Generating Operators for the Generalized
Zakharov-Shabat System and its Gauge
Equivalent System in the \( sl(3, \mathbb{C}) \) Case*

by

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Abstract

We consider the $\Lambda$-operator approach to the soliton equations related to the generalized Zakharov-Shabat linear problem $L$ and its gauge equivalent linear problem $\tilde{L}$. For the case when $L$ is defined on the algebra $g_0 = sl(3, \mathbb{C})$ we calculate the generating operators $\hat{\Lambda}_\pm$ for $\tilde{L}$ and thus exhibit the hierarchy of soliton equations related to it as well as their conservation laws and the hierarchy of Hamiltonian structures.
1 Introduction

In the past decades considerable attention has been paid to the so-called soliton equations — evolution equations solvable through the inverse scattering method. The theory of these equations now is an amalgamation of numerous mathematical techniques which are difficult even to list. The interest in soliton equations is due both to their remarkable mathematical properties and to the fact that they describe important physical systems. It is enough to mention such equations as the nonlinear Schrödinger equation (NSE), the Korteweg-de-Vries equation (KdV) and sine-Gordon equation, see [4].

Consequently the approaches to the subject are quite different but one can divide them into two large classes — the approaches that explain the general properties such as the existence of hierarchies of conservation laws and symplectic (or Poisson) structures on one hand and approaches designed to obtain exact solutions on the other.

The so-called generating operator (or A-operator) method which we are going to introduce belongs to the first of the above classes. It is remarkable because it gives important interpretation of the inverse scattering transform as generalized Fourier transform. The method appeared in the famous paper by Ablowitz et al. [1] and was applied to the investigation of the so-called Zakharov—Shabat system:

\[ L\Psi = \left[ i \frac{\partial}{\partial x} + \begin{pmatrix} 0 & q_+(x) \\ q_-(x) & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \Psi = 0. \] (1)

Here \( q_\pm(x) \) are smooth complex-valued functions of \( x \in \mathbb{R} \), \( \lambda \) is the spectral parameter. The matrix-valued function:

\[ q(x) = \begin{pmatrix} 0 & q_+(x) \\ q_-(x) & 0 \end{pmatrix}, \] (2)

is usually called the potential of the linear problem (1).

The hierarchy of equations related to (1) consists of the equations having the following Lax representation:

\[ [L, M] = 0, \] (3)

\[ M = i \frac{\partial}{\partial t} + \lambda^n r_1 + \cdots + r_n, \] (4)

where \( r_1, r_2, \ldots, r_n \) are 2 x 2 traceless matrix functions whose entries depend on \( q_\pm(x) \) and their \( x \)-derivatives.

Remark 1.1. One should have in mind that in (3) \( q_\pm(x) \) depend also on \( t \) (the time) as a parameter.
After its first success the method was also applied to hierarchies connected with other linear problems and it took several years to overcome some additional difficulties [5]—[8], [15, 16, 19, 22]. We must mention also the beautiful geometrical theory interpreting the \( \Lambda \)-operators as tensors with very special properties (Nijenhuis tensors) on the infinite dimensional manifold of potentials [17, 18, 20]. For our purposes we must present the \( \Lambda \)-operator theory for the so-called generalized Zakharov—Shabat system:

\[
\left( i \frac{\partial}{\partial x} + q(x) - \lambda J \right) \Psi = \hat{L} \Psi = 0.
\]  

(5)

Here \( q(x) \) and \( J \) belong to the fixed simple Lie algebra \( \mathcal{G} \) in some finite dimensional irreducible representation; (1) is obtained for \( \mathcal{G} = \mathfrak{sl}(2, \mathbb{C}) \).

The element \( J \) is real and regular, that is the kernel of \( \text{ad}_X \) \((\text{ad}_X X \equiv [J, X], X \in \mathcal{G})\) is the Cartan subalgebra \( \mathcal{H} \subset \mathcal{G} \). The potential \( q(x) \) belongs to the orthogonal completion \( \mathcal{H}^\perp \) of \( \mathcal{H} \) with respect to the Killing form:

\[
(X, Y) = \text{tr} \, \text{ad}_X \text{ad}_Y, \quad X, Y \in \mathcal{G},
\]  

(6)

\( q(x) \) is smooth and tends to zero as \( x \to \pm \infty \). We shall restrict ourselves below to the case when \( q(x) \) is of Schwartz type. This is enough for our purposes.

