - \frac{1}{\alpha_x^N} \right) H^B(E^*) \frac{dE^*}{E^*} \\
&= \frac{1}{N} \left(\frac{m}{Q} \right)^{N-1} \left[\int \frac{F^B(x, Q)}{\gamma_x \beta_x} \left(\alpha_x^N - \frac{1}{\alpha_x^N} \right) dx \right] \\
&\cdot \left[\int \left(\frac{2E^*}{m} \right)^{N-1} H^B(E^*) dE^* \right]. \tag{3.20}
\end{aligned}$$

Using eq. (3.19) we obtain

$$\tilde{F}(N, Q) = \tilde{H}^B(N) \int F^B(x, Q) x^{N-1} \frac{(1 + \beta_x)^N - (1 - \beta_x)^N}{N 2^N \beta_x} dx \tag{3.21}$$

where we have defined the decay moment function

$$\tilde{H}^B(N) = \int \left(\frac{2E^*}{m} \right)^{N-1} H^B(E^*) dE^*. \tag{3.22}$$

The longitudinal, transverse and asymmetric fragmentation functions $F_P(y, Q)$ are easily treated by introducing in the integrand of eq. (3.18) the appropriate weight factors $W_P(\cos \theta)$ given in eq. (2.3). The moments of the longitudinal fragmentation function are given by

$$\begin{aligned}
\tilde{F}_L(N, Q) &= 2 \tilde{H}^B(N) \int dx x^{N-1} \left\{ \frac{(1 + \beta_x)^N - (1 - \beta_x)^N}{2^N \beta_x} \frac{F_T^B(x, Q)}{2N} \right. \\
&+ \left(F_L^B(x, Q) - \frac{1}{2} F_T^B(x, Q) \right) \left[\frac{(1 + \beta_x)^N (1 - N \beta_x) - (1 - \beta_x)^N (1 + N \beta_x)}{N (N - 1) (N - 2) 2^{N-1} \beta_x^3} \right. \\
&+ \left. \left. \frac{(1 + \beta_x)^N - (1 - \beta_x)^N}{(N - 2) 2^N \beta_x} \right] \right\}, \tag{3.23}
\end{aligned}$$

while those of the asymmetric part are

$$\tilde{F}_A(N, Q) = -2 \tilde{H}^B(N) \int dx x^{N-1} F_A^B(x, Q)$$

$$\left[\frac{(1 + \beta_x)^N (1 - N\beta_x) - (1 - \beta_x)^N (1 + N\beta_x)}{N(N-1)2^N \beta_x^2} \right]. \quad (3.24)$$

Let us now comment on the results obtained so far, beginning with eq. (3.21). The first few moments are

$$\begin{aligned} \tilde{F}(1, Q) &= \tilde{H}^B(1) \int F^B(x, Q) dx \\ \tilde{F}(2, Q) &= \frac{1}{2} \tilde{H}^B(2) \int F^B(x, Q) dx \\ \tilde{F}(3, Q) &= \frac{1}{3} \tilde{H}^B(3) \int F^B(x, Q) \left[x^2 - \frac{m^2}{Q^2} \right] dx, \end{aligned} \quad (3.25)$$

and in general, for $N > 1$

$$\tilde{F}(N, Q) = \frac{1}{N} \tilde{H}^B(N) \left[\tilde{F}^B(N, Q) - \tilde{F}^B(N-2, Q)(N-2) \frac{m^2}{Q^2} + \mathcal{O}(m^4/Q^4) \right]. \quad (3.26)$$

We see that there are no power corrections to the multiplicity and energy moments, and that power corrections to higher moments are always suppressed by at least two powers of m/Q . Furthermore, it is always possible to express power corrections in terms of integer moments (with $N \geq 1$) of the heavy hadron fragmentation function F^B . It is therefore possible to correct for these effects, by using a convenient parametrization of this fragmentation function together with data for the inclusive energy spectrum in B decays. At LEP energies m^2/Q^2 is 0.2%, to be multiplied by the b fraction, but effects can be substantially higher at lower energies, so that, when appropriate, they should be corrected for.

Next we turn to the corrections to F_L . As for the total cross section, we find from eq. (3.23) that for $N > 2$ the power corrections to integer moments can be expressed in terms of previous integer moments, with $N \geq 1$. For the first and second moments, however, things are different, because of the presence of the vanishing denominators $N-1$ and $N-2$. Taking the limit for $N \rightarrow 2$, for example, we find

$$\begin{aligned} \frac{\sigma_L}{\sigma_{tot}} = \tilde{F}_L(2, Q) &= \tilde{H}^B(2) \int dx x \left[F_L^B(x, Q) + \left(F_L^B(x, Q) - \frac{1}{2} F_T^B(x, Q) \right) \right. \\ &\cdot \left. \left(-\frac{2^2 m^2}{x^2 Q^2} + \frac{1}{2\beta_x^3} \left\{ \log \frac{1 + \beta_x}{1 - \beta_x} - 2\beta_x \right\} \frac{2^4 m^4}{x^4 Q^4} \right) \right]. \end{aligned} \quad (3.27)$$

We see that the power corrections are no longer given in terms of integer moments with $N \geq 1$. Let us assume that $F^B(x, Q)$ has a $1/x$ behaviour for small x . In fact x cannot reach zero; we must have $x > 2m/Q$. The x integration is then linearly divergent, and gives a contribution of order Q/m . The power corrections are therefore formally of the order m/Q .

In practice, even at LEP energies, the behaviour of $F^B(x, Q)$ does not yet show the small- x growth due to gluon splitting into heavy quark pairs. On the contrary, the fragmentation function is small, of order α_s , at small x , at least in the case of bottom quarks. The x integration in eq. (3.27) is then only logarithmically divergent, and we expect corrections of the order of

$$\alpha_s \frac{m^2}{Q^2} \log \frac{m}{Q}, \quad (3.28)$$

to be compared with the perturbative value $\sigma_L = \alpha_s/\pi$, and the leading dynamical correction $2m^2/Q^2$. We therefore expect that in practice heavy quark decay effects will not spoil the possibility of determining α_s from the longitudinal fragmentation function.