The theory for (5) was first developed for the case \( \mathcal{G} = \mathfrak{sl}(n, \mathbb{C}) \) and is presented in its final form in [9].

The case when \( J \) is not real is rather difficult. For \( \mathcal{G} = \mathfrak{sl}(n, \mathbb{C}) \) in the typical representation (5) it is studied thoroughly in [2]. The corresponding adjoint solutions (see below) are constructed in an elaborate fashion and the \( \Lambda \)-operator theory will be presented elsewhere.

There is another important trend in the theory of equations related to (1) and (5). As it is well known there is a gauge equivalence between the hierarchies of equations related to (1) and to the following linear problem:

\[
\left( i \frac{\partial}{\partial x} - \lambda S(x) \right) \Psi = \hat{L} \Psi = 0, \quad S(x) \in \mathfrak{sl}(2, \mathbb{C}), \quad \lim_{x \to \pm \infty} S(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^0(x) = \mathbb{I}.
\]  

(7)

In their famous paper [21] Zakharov and Takhtadjan showed that \( L \) (1) and \( \hat{L} \) (7) are gauge equivalent, that is:

\[
\hat{L} = \Psi_0^{-1} L \Psi_0.
\]  

(8)

Here \( \Psi_0 \) is the fundamental solution of (1) for \( \lambda = 0 \) having the properties:

\[
\lim_{x \to -\infty} \Psi_0(x) = \mathbb{I}, \quad \lim_{x \to \infty} \Psi_0(x) = T(0)^{-1}
\]  

(9)
As a result the hierarchies of equations related to $L$ and $\widetilde{L}$ are also equivalent. Zakharov and Takhtadjan showed in particular the equivalence between the NSE:

$$i\varphi_1 + \varphi_{xx} + 2\varphi |\varphi|^2 = 0,$$

(\varphi is a complex-valued function) and the Heisenberg ferromagnet equation:

$$S_t = \frac{1}{2i}[S, S_{xx}],$$

with $S(x)$ like in (7) and $S^\dagger = -S$ ($^\dagger$ means hermitian conjugation).

In [10, 11] we have shown how one can calculate the equivalent equations in both hierarchies and have calculated the generating operator for (7). We generalized these considerations, see [12], but until I have read [3] I believed that the expressions for the algebras of rank $r \geq 2$ are too cumbersome and consequently have no physical applications. In [3] however, a system, gauge equivalent to (5) for the case $\mathcal{G} = sl(3, \mathbb{C})$ was considered. It turns out that one of the soliton equations related to it describes magnetic chains. This prompted my writing this article. My intention was to show how the general theory is applied in this particular case and to calculate the generating operators and thus the hierarchies of conservation laws and symplectic structures.

### 2 Generalized Fourier Transform for (5)

In this section we introduce the basic facts about the scattering theory of (5), the adjoint solutions and the corresponding $\Lambda$-operator and consider their applications to the soliton equations related to the linear problem (5).

We need some facts about simple Lie algebras and their representations. For the sake of convenience all these facts are listed in the Appendix. In addition all the notations we use here are universally accepted.

Let us introduce the Jost solutions $\phi$ and $\psi$ for the system (5). As usual they are defined by their asymptotics:

$$\psi \to e^{-i\lambda Jx}, \quad \phi \to e^{-i\lambda Jx}, \quad \lambda \in \mathbb{R}. \quad (12)$$

Then the scattering matrix is defined in a standard way:

$$T(\lambda) = \psi^{-1}\phi, \quad \lambda \in \mathbb{R}. \quad (13)$$

It is well known that $\phi$ and $\psi$ have no good analytical properties in $\lambda$, but there is a procedure to construct from these fundamental solutions another ones which are analytical in the upper (lower) half plane in $\lambda$ [9]:

$$\chi^+(x, \lambda) = \psi(x, \lambda) T^+(\lambda) D^+(\lambda) = \Phi(x, \lambda) S^+(\lambda), \quad \lambda \in \mathbb{R},$$

$$\chi^-(x, \lambda) = \psi(x, \lambda) T^+(\lambda) D^+(\lambda) = \Phi(x, \lambda) S^-(\lambda), \quad \lambda \in \mathbb{R}. \quad (14)$$

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In (14) \( T^\pm(\lambda), S^\pm(\lambda), D^\pm(\lambda) \) are the factors of the following Gauss decompositions of \( T(\lambda) \)

\[
T(\lambda) = T^-(\lambda)D^+(\lambda)S^- = T^+(\lambda)D^-(\lambda)S^+.
\] (15)

Here and below we are using the notation \( \hat{A} \equiv A^{-1} \) and

\[
S^\pm(\lambda) = \exp \sum_{\alpha \in \Delta_+} t_{\pm \alpha}^\pm(\lambda) E_{\pm \alpha}, \quad T^\pm(\lambda) = \exp \sum_{\alpha \in \Delta_+} t_{\pm \alpha}^\pm(\lambda) E_{\pm \alpha},
\]

\[
D^\pm(\lambda) = \exp \sum_{j=1}^n d_j^\pm(\lambda) H_j.
\] (16)

\( \{E_\alpha, H_j\} \) is the Cartan-Weyl basis for the algebra \( \mathcal{G} \) with rank \( r \), \( \Delta_+ \) is the system of positive roots (see the Appendix). We are using the particular ordering (131).

It can be shown [9] that these decompositions can be performed if the functions

\[
D^+_j(\lambda) = \langle \omega_j \mid T(\lambda) \mid \omega_j \rangle, \quad D^-_j(\lambda) = \langle \bar{\omega}_j \mid T(\lambda) \mid \bar{\omega}_j \rangle, \quad j = 1, 2, \ldots, r,
\] (17)

(see the Appendix) have no zeroes on the real axis. The functions \( \chi^+, D^+_j \) and \( \chi^-, D^-_j \) allow analytic continuation in the upper (lower) half-plane \( \lambda \).

The zeroes of \( D^+_j \) and \( D^-_j \) define the discrete spectrum of \( L \), see [9], and the continuous spectrum fills up the real axis.

All the constructions are simplified if \( \mathcal{G} = sl(n, \mathbb{C}) \) is considered in the typical representation. Then \( \mathcal{H} \) is the algebra of diagonal traceless matrices, \( (X, Y) = 2n \text{tr} XY \). If the eigenvalues of \( J = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) are ordered to decrease, that is if \( \lambda_i > \lambda_j \) for \( i < j \), then the root vectors are strictly triangular matrices with only one element different from zero. \( T^+, S^+ \) are upper triangular and \( T^-, S^- \) are lower triangular with units on the diagonal. \( D^+ \) and \( D^- \) are diagonal and their elements are ratios of the upper (lower) principal minors of \( T(\lambda) \).

For arbitrary \( \mathcal{G} \) using the same method as in [2] for the case \( \mathcal{G} = sl(n, \mathbb{C}) \) it can be shown that the set of potentials for which \( D^\pm(\lambda), j = 1, 2, \ldots, r \) have only a finite number of zeroes located in the upper (lower) half-plane is dense in the space of all Lebesgue integrable potentials. Such potentials we shall call generic. Below we consider \( q(x) \) to be generic and a Schwartz-type function of \( x \). For the sake of brevity we shall omit the discrete spectrum terms in all formulae below. Of course the discrete spectrum can be taken into account but the general results about the corresponding hierarchies are not affected when it is omitted. If however, one wants to be rigorous then one must assume that \( q(x) \) has small \( L_1 \)-norm. Then there is no discrete spectrum, see [2].

It is well known that the idea of inverse scattering implies that we can reconstruct the potential function \( q(x) \) using some spectral and asymptotical information called minimal set of scattering data. For the system \( (5) \) it can be shown that a minimal set of scattering data may be each one of the following...
sets:

1) \( \{s^+_\sigma(\lambda), s^-_\sigma(\lambda) ; \lambda \in \mathbb{R}, \sigma \in \Delta_+ \} \),

2) \( \{t^+_\sigma(\lambda), t^-_\sigma(\lambda) ; \lambda \in \mathbb{R}, \sigma \in \Delta_+ \} \),

3) \( \{\rho_{\Delta_+}\sigma(\lambda), \rho_{\Delta_-}\sigma(\lambda) ; \lambda \in \mathbb{R}, \sigma \in \Delta_+ \cap B \subset \mathcal{H} \) regular\},