For the asymmetry, we find from eq. (3.24) that mass corrections are of order m^2/Q^2 for $N > 1$, while for $N = 1$ we obtain

$$\tilde{F}_A(2, Q) = 2 \tilde{H}^B(1) \int dx F_A^B(x, Q) \frac{1}{\beta_x^2} \left(\beta_x - \frac{2m^2}{x^2 Q^2} \log \frac{1 + \beta_x}{1 - \beta_x} \right). \quad (3.29)$$

Thus if the heavy quark asymmetry F_A^B is of order α_s at small x , the kinematical correction due to heavy quark decay is of order $\alpha_s m/Q$, i.e. of the same order as the dynamical correction discussed in subsect. 3.1.1.

Since the asymmetry sum rules (2.25) receive no corrections of order α_s in the massless limit, heavy quark production and decay provide the main source of corrections to them.

3.2. Hadronization – simple models

The power corrections that arise from the non-perturbative process in which partons are converted to hadrons are much less well understood than the mass effects

discussed above. Nowadays they are traditionally estimated using Monte Carlo programs, which combine an approximate perturbative treatment to all orders (“parton showering”) with rather complex models of hadron formation from the resulting multiparton states. We present some results from this type of approach in sect. 3.3. However, it is valuable to consider first simple models, which are certainly incorrect in detail but permit some physical insight, and may even in some cases provide plausible estimates.

The simplest type of hadronization model is one in which a parton (more correctly, a colour-connected pair of partons) produces a set of light hadrons which occupy a tube in (y, p_t) -space, where y is rapidity and p_t is transverse momentum. If the hadron density in this space is $\rho(p_t)$, the energy and momentum of a tube of length Y are, neglecting hadron masses,

$$\begin{aligned} E &= \int_0^Y dy \, d^2 p_t \rho(p_t) p_t \cosh y = \lambda \sinh Y \\ P &= \int_0^Y dy \, d^2 p_t \rho(p_t) p_t \sinh y = \lambda (\cosh Y - 1) \sim E - \lambda, \end{aligned} \quad (3.30)$$

where $\lambda = \int d^2 p_t \rho(p_t) p_t$ sets the hadronization scale. Notice that the “jet momentum” P receives a negative power correction of relative order $\lambda/E = 2\lambda/Q$ for a two-jet configuration. Thus one generally expects hadronization power corrections to scale like $1/Q$. Assuming 3 particles per unit rapidity, with mean transverse momentum of about 300 MeV/c, we obtain $\lambda \sim 1$ GeV, which is consistent with the order of magnitude obtained from studies of event shapes.

In the case of the single-particle inclusive cross section (2.1), the simplest form of power correction that one could imagine would be a rescaling of x by a factor $a(Q) = 1 + \mathcal{O}(1/Q)$. Taking into account the energy sum rule (2.7), the simplest hypothesis is in fact that

$$F(x, Q) = \sum_h \frac{1}{\sigma_{tot}} \frac{d\sigma^h}{dx} = [a(Q)]^2 F_{\text{pert}}[a(Q)x, Q] \quad (3.31)$$

where $F_{\text{pert}}(x, Q)$ is the prediction in the absence of power corrections, i.e. with perturbative logarithmic evolution in Q . Then the average hadron multiplicity would be

$$\langle n(Q) \rangle = \int dx \, F(x, Q) = a(Q) \langle n(Q) \rangle_{\text{pert}}. \quad (3.32)$$

As an explicit model we may try

$$a(Q) = (1 + B/Q)^{-1} , \quad (3.33)$$

corresponding to replacing Q by $Q + B$ in eq. (1.2), so that the particle multiplicity is strongly suppressed for $Q \lesssim B$, relative to a perturbative extrapolation from higher energies. Since the total energy is shared amongst fewer particles, the effect of this is to harden the spectrum, relative to the asymptotic prediction. We shall see that a simple model of this type, combined with a next-to-leading treatment of perturbative scaling violation, can give a good description of Monte Carlo and experimental data.

If we suppose that the hadronic x distribution is obtained simply by convoluting the distribution of parton energies E with the ‘tube’ hadronization model discussed above, then a uniform rapidity distribution along the tube would give only $1/Q^2$ corrections. However, any non-uniformity that extends up to a fixed rapidity would produce a $1/Q$ correction similar to that in eq. (3.33).

Although the prediction for the e^+e^- total cross section is supposed to be free from $\mathcal{O}(1/Q)$ (and $\mathcal{O}(1/Q^2)$) power corrections, this is not expected to be the case for the integrated transverse and longitudinal contributions separately, since they can be mixed by hadronization. For illustration, let us consider again the ‘tube’ model. If we suppose that the tube axis, making an angle θ_0 with the electron beam, has the perfectly transverse $(1 + \cos^2 \theta_0)$ distribution, then we have from eq. (2.2)

$$\begin{aligned} \frac{d\sigma_L^h}{dx} &= \int d\cos\theta \, d\cos\theta_0 \, d\cos\theta^* \frac{d\phi^*}{2\pi} \frac{3}{8} (2 - 5\cos^2\theta) (1 + \cos^2\theta_0) \\ &\quad \cdot \delta(\cos\theta - \cos\theta_0 \cos\theta^* + \sin\theta_0 \sin\theta^* \cos\phi^*) \frac{d^2\sigma^h}{dx \, d\cos\theta^*} \\ &= \frac{1}{2} \int d\cos\theta^* \sin^2\theta^* \frac{d^2\sigma^h}{dx \, d\cos\theta^*} \end{aligned} \quad (3.34)$$

where (θ^*, ϕ^*) specifies the direction of motion of the produced hadron with respect to the tube axis. For the uniform rapidity distribution that led to Eqs. (3.30), writing $\sin\theta^* = 1/\cosh y$ we find

$$\frac{\sigma_L}{\sigma_{tot}} = \frac{1}{Q} \int_0^Y dy \, d^2p_t \rho(p_t) \frac{p_t}{\cosh y} \sim \pi \frac{\lambda}{Q} . \quad (3.35)$$

Since σ_L/σ_{tot} is itself $\mathcal{O}(\alpha_s)$, this would represent a very large relative correction

($\pi^2\lambda/\alpha_s Q \sim 100\%$ at $Q = m_Z$). As we discuss in sect. 3.3, from Monte Carlo studies using a more sophisticated hadronization model we do find a $\mathcal{O}(1/Q)$ correction, but it is not nearly as large as that suggested by this simple model.