4) \( \{\rho_{\Delta_-}\sigma(\lambda), \rho_{\Delta_+}\sigma(\lambda) ; \lambda \in \mathbb{R}, \sigma \in \Delta_+ \cap B \subset \mathcal{H} \) regular\},

where

\[
\rho_{\Delta_+}\sigma(\lambda) \equiv -\langle \hat{\chi}_B, BS\hat{\chi}_B, E_{\sigma} \rangle = \langle \hat{\chi} \hat{\chi}_B, E_{\sigma} \rangle^\infty_{-\infty},
\]

\[
\rho_{\Delta_-}\sigma(\lambda) \equiv \langle \hat{\chi}_B, BT\hat{\chi}_B, E_{\sigma} \rangle = \langle \hat{\chi} \hat{\chi}_B, E_{\sigma} \rangle^\infty_{-\infty},
\]

Remark 2.1. In fact we must add here the locations of the zeroes of \( D_j^{\pm}(\lambda) \), \( j = 1, 2, \ldots, r \) and the residues of \( \chi^\pm \) at these points in order to reconstruct a generic potential \( q(x) \), but as we mentioned above, we omit the discrete spectrum.

The fact that III and IV in (18) are minimal sets of scattering data is a starting point for the \( \Lambda \)-operator approach. Let us put the right hand sides of (19) in a more convenient form:

\[
(E_o, \hat{\chi}^B \hat{\chi}_B^B)^{\infty}_{-\infty} = \int_{-\infty}^{\infty} \frac{d}{dx} (\chi^B E_o \chi^B, B) dx = \frac{1}{i} \int_{-\infty}^{\infty} \langle [B, q], \chi^B E_o \chi^B \rangle dx
\]

Since \([B, q] \in \mathcal{H}^\perp\) then if one knows the functions

\[
\pi_0 (\chi^B E_o \chi^B) \equiv \epsilon^B_0 (x, \lambda),
\]

where \(\pi_0\) is the orthogonal projector on \( \mathcal{H}^\perp \), calculating the integrals of the type:

\[
\int_{-\infty}^{\infty} \langle [B, q], \epsilon^B_\sigma(x, \lambda) \rangle dx, \quad \sigma \in \Delta, \quad \lambda \in \mathbb{R},
\]

one gets the minimal set of scattering data. Therefore the functions \(\epsilon^B_\sigma(x, \lambda)\) play important role in the scattering theory. They are called adjoint or squared solutions.

A more detailed investigation shows that we have the following:

Theorem 2.2. Let \( q(x) \) be a generic potential and function of Schwartz type on the line. Then the following expansion formulae hold (for \( \epsilon = +1 \) and \( \epsilon = -1 \)):

\[
Z(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \sum_{\sigma \in \Delta_+} \zeta^\sigma (\lambda) \epsilon^\sigma_\sigma (x, \lambda) - \zeta^-\sigma (\lambda) \epsilon^-\sigma_\sigma (x, \lambda) \right\} d\lambda,
\]

\[
\zeta^\sigma (\lambda) = \int_{-\infty}^{\infty} \langle \epsilon^-\sigma_\sigma (x, \lambda), [J, Z(x)] \rangle dx.
\]
Corollary 2.4. The following expansions hold:

a) \( \text{adj}^{-1} [B, q] = \frac{i}{2\pi} \int_{-\infty}^{\infty} \left\{ \sum_{\alpha \in \Delta_+} \left[ \rho_{B, a}^+(\lambda)e_\alpha^+ - \rho_{B, a}^-(\lambda)e_\alpha^- \right] \right\} d\lambda, \)

b) \( \text{adj}^{-1} [B, q] = \frac{i}{2\pi} \int_{-\infty}^{\infty} \left\{ \sum_{\alpha \in \Delta_+} \left[ \sigma_{B, a}^+(\lambda)e_\alpha^+ - \sigma_{B, a}^-(\lambda)e_\alpha^- \right] \right\} d\lambda, \)

(28)
All these expansions were obtained in [9].

Now we introduce the generating operators as operators for which the sets $F_p$ and $F_q$ are eigenfunctions. These operators are not difficult to calculate. Indeed the function of the type $\chi E_{\alpha\beta} W$ satisfies the following equation:

$$\frac{\partial W}{\partial x} + [q - \lambda J, W] = 0.$$  

(30)

This is simply (5) in the adjoint representation and hence the name adjoint solution.

Projecting this equation on $\mathcal{H}$ and $\mathcal{H}^\perp$, and taking into account:

$$\lim_{x \to \pm \infty} (I - \pi_0)W = 0,$$

we obtain

$$\Lambda_\pm(Y(z)) = \text{ad}_0^{-1} \left[ i\frac{\partial}{\partial x} + \pi_0 \text{ad}_y Y(z) + \text{ad}_y \int_{-\infty}^{x} (I - \pi_0) \text{ad}_y Y(y) dy \right].$$  

(32)

**Remark 2.5.** We note that if $a \in \mathbb{A}^+$ then $\text{ad}_0^{-1} a$ is correctly defined.

Now we can recapitulate the above facts. It is clear that passing from $q(x)$ to the scattering data is quite analogous to the Fourier transform. The operators $\Lambda_\pm$ play the role of the operator $i\frac{\partial}{\partial x}$ for the Fourier transform. Though $\Lambda_\pm$ are rather complex they have beautiful spectral properties, actually (23) is the spectral decomposition for $\Lambda_\pm$.

We cannot go into more details and so we just list the most important results for the hierarchy of soliton equations related to (5). Most of them are direct consequences of the Corollary 2.4.

1. All the equations having Lax representation:

$$[L, M] = 0,$$

(34)

$$M = i \frac{\partial}{\partial t} + \sum_{k=0}^{N} \lambda^k M_k, \quad M_N \in \mathcal{H}, \quad M_N = \text{constant.}$$  

(35)

can be written into one of the following equivalent forms:

a) $\text{iad}_0^{-1} \frac{\partial q}{\partial t} + \Lambda_0^+ (\text{ad}_0^{-1} [M_N, q]) = 0,$

b) $\text{iad}_0^{-1} \frac{\partial q}{\partial t} + \Lambda_0^-(\text{ad}_0^{-1} [M_N, q]) = 0,$  

(36)
II The equations (36) in terms of the scattering data run as follows:

\[ \frac{\partial}{\partial t} \rho_{B, \pm \alpha} \pm \alpha(M_N) \lambda^N \rho_{B, \pm \alpha} = 0, \quad \alpha \in \Delta_+, \]

\[ \frac{\partial}{\partial t} \sigma_{B, \pm \alpha} \pm \alpha(M_N) \lambda^N \sigma_{B, \pm \alpha} = 0, \quad \alpha \in \Delta_+, \]  

(37)

and one recovers the remarkable fact that in terms of the scattering data the equations (36) become linear.

Remark 2.6. It is not difficult to see that all these considerations can be applied to a larger class than (36), to equations of the form:

\[ \text{ad}_{J}^{-1} \frac{\partial q}{\partial t} + \sum_{j=1}^{r} c_j \Lambda_{\pm j}^{N_j} \left( \text{ad}_{J}^{-1} [H_j, q] \right) = 0, \]  

(38)

where \( c_j \) are complex constants and \( N_j \)-natural numbers.

III The equations (36) have the following series of conservation laws

\[ d_j^{(s)} = \frac{1}{s} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{s} \langle [J, k], \Lambda_{\pm j}^{N_j} \text{ad}_{J}^{-1} [H_j, q] \rangle dy \right) dx, \]  

(39)

where \( j = 1, 2, \ldots, r, s = 1, 2, \ldots \) and \( \{H_j^y\} \) is the basis in \( \mathcal{H} \), biorthogonal to \( \{H_j\} \) with respect to the Killing form, \( \langle H_j^y, H_k \rangle = \delta_{jk} \).

IV The equations (33) are Hamiltonian with respect to the following hierarchy of symplectic forms:

\[ \Omega^{(m)}(\delta q_1, \delta q_2) = \int_{-\infty}^{+\infty} \langle [J, q_1], \Lambda^{m} \text{ad}_{J}^{-1} \delta q_2 \rangle dx, \]  

(40)

\[ \Lambda \equiv \frac{1}{2} (\Lambda_+ + \Lambda_-). \]  

(41)

In other words (36) can be written in the form:

\[ - \Omega^{(m)} \left( \cdot, \frac{\partial q}{\partial t} \right) + d \left( \sum_{j=1}^{r} C_{N_j} d_j^{(N+m+1)} \right) = 0, \]  

(42)

where the constants \( C_{N_j} \) are the coefficients of the expansion:

\[ M_N = \sum_{j=1}^{r} C_{N_j} H_j. \]  

(43)
We must note also that all the equations (38) are local ones, i.e. only \(q(x)\) and its \(x\)-derivatives enter in the expressions. The same is true for the densities of the conservation laws (39). This means that the integrand of (39) is always a total derivative with respect to \(x\). This fact has no simple proof. For the case of the Zakharov–Shabat system it is proved in [13].