Consider finally the effect of hadronization on flavour asymmetry sum rules of the form (2.25). Intuitive reasoning would lead us to conclude that it is unlikely that $\hat{Q}_f$ is process independent. Therefore standard factorization (which implies process independence of the fragmentation functions) should fail for the first moment of the asymmetry. Consider a fictitious production process, similar to e^+e^- annihilation, in which the initial vector coupling is replaced by a pointlike coupling of three quark fields of the same flavour f in a colour singlet state. Let us denote by $Q_h^{(f)}$ the total number of quarks minus the total number of antiquarks of type f in hadron h , and take $Q_i^{(f)}$ to be 1 if parton i is a quark of flavour f , -1 if it is an antiquark of flavour f , and zero if it is any other parton. From flavour conservation we must obviously have

$$\sum_h \int_0^1 dx Q_h^{(f)} \frac{1}{\sigma^{(f)}} \frac{d\sigma_h^{(f)}}{dx} = 3. \quad (3.36)$$

From factorization, on the other hand, we have

$$\sum_h \int_0^1 dx Q_h^{(f)} \frac{1}{\sigma^{(f)}} \frac{d\sigma_h^{(f)}}{dx} = \sum_i \int_0^1 dx \frac{1}{\sigma^{(f)}} \frac{d\hat{\sigma}_i^{(f)}}{dx} \hat{Q}_i^{(f)} \quad (3.37)$$

where

$$\hat{Q}_i^{(f)} = \int_0^1 dx \sum_h Q_h^{(f)} D_i^h(x, \mu) \quad (3.38)$$

and $d\hat{\sigma}_i^{(f)}/dx$ is the short distance cross section (or coefficient function) for the inclusive production of parton i from our pointlike interaction of three quarks of flavour f . Perturbation theory also guarantees that

$$\sum_i \int_0^1 dx \frac{1}{\sigma^{(f)}} \frac{d\hat{\sigma}_i^{(f)}}{dx} Q_i^{(f)} = 3. \quad (3.39)$$

From equations (3.36), (3.37) and (3.39) we get immediately

$$\sum_i C_i^{(f)} \hat{Q}_i^{(f)} = \sum_i C_i^{(f)} Q_i^{(f)} \quad (3.40)$$

where

$$C_i^{(f)} = \int_0^1 dx \frac{1}{\sigma^{(f)}} \frac{d\hat{\sigma}_i^{(f)}}{dx}. \quad (3.41)$$

Charge conjugation invariance tells us that $\hat{Q}_i^{(f)} = Q_i^{(f)} = 0$ if $i = g$. Therefore eq. (3.40) states the identity of the projections of the two $2n_f$ component vectors $Q_i^{(f)}$ and $\hat{Q}_i^{(f)}$ onto the vectors $C_i^{(f)}$. Since f is arbitrary, there are $2n_f$ such vectors. Furthermore the $C_i^{(f)}$ vectors must be linearly independent. In fact we must have

$$C_i^{(f)} = 3\delta_{fi} + \mathcal{O}(\alpha_s). \quad (3.42)$$

It follows then that

$$\hat{Q}_i^{(f)} = Q_i^{(f)} \quad (3.43)$$

for all i and f . In other words, the average flavour content of the final hadron should equal the flavour content of the parton from which the hadron has fragmented.

Although not impossible, the above conclusion goes against common sense. It is very difficult to construct a hadronization model, compatible with confinement, which supports this conclusion. Let us now, for definiteness, discuss the sum rule for the electric charge. Factorization would lead us to conclude that $\hat{Q}_f = e_f$. This prediction corresponds to assigning a mean charge of e_f to a quark jet of flavour f . In order to hadronize, however, the jet must acquire a net additional antiquark (or diquark) and its mean charge is shifted. The sum rule therefore receives a finite hadronization correction which may not vanish asymptotically.

As a simple model, suppose that the additional antiquark is of flavour $\bar{u}$, $\bar{d}$ or $\bar{s}$, with probabilities $\frac{1}{2}(1 - \epsilon)$, $\frac{1}{2}(1 - \epsilon)$ and ϵ , respectively. Then the mean jet charge is

$$\hat{Q}_f = e_f - \frac{1}{2}(1 - \epsilon)(e_u + e_d) - \epsilon e_s = e_f - \frac{1}{6} + \frac{1}{2}\epsilon. \quad (3.44)$$

Thus in this model the perturbative prediction $\hat{Q}_f = e_f$ should be multiplied by a factor of $3(1 + \epsilon)/4$ for up-type quarks and $3(1 - \epsilon)/2$ for down-type quarks.

3.3. Hadronization – Monte Carlo

To investigate the possible size of hadronization effects more quantitatively, we generated large samples of HERWIG^[12] Monte Carlo events at a range of c.m. energies ($Q = 22, 35, 57, 91, 150$ and 240 GeV) and looked for power corrections of the types suggested by the simple models discussed above. We generated separate samples of

50 000 events of each initial flavour $f = u, d, s, c, b$ at each energy, as well as a similar sample of the hypothetical process $e^+e^- \rightarrow gg$ at 91 GeV, in order to have the HERWIG prediction for the gluon fragmentation function at that energy.