3 The Generalized Fourier Transform for the Gauge Equivalent System

In this section we consider the linear problem:

\[
\hat{L}\Psi = \left(\gamma \frac{\partial}{\partial x} - \lambda S(x)\right)\Psi = 0, \quad S(x) \in \mathcal{O}_J, \quad \lim_{x \to \pm \infty} S(x) = J. \quad (44)
\]

Here \(\mathcal{O}_J\) stands for the orbit of \(J\) under the adjoint action of the group \(G\) corresponding to \(\mathcal{G}: \mathcal{O}_J = \{\tilde{X}: \tilde{X} = gJg^{-1}, \ g \in G\}\). We shall show that the main results concerning the hierarchies of integrable equations related to (44) can be obtained from their analogs for the system (5). For \(\mathcal{G} = sl(2, \mathbb{C})\) all this program was performed in [11]. The results for the general case are obtained by V. S. Gerdjikov and the author but as mentioned before until now we did not have a sufficiently good example to apply the general theory.

First of all, following Zakharov and Takhtadjan let us remark that if \(\Psi_0\) is the Jost solution for \(\lambda = 0\) to (5) and if in addition:

\[
\lim_{x \to \pm \infty} \Psi_0(x) = T^{-1}(0) = \exp H, \quad H \in \mathcal{H}, \quad (45)
\]

then from \(\hat{L}\Psi = 0\) it follows that \(\hat{L}\Psi = 0\) with \(\Psi = \Psi_0^{-1}\Psi\) and \(S(x) = \Psi_0^{-1}J\Psi_0\).

Since (45) is important we cast it into other equivalent forms – a), b), c), d):

a) \(T^+(0) = S^-(0) = \mathbb{I}\),

b) \(T^-(0) = S^+(0) = \mathbb{I}\),

c) \(\rho^+_B,\alpha(0) = \rho^-_{B,\alpha}(0) = 0, \quad \alpha \in \Delta_+, B \in \mathcal{H}\),

d) \(\sigma^+_B,\alpha(0) = \sigma^-_{B,\alpha}(0) = 0, \quad \alpha \in \Delta_+, B \in \mathcal{H}\), \quad (46)

For the system (44) it is natural to introduce the Jost solutions \(\hat{\Phi}\) and \(\hat{\Psi}\) in the same manner as for (5):

\[
\hat{\Psi} \xrightarrow{x \to \pm \infty} e^{-i\lambda Jx}, \quad \hat{\Phi} \xrightarrow{x \to \pm \infty} e^{-i\lambda Jx}, \quad (47)
\]

Below we shall denote the analogs of the quantities for \(\hat{L}\) with the same letter with tilde.

We have the scattering matrix

\[
\hat{T}(\lambda) = \hat{\Psi}^{-1} \hat{\Phi}, \quad (48)
\]
and also the obvious identities:
\[ \hat{\Phi} = \Phi_0^{-1}\Phi T^{-1}(0), \quad \hat{\Psi} = \Psi_0^{-1}\Psi, \quad \hat{T}(\lambda) = T(\lambda)T^{-1}(0). \quad (49) \]

The corresponding factors in the Gauss decompositions have the form:
\[ \hat{T}^\pm = T^\pm, \quad \hat{S}^\pm = D^\pm(0)S^\pm D^\pm(0), \quad \hat{D}^\pm(\lambda) = D^\pm(\lambda)D^\pm(0), \quad (50) \]

Since \( D^+(0) = D^-(0) \) we shall write below \( D(0) = \exp \sum_{j=1}^{r} d_j H_j, \quad d_j \equiv d_j^+(0) \).

The fundamental analytical solutions are then easily obtained:
\[ \hat{\chi}^\pm = \Psi_0^{-1} \chi^\pm D(0). \quad (51) \]

Furthermore
\[ \hat{\chi}^\pm E_\alpha \hat{\chi}^\pm = \hat{\Psi}_0 \chi^\pm D(0)E_\alpha \hat{D}(0)\chi^\pm \Psi_0 \]
\[ = Ad(\hat{\Psi}_0)\chi^\pm E_\alpha \chi^\pm \exp \sum_{j=1}^{r} \alpha(H_j)d_j. \quad (52) \]

We use here the notation \( Ad(g)X = gXg^{-1} \) for \( X \in \mathcal{G} \) and \( g \in G \) - the Lie group corresponding to \( \mathcal{G} \).

Instead of the Cartan subalgebra \( \mathcal{H} \) we have now the Cartan subalgebra \( \bar{\mathcal{H}} = Ad(\hat{\Psi}_0)\mathcal{H} = \ker \text{ad} S \) and instead of the projection \( \pi_0 \) onto \( \mathcal{H}^+ \) the projection \( \bar{\pi}_0 \) onto \( \bar{\mathcal{H}}^+ \). Clearly:
\[ \bar{\pi}_0 = Ad(\hat{\Psi}_0)\pi_0 Ad(\Psi_0). \quad (53) \]

It is natural to introduce
\[ \hat{\epsilon}_\alpha^\pm = \bar{\pi}_0 \left( \chi^\pm E_\alpha \chi^\pm \right), \quad (54) \]
and to expect some completeness properties analogous to Theorem 2.2 in the previous section. Indeed, one has

**Theorem 3.1.** Let \( \tilde{\mathcal{S}} \) be the image of \( \mathcal{S} \) under the map \( Ad(\hat{\Psi}_0) \). Then for every \( \tilde{Z}(x) \in \tilde{\mathcal{S}} \) the following expansion formulae hold:

\[ \tilde{Z}(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[ \sum_{\alpha \in \Delta^+} \left( \hat{\epsilon}_\alpha^+(x, \lambda) \hat{\zeta}_\alpha^+(\lambda) - \hat{\epsilon}_\alpha^-(x, \lambda) \hat{\zeta}_\alpha^-(\lambda) \right) \right] d\lambda, \quad (55) \]

\[ \epsilon = \pm 1, \quad \hat{\zeta}_\alpha^\pm(\lambda) \equiv \int_{-\infty}^{\infty} \left( \hat{\epsilon}_\alpha^\pm(x, \lambda) [S, \tilde{Z}] \right) dx. \]

**Proof.** The proof is obtained readily using the invariance of the Killing form under the inner automorphisms:
\[ (Ad(g)X, Ad(g)Y) = (X, Y), \quad X, Y \in \mathcal{G}, \quad g \in G, \quad (56) \]
and the equation (52).

The theory of the soliton equations related to (44) proceeds now in the same manner as the theory for the system (5).

We exploit the following relations:

\[ i \lambda \int_{-\infty}^{\infty} \langle [S, B], \dot{\chi} E_\alpha \dot{\chi} \rangle dz = \langle B, \dot{\chi} E_\alpha \dot{\chi} \rangle \bigg|_{-\infty}^{\infty}, \quad \lambda \in \mathbb{R}, B \in \mathcal{H}, \alpha \in \Delta, \quad (57) \]

and

\[ -i \lambda \int_{-\infty}^{\infty} \langle \delta S, \dot{\chi} E_\alpha \dot{\chi} \rangle dz = \langle E_\alpha, \dot{\chi} \delta \dot{\chi} \rangle \bigg|_{-\infty}^{\infty}, \quad \lambda \in \mathbb{R}, \alpha \in \Delta, \quad (58) \]

Then taking into account that one can write:

\[ \delta S = [S, \text{ad}_S^{-1} \delta S], \quad \{S, \pi_0 B\} = [S, B], \quad (59) \]

and from Theorem 3.1 one has the following analog of the corollary in the previous section.

**Corollary 3.2.** The following expansion formulæ hold:

\[ \pi_0 B = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \left\{ \sum_{\alpha \in \Delta_+} \left[ \tilde{\rho}^+_{B, -\alpha} \dot{\chi}^{+\alpha} - \tilde{\rho}^-_{B, -\alpha} \dot{\chi}^{-\alpha} \right] \right\} d\lambda, \quad (a) \]

\[ \pi_0 B = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \left\{ \sum_{\alpha \in \Delta_+} \left[ \tilde{\sigma}^+_{B, \alpha} \dot{\chi}^{+\alpha} - \tilde{\sigma}^-_{B, \alpha} \dot{\chi}^{-\alpha} \right] \right\} d\lambda, \quad (b) \]