3.3.1. Scaling violation

We first studied the moments

$$\tilde{F}(N, Q) = \int_0^1 F(x, Q) x^N \frac{dx}{x} \quad (3.45)$$

where $F(x, Q)$ is the total fragmentation function, as defined in eq. (3.31). The Monte Carlo moments for $N = 2, \dots, 11$ are shown by the points in fig. 5. They were compared with the moments $\tilde{F}_p(N, Q)$ obtained by next-to-leading perturbative evolution upwards and downwards from $Q = 91$ GeV (dashed curves), using the same value of $\Lambda_{\overline{\text{MS}}}$ as in the Monte Carlo simulation (180 MeV) and the same input distributions at $Q = 91$ GeV, including the gluon fragmentation function, generated as explained above. The simulation does not use the full next-to-leading formalism, but the QCD scale used can be shown^[13] to correspond to $\Lambda_{\overline{\text{MS}}}$ in the high- x region, which is emphasised in the higher moments.

The Monte Carlo evolution agrees quite well with the perturbative expectations above $Q = 50$ GeV, but shows a significantly faster variation, especially in the higher moments, at lower energies. The form of the variation is consistent with a $1/Q$ power correction. We fitted the Monte Carlo moments to the expression

$$\tilde{F}(N, Q) = \tilde{F}_p(N, Q) \left(\frac{1 + B/Q}{1 + B/m_Z} \right)^{N-2}, \quad (3.46)$$

which would be expected on the basis of the simple rescaling hypothesis (3.31), and found good agreement for $B = 0.5$ GeV, as shown by the solid curves. Inclusion of additional higher power corrections would improve the agreement, but the effect was too small for such terms to be reliably estimated.

We conclude that the dominant Monte Carlo hadronization effects in the total inclusive cross section are of the $1/Q$ form, and that the rescaling model provides a simple parametrization of them.

Figure 5: Moments of inclusive x distribution. Points: HERWIG Monte Carlo. Solid curves: best fit of form (3.46). Dashed curves: perturbative evolution only.

3.3.2. Longitudinal cross section

We also studied the possible magnitude of hadronization corrections to the integrated longitudinal cross section, defined by eq. (2.8) summed over all final-state particles h . The results are shown in fig. 6, separated according to the quark flavour at the primary vertex, in order to illustrate quark mass effects.

It should be noted that the HERWIG Monte Carlo program generates the primary quarks according to the massless Born approximation, and therefore the mass corrections to the matrix elements discussed in sect. 3.1.1 are not fully taken into account. Mass effects due to hadronization and heavy quark decays are included. From the models of sect. 3.1 and 3.2 we expect the approximate behaviour

$$\sigma_L/\sigma_{tot} = \alpha_s/\pi + a \alpha_s^2 + b/Q + c (m_q/Q)^2. \quad (3.47)$$

A parametrization of this form, with $a = 1.2$, $b = 0.21$ GeV and $c = 1.2$, shown by the curves, gives a good fit to the Monte Carlo results ($\chi^2/\text{d.o.f.} = 28/27$). We should emphasize that the perturbative term of order α_s^2 generated by the program does not

necessarily have the correct coefficient, because the program does not take account of all sources of higher-order corrections.

Figure 6: Longitudinal cross section. Points: HERWIG Monte Carlo. Curves: best fit of form (3.47).

The overall non-perturbative hadronization contribution to σ_L from the last two terms in eq. (3.47) is about 6% at $Q = 91$ GeV. This is comparable to the statistical error for the number of events generated. The $\mathcal{O}(1/Q)$ term in particular is much smaller than that suggested by the simple ‘tube’ hadronization model of the previous section. This point should be investigated in other models. If the Monte Carlo estimate is reliable, then hadronization corrections are unlikely to be the main source of uncertainty in comparing measurements of σ_L with perturbative predictions at LEP energies.

We note that a 6% uncertainty in σ_L would correspond to an uncertainty of the order of 20% in the $\mathcal{O}(\alpha_s^2)$ coefficient a . The calculation of this coefficient would therefore be worthwhile.

3.3.3. Charge asymmetry sum rule

We also used HERWIG Monte Carlo data to study hadronization effects in the asymmetry sum rule (2.25) for the electric charge, again separating the different quark flavour contributions. The results, normalized to the perturbative prediction $\hat{Q}_f = e_f$, are shown in fig. 7. We see that the values are shifted from the perturbative prediction, by amounts that are approximately energy-independent, being about 0.85 and 1.3 for up- and down-type quarks, respectively. Thus the effect of the cluster hadronization model used in HERWIG is similar to that of the simple model of single quark transfer considered in sect. 3.2, with a strange quark contribution $\epsilon \sim 0.13$ in eq. (3.44).

We conclude from this that studies of flavour asymmetries, preferably with tagged quark flavours, could yield useful information on the hadronization process. It can be seen from the error bars in fig. 7 (corresponding to 5×10^4 events of each flavour at each energy) that such studies are unfortunately more difficult on the Z^0 resonance, where asymmetries are small.

Figure 7: Charge asymmetry sum rule. Points: HERWIG Monte Carlo. Curves: predictions (3.44) for up-type (dashed) and down-type (dot-dashed) quarks.

4. Phenomenology

4.1. Gluon fragmentation function and $F_L(x)$

A complication in the analysis of scaling violation in e^+e^- annihilation is that we need to know the fragmentation function of the gluon into particles of the detected type h . In leading order, sensitivity to gluon fragmentation comes about through the mixing between gluon and singlet quark contributions in the evolution equations. In next-to-leading order there is also, in most factorization schemes, a direct gluon contribution via the coefficient functions $C_{P,g}$. The direct contribution corresponds to the fact that in next-to-leading order the initial parton configuration can be $q\bar{q}g$ instead of simply $q\bar{q}$.

There are two model-independent ways to extract the gluon fragmentation function directly from e^+e^- annihilation data. One way is to study three-jet final states in which the gluon jet is identified, e.g. by secondary vertex tagging of the other jets [16]. This method is probably best at larger values of x , where the association of final-state particles with jets is fairly unambiguous. We adopted this method for the analysis of scaling violation presented in sect. 4.2.

A complementary method, which has not yet been applied but which we would like to advocate here, would be to use the longitudinal component of the cross section, discussed in sect. 2.1. From eqs. (2.6) and (2.15) we may write the longitudinal fragmentation function as

$$F_L^h(x) = \frac{\alpha_s}{2\pi} C_F \int_x^1 \frac{dz}{z} \left[F_T^h(z) + 4 \left(\frac{z}{x} - 1 \right) D_g^h(z) \right] + \mathcal{O}(\alpha_s^2). \quad (4.1)$$

Note that one could equally well write the total fragmentation function F_{tot}^h in place of the transverse contribution F_T^h on the right-hand side, to the given accuracy. To solve for D_g^h , various approaches could be used, depending on the precision of the data. The simplest might be to parametrize $D_g^h(z)$ and fit to eq. (4.1) using the data on F_L^h and F_T^h . For sufficiently precise data, one could use the formal solution obtained by differentiation with respect to x :

$$D_g^h(x) = \frac{1}{4} \frac{d}{dx} (x F_T^h) + \frac{\pi}{2 C_F \alpha_s} \frac{d}{dx} \left(x^2 \frac{d F_L^h}{dx} \right). \quad (4.2)$$

Alternatively, one could take moments as in eq. (2.31): defining

$$\tilde{F}_P^h(N) = \int_0^1 dx x^{N-1} F_P^h(x) , \quad (4.3)$$

the inverse transform is

$$F_P^h(x) = \frac{1}{2\pi i} \int_C dN x^{-N} \tilde{F}_P^h(N) \quad (4.4)$$

where the complex contour C is to the right of all singularities of $\tilde{F}_P^h(N)$. Integrating by parts, eq. (4.2) gives

$$D_g^h(x) = \frac{1}{2\pi i} \int_C dN x^{-N} (N-1) \left[\frac{\pi N}{2C_F \alpha_S} \tilde{F}_L^h(N) - \frac{1}{4} \tilde{F}_T^h(N) \right] . \quad (4.5)$$

To see what results might be obtainable from an analysis of the longitudinal inclusive cross section at $Q = m_Z$, we have followed the opposite procedure of predicting this quantity from eq. (4.1). We used the ALEPH^[15] data on $F_{tot}(x)$ and the OPAL^[16] data on $D_g(x)$ from tagged three-jet events, both summed over all charged particles h . The published OPAL data are uncorrected for detector effects and are not absolutely normalized. We applied detector corrections estimated from Monte Carlo studies^[17], and normalized by assuming a charged energy fraction of 0.6 for gluon jets. Our purpose was not to make a precise prediction, but rather to provide a semi-quantitative idea of what $F_L(x)$ should look like. The results are shown in fig. 8.^{||}

We see that the longitudinal component is expected to be much smaller, and more rapidly decreasing with x , than the overall x distribution $F_{tot}(x)$. Furthermore it is dominated by the quark contribution at large x . The gluon contribution becomes dominant below $x \sim 0.1$, where $F_L(x)$ is largest. Thus it should be possible to obtain useful information from $F_L(x)$ on the gluon fragmentation function at small x , where the results of three-jet analyses are not so reliable.

The predictions shown by the curves in fig. 8 do not include hadronization effects, which as we discussed in sect. 3.1 could be significant. The points with error bars show the results obtained from 2.5×10^5 HERWIG Monte Carlo events using the

^{||}Results obtained by taking $D_g(x)$ from the HERWIG simulation of the hypothetical process $e^+e^- \rightarrow gg$, rather than from the OPAL three-jet data, were very similar. Although the HERWIG gluon fragmentation function is somewhat softer, it agrees with the OPAL data at small x , where the gluon contribution to F_L is dominant.

Figure 8: Longitudinal inclusive charged particle x distribution. Points: HERWIG Monte Carlo. Curves: predictions from eq. (4.1).

full angular range and the weight factor $W_L = 2 - 5 \cos^2 \theta$ as explained in sect. 2.1. Since the Monte Carlo program contains a hadronization model, the agreement with the perturbative prediction suggests that hadronization corrections are small at this energy, in agreement with the results on the integrated longitudinal cross section discussed in sect. 3.3.

The indication that hadronization corrections to $F_L(x)$ are smallest at low x is encouraging, since that is the region in which we have the best chance of extracting the gluon fragmentation function.

4.2. Scaling violation in $F_{tot}(x)$

In this section we present a study of scaling violation in the total inclusive e^+e^- cross section based upon available data. This analysis will be rather incomplete; in fact more experimental input is needed in order to perform a serious study. We only present it as an example of how the analysis could be performed, and in order to pinpoint the most important uncertainties and sources of errors one has to deal with.

In order to study scaling violation in the total inclusive cross section, using the formalism presented in sect. 2.2, we require in principle the fragmentation functions for all quark flavours, and for gluons, as initial conditions for the evolution equations. In the energy region of interest, the contributions of the different flavours vary strongly with centre-of-mass energy, owing to the presence of the Z^0 resonance and its interference with the virtual photon contribution. This energy dependence is contained in the functions $\sigma_{0,f}(s)$ and $A_f(s)$ given by eqs. (2.12) and (2.13). If different flavours have different fragmentation functions, the variation of flavour composition will lead to energy-dependent inclusive cross sections, even in the absence of QCD scaling violation.

Monte Carlo simulation suggests that after summation over all particles, or over all charged particles, the fragmentation functions of the light (u, d, s) quarks can be taken to be identical. We will assume this to hold in reality. The flavour composition problem then concerns only the heavy (i.e. charm and bottom) quark contributions. The heavy quark fragmentation functions can be determined experimentally, for example using secondary vertices and leptons to tag heavy quark events. Since these two tagging techniques have different efficiencies for charm and bottom quarks, it should be possible to disentangle their contributions, which is important in view of their very different energy dependence.

At present, data on the fragmentation of identified heavy quark events into all hadrons, or all charged hadrons, are not yet available. For this study of scaling violation, we have therefore taken these fragmentation functions from Monte Carlo simulations, again using the program HERWIG^[12]. In the “annihilation scheme” defined in sect. 2.2, the inclusive cross section obtained from simulated events with primary quarks of flavour f can be taken to measure the fragmentation function D_f^h directly. The gluon fragmentation function was obtained from OPAL three-jet data as explained in sect. 4.1. The sensitivity of scaling violation to the input gluon and heavy quark fragmentation functions was discussed in ref. [14].

As discussed in sect. 2.2, in order to use the Mellin transform method to solve the evolution equations, we first fit the input data to functions of the form (2.44), which have simple Mellin transforms. We used the following parametrization

$$D_i(x, Q_0) = Ax^{a-1}(1-x)^b \sum_{i=0}^4 c_i x^i \quad (4.6)$$

	A	r_1	r_2	r_3	r_4	a	b
t	1.1964	-4.1735	8.6513	-8.9914	3.7514	0.01684	2
c	0.71669	-4.1806	6.8754	-5.3553	1.679	-0.08502	1.4484
b	0.66673	3.10480	-22.509	61.022	-53.71	-0.23818	6.5217
g	1.2933	18.223	-84.48	114.32	-43.415	-0.87368	2.1313

Table 1: Coefficients for the fits to the charge fragmentation function at $Q = 91.2 \text{ GeV}$. The labels g , b , c and t stand for gluon, bottom, charm, and the total fragmentation function.

where

$$\begin{aligned}
c_i &= \frac{r_i}{\sum_{j=0}^4 B(a+j+1, b+1) r_j} \\
r_0 &= 1 \\
B(x, y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\end{aligned} \tag{4.7}$$

With the above normalization we have

$$\int_0^1 D_i(x, \mu_0) x dx = A. \tag{4.8}$$

We fitted the Herwig Monte Carlo results for Z^0 hadronic decays for the charm and bottom fragmentation function, and the gluon fragmentation function was fitted from the OPAL data (see Sec. 4.1). The quality of the fits is shown in fig. 9. The total fragmentation function is obtained by fitting the ALEPH^[15] and DELPHI^[3] data. The fit is displayed in fig. 10. The two sets of data are highly consistent. The light quark fragmentation function is then obtained from the total, charm and bottom fragmentation functions, by solving eq. (2.60) at $s = M_Z^2$, assuming equality of the u , d and s fragmentation functions. The coefficients of the fits are given in table 1. Observe that, because of the chosen form of the fits, it is difficult to reproduce well the small x behaviour of the fragmentation function. On the other hand, we will always impose a cut of $x > 0.1$ when performing evolution. As mentioned in sect. 2.2 the evolution of the fragmentation function at a given x is influenced only by the value of the function at larger x ; therefore, the value assumed by the fits for $x < 0.1$

does not influence the evolution of the fragmentation function in the remaining x range. Similarly, since the small x region is not well described by the fit, the values of A given in the table do not reflect correctly the fraction of momentum carried by charged particles.

Figure 9: Fits of the fragmentation functions in Z^0 decays.

Figure 10: Fit to the total fragmentation function measured by the DELPHI and ALEPH experiments.

We evolved the fragmentation functions backwards to the TASSO energies of 44,

35 and 22 GeV (ref. [18]). In view of the possible importance of bottom quark mass corrections at lower energies, as discussed in sect. 3.1, we did not use the TASSO data at 14 GeV. Besides standard evolution, we included an effect of non-perturbative origin, scaling like the inverse of the centre-of-mass energy, of the form given in eq. (3.46). We therefore proceeded in the following way. Defining

$$\tilde{D}_i(N, Q_0) = \int_0^1 D_i(x, Q_0) x^{N-1} dx \quad (4.9)$$

with $Q_0 = 91.2$ GeV, we computed the evolved fragmentation function

$$\tilde{D}(N, Q) = \left(\frac{1 + B/Q}{1 + B/Q_0} \right)^{N-2} \tilde{P}^{(AS)}(N, R, \mu, \mu_0) \tilde{D}(N, Q_0). \quad (4.10)$$

with $\mu_0/Q_0 = \mu/Q = R$, and obtain the χ^2 of the fit to the TASSO data in the range $0.1 < x < 0.8$. We accounted for the normalization uncertainty of the TASSO data as follows: the normalization factor for each energy was treated as an extra random variable in the χ^2 computation, with mean value equal to 1 and an error given by the normalization error quoted by the experiment. The results of the computations are given in figs. 11, 12 and 13, as contours of constant χ^2 in the B, Λ_5 plane, for different choices of the scale factor R . The χ^2 values displayed in the figures include

Figure 11: Contours of equal χ^2 for the fits of the charged fragmentation function to the TASSO data for $R = 1$.

Figure 12: Contours of equal χ^2 for the fits of the charged fragmentation function to the TASSO data, for $R = 1/2$.

Figure 13: Contours of equal χ^2 for the fits of the charged fragmentation function to the TASSO data for $R = 2$.

only the χ^2 of the fits to TASSO data (39 data points in the interval $0.1 < x < 0.8$). We in fact assumed that in comparison the LEP data have negligible errors. As one can see, the fits require large values for the B coefficient, or large values of Λ_5 . From figs. 12 and 13 we also see that there is no reasonable choice of the scale that gives small values of the B coefficient.

Figure 14: Ratio of charged particle x distributions at 91 and 35 GeV. Points: LEP data divided by TASSO data. Curves: prediction for $\Lambda_5 = 500$ MeV, $B = 4$ GeV (solid); $\Lambda_5 = 250$ MeV, $B = 0$ (dashed); $\Lambda_5 = 250$ MeV, $B = 0$ GeV with modified c and b fragmentation (dot-dashed). Renormalization scale factor $R = 1$ in all cases.

Figure 14 shows the ratio between the predictions at $\sqrt{s} = 91$ and 35 GeV, compared with the ratio obtained from TASSO data and the the fit to ALEPH and DELPHI data. For $\Lambda_5 = 500$ MeV and $B = 4$ GeV, in the vicinity of the best fit, there is good agreement throughout the fitted range. For $\Lambda_5 = 250$ MeV and $B = 0$, there is too little scaling violation.

Our result depends upon using the Monte Carlo data for the b and c fragmentation functions. We therefore attempted to fit the TASSO data by using the fits of table 1, and minimized the χ^2 by letting the b coefficients (the exponent of the $(1-x)$ factor) for charm and bottom float freely, keeping $\Lambda_5 = .25$ GeV and $B = 0$. We then obtained a good fit to the data, with $\chi^2 = 50.7$ for 37 degrees of freedom, for the b parameter of table 1 equal to 0 for charm and 6.998 for bottom. As shown by the dot-dashed curve in fig. 14, the amount of scaling violation is similar to that seen in the data, although still somewhat smaller at high x . In this case, however, the c and b parametrizations (the c parametrization in particular) are no longer a good fit to

the Monte Carlo data, as displayed in fig. 15.

Figure 15: The c and b parametrization compared to the Monte Carlo result. The dashed lines are the best fit to the Monte Carlo points, and the solid lines are obtained by letting the b exponents for charm and bottom vary at $\Lambda_5 = .25$, $B = 0$ in order to get the best fit to the TASSO data, as described in the text.

We also report in fig. 16 the χ^2 contours obtained with this parametrization of the b and c fragmentation, for $R = 1$.

In spite of the fact that they no longer fit the Monte Carlo results so well, the b and c fragmentation functions displayed in fig. 15 are quite plausible. Since the final results change so drastically when we use these parametrizations, it seems that experimental studies of the heavy quark fragmentation functions are a fundamental prerequisite for a serious study of scaling violation between LEP and lower energy e^+e^- experiments.

5. Conclusions

In this paper we have discussed in some detail the wide range of theoretical and phenomenological issues that arise in the study of the single-particle inclusive cross section in e^+e^- annihilation. Here we summarize our main results and identify areas in which further experimental and theoretical study would be worthwhile.

Figure 16: Contours in χ^2 for the parametrization of the b and c fragmentation function that give the best fit to the TASSO data for $\Lambda_5 = 250$ MeV and $B = 0$.

We considered first the decomposition of the cross section into transverse, longitudinal and asymmetric contributions, which has not yet been performed experimentally. Data on the longitudinal cross section in particular would provide new tests of perturbative QCD as well as new information on gluon fragmentation. We presented a prediction of this quantity based on gluon fragmentation data from three-jet events; such data are probably less reliable at small x , where longitudinal cross section data would be particularly effective in constraining the gluon fragmentation function. On the theoretical side, a next-to-leading order calculation of σ_L , the second moment of the longitudinal cross section, appears feasible and would provide a new method for measuring α_s . Experimental studies of the asymmetric part of the cross section would provide information on the non-perturbative hadronization process, especially the relationship between the hadron- and parton-level quantum numbers of jets.

Studies of scaling violation, using the next-to-leading order formalism presented here, can also be used to test QCD and constrain hadronization models. In addition to the logarithmic scaling violation associated with QCD evolution of the fragmentation functions, various other sources of energy-dependence need to be taken into account. The best understood of these are heavy quark mass effects. We presented full results on dynamical mass corrections in heavy quark production up to order α_s . Generally speaking, such corrections are of order m^2/Q^2 (modulo logarithms) and are therefore

small at $Q \sim M_Z$, although they can be significant at lower energies. An exception is the first moment of the asymmetric contribution, which has a correction of order m/Q . This should be taken into account in the asymmetry studies suggested above.

We also studied the kinematical mass effects which arise when the observed hadron distributions include decay products from heavy quark decays. Here again the corrections are of order m^2/Q^2 in most cases. There is a correction to σ_L which is formally of order m/Q asymptotically, but we expect the behaviour at present energies to be m^2/Q^2 (modulo logarithms) in this case too.

The non-scaling effects of hadronization are more difficult to estimate because we have no good theory of such non-perturbative processes. We showed that simple models and Monte Carlo studies suggest that there may be corrections of order λ/Q , where λ is related to the hadronization scale. As far as we know, there are no theoretical grounds for excluding non-perturbative corrections of this order. Thus hadronization effects could make an important contribution to scaling violation at present energies. In an illustrative analysis of fragmentation data between 22 and 91 GeV, we found a strong correlation between the assumed magnitude of $1/Q$ corrections and the value obtained for the QCD scale $\Lambda_{\overline{\text{MS}}}$.

The change in flavour composition when going from TASSO to LEP data can also be a cause of fake scaling violation. In general we expect that the fragmentation function will be softer for b than for c . Since b 's are more abundantly produced at the Z^0 peak, this effects induces stronger apparent scaling violation. We have found that this problem is particularly severe. It is possible to get very different fitted values of Λ and of the non-perturbative corrections by assuming different forms for the charm and bottom fragmentation functions. It is therefore important to measure these fragmentation functions at LEP, in order to get rid of this uncertainty. In the meantime, it is not possible to quote a measured value of $\Lambda_{\overline{\text{MS}}}$ until the form and magnitude of the heavy flavour contributions, together with the hadronization corrections, have been better understood.

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APPENDIX A: Single inclusive cross sections for heavy quarks at order α_s

For the soft-virtual terms

$$\begin{aligned}
S_T^{(v)}(\rho) &= \frac{1}{2} \left\{ (2 - \rho) \left[4 \log \frac{4}{\rho} \log \frac{1 + \beta}{1 - \beta} - 4 \text{Li}_2 \left(-\frac{1 - \beta}{2\beta} \right) - 2 \log^2 \frac{2\beta}{1 - \beta} \right. \right. \\
&\quad \left. \left. + \frac{4}{3} \pi^2 + \log^2 \frac{1 + \beta}{1 - \beta} + \text{Li}_2 \left(-\frac{4\beta}{(1 - \beta)^2} \right) - \text{Li}_2 \left(\frac{4\beta}{(1 + \beta)^2} \right) \right] \right. \\
&\quad \left. + (10 - 8\rho) \log \frac{1 + \beta}{1 - \beta} - 4\beta - 8\beta \log \frac{4}{\rho} \right\} \\
S_L^{(v)}(\rho) &= \frac{\rho}{2} S_T^{(v)}(\rho) - \frac{\rho\beta^2}{2} \log \frac{1 + \beta}{1 - \beta} \\
S_T^{(a)}(\rho) &= \beta^2 S_T^{(v)}(\rho) + 2\rho\beta^2 \log \frac{1 + \beta}{1 - \beta} \\
S_L^{(a)}(\rho) &= 0 \\
S_A(\rho) &= \beta S_T^{(v)}(\rho) + \rho\beta \log \frac{1 + \beta}{1 - \beta}. \tag{A.1}
\end{aligned}$$

For the real terms

$$\begin{aligned}
R_T^{(v)}(x, \rho) &= \frac{2}{\sqrt{x^2 - \rho}} \left[\rho(2 - \tau_x^2) + 4 \frac{\tau_x^2(1 + \tau_x)^3}{(4\tau_x + \rho)^2} + \tau_x(4 + \tau_x) \left(1 - \frac{2\tau_x(1 + \tau_x)}{4\tau_x + \rho} \right) \right. \\
&\quad \left. - 2 \right] + \frac{\rho^2(2 - \tau_x^2) + \rho(2x^3 - 7x^2 - 1) + 2x^2(1 + x^2)}{2(x^2 - \rho)} \xi(x, \rho) \\
R_L^{(v)}(x, \rho) &= \frac{2}{\sqrt{x^2 - \rho}} \left[-\rho(1 - \rho) - \tau_x(\tau_x - 2\rho) + \frac{2\tau_x^2(1 + \tau_x)}{4\tau_x + \rho} \right] \\
&\quad + \frac{\rho^3 + \rho^2(4\tau_x - 3) + \rho(3x^2 - 1)}{2(x^2 - \rho)} \xi(x, \rho) \\
R_T^{(a)}(x, \rho) &= \frac{4}{\sqrt{x^2 - \rho}} \left[-\rho^2 + 2\rho x + \tau_x^3 + \frac{3}{2} \tau_x^2 + 2\tau_x - 1 \right. \\
&\quad \left. + \frac{2\tau_x^2(1 + \tau_x)^3}{(4\tau_x + \rho)^2} - \frac{\tau_x^2(1 + \tau_x)(5\tau_x + 4)}{4\tau_x + \rho} \right] \\
&\quad + \frac{-2\rho^3 + 8\rho^2 x + \rho x^2(2\tau_x - 9) - \rho + 2x^2(1 + x^2)}{2(x^2 - \rho)} \xi(x, \rho)
\end{aligned}$$

$$\begin{aligned}
R_L^{(a)}(x, \rho) &= \frac{2\tau_x^2}{\sqrt{x^2 - \rho}} \left[\rho + x^2 - 5 - \frac{8\tau_x(1 + \tau_x)^3}{(4\tau_x + \rho)^2} - \frac{2(1 + \tau_x)}{4\tau_x + \rho} (\tau_x^2 - 8\tau_x - 2) \right] \\
&\quad + \frac{\tau_x^2 \rho (\rho + \tau_x^2 - 2)}{2(x^2 - \rho)} \xi(x, \rho) \\
R_A(x, \rho) &= 4\rho - 4x + 8 \frac{\tau_x^2(1 + \tau_x)^2}{(4\tau_x + \rho)^2} - 4 \frac{\tau_x^2(3 + \tau_x)}{4\tau_x + \rho} \\
&\quad - \frac{-\rho^2 + 3\rho x - x(1 + x^2)}{\sqrt{x^2 - \rho}} \xi(x, \rho)
\end{aligned} \tag{A.2}$$

where

$$\tau_x = 1 - x, \quad \xi(x, \rho) = \log \frac{\rho - 2x - 2\sqrt{x^2 - \rho}}{\rho - 2x + 2\sqrt{x^2 - \rho}}. \tag{A.3}$$

For the gluon single inclusive cross section we have

$$\begin{aligned}
G_T^{(v)}(x, \rho) &= 2 \left[\frac{1 + (1 - x)^2}{x} + \rho \frac{1 - x}{x} - \frac{\rho^2}{2x} \right] \left(\log \frac{1 + \beta_x}{1 - \beta_x} - \beta_x \right) - 4 \frac{1 - x}{x} \beta_x - \frac{\rho^2 \beta_x}{x} \\
G_L^{(v)}(x, \rho) &= -\frac{2\rho}{x} \log \frac{1 + \beta_x}{1 - \beta_x} + 4\beta_x \frac{1 - x}{x} \\
G_T^{(a)}(x, \rho) &= 2 \left[\frac{1 + (1 - x)^2}{x} - \rho \frac{1 - x}{x} + \frac{\rho^2}{2x} \right] \left(\log \frac{1 + \beta_x}{1 - \beta_x} - \beta_x \right) - 4 \frac{1 - x}{x} \beta_x + \frac{\rho^2 \beta_x}{x} \\
G_L^{(a)}(x, \rho) &= \rho \frac{\rho + x^2 + 2x - 4}{x} \log \frac{1 + \beta_x}{1 - \beta_x} + 2\beta_x(2 + \rho) \frac{1 - x}{x}
\end{aligned} \tag{A.4}$$

where $\beta_x = \sqrt{1 - \rho/(1 - x)}$.

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