\[ \text{a)} \quad \text{ad}_S^{-1} \delta S = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \left\{ \left[ (\dot{S}^{+}\delta S^{+}, E_{-\alpha}) \dot{\chi}^{+\alpha} - (\dot{S}^{-}\delta S^{-}, E_{\alpha}) \dot{\chi}^{-\alpha} \right] \right\}, \]

\[
\text{b)} \quad \text{ad}_S^{-1} \delta S = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \left\{ \sum_{\alpha \in \Delta_+} \left[ (\dot{D}^{+}\delta \dot{D}^{+}(\bar{\delta} \delta \dot{D}^{+}), E_{\alpha}) \dot{\chi}^{+\alpha} - (\dot{D}^{-}\delta \dot{D}^{-}(\bar{\delta} \delta \dot{D}^{-}), E_{-\alpha}) \dot{\chi}^{-\alpha} \right] \right\}, \]

where the coefficients \( \tilde{\rho} \) and \( \tilde{\sigma} \) are equal to

\[ \tilde{\rho}^\pm_{B, \mp \alpha} = (\dot{\chi}^\mp B \dot{\chi}^\pm, E_{\mp \alpha}), \quad \alpha \in \Delta_+, \lambda \in \mathbb{R}, \quad (62) \]

\[ \tilde{\sigma}^\pm_{B, \pm \alpha} = (\dot{\chi}^\pm B \dot{\chi}^\mp \dot{\chi}^\pm, E_{\pm \alpha}), \quad \alpha \in \Delta_+, \lambda \in \mathbb{R}, \quad (63) \]
It is easy to prove that the coefficients $\hat{\rho}$ and $\hat{\sigma}$ are connected in the following way with the coefficients $\rho$ and $\sigma$ introduced in the previous section:

\begin{align*}
a) \quad & \hat{\rho}^\pm_{B,\pm\alpha} = \rho^\pm_{B,\pm\alpha} \exp \left( \pm \sum_{j=1}^{r} \alpha(H_j)d_j \right), \\
b) \quad & \hat{\sigma}^\pm_{B,\pm\alpha} = \sigma^\pm_{B,\pm\alpha} \exp \left( \mp \sum_{j=1}^{r} \alpha(H_j)d_j \right). \quad (64)
\end{align*}

Therefore the coefficients $\hat{\rho}$ and $\hat{\sigma}$ vanish at $\lambda = 0$ (compare with (46 c), d)) and there are no singularities in the integrands of (60)–(61).

It can be proved also that if $B$ is regular then $\hat{\rho}$ and $\hat{\sigma}$ can be used as a minimal set of scattering data for the linear problem (44).

Now we can introduce the generating operators, i.e. the operators for which the sets:

\begin{align*}
\hat{F}_\rho &= \{ \hat{\varepsilon}^+_\alpha(x,\lambda), \hat{\varepsilon}^-_\alpha(x,\lambda), \quad \alpha \in \Delta_+, \lambda \in \mathbb{R} \}, \\
\hat{F}_\sigma &= \{ \hat{\varepsilon}^+_\alpha(x,\lambda), \hat{\varepsilon}^-_\alpha(x,\lambda), \quad \alpha \in \Delta_+, \lambda \in \mathbb{R} \}, \quad (65, 66)
\end{align*}

are eigenfunctions.

First of all it is clear that

\[ \hat{\Lambda}_\pm = \text{Ad}(\hat{\Psi}_0)\hat{\Lambda}_\pm \text{Ad}(\Psi_0). \quad (67) \]

This identity can be used to calculate $\hat{\Lambda}_\pm$. Another possibility is exploited in section 4, see also [11, 12]. We must stress that $\hat{\Lambda}_\pm$ must be expressed only through $S(x)$ and therefore (67) cannot be applied directly. However we need now only the fact that $\hat{\Lambda}_\pm$ exists and will pass to the question about the connection between the hierarchies associated with (44) and (5).

**Proposition 3.3.** The following formulae hold:

\[ \hat{\Lambda}_\pm (\hat{\pi}_0 B) = \text{Ad}(\hat{\Psi}_0) \text{ad}^{-1}_J[B, q], \quad (68) \]

\begin{align*}
a) \quad & \hat{\Lambda}_+ (\text{ad}^{-1}_J \delta S) = -\text{Ad}(\hat{\Psi}_0) \left\{ \text{ad}^{-1}_J \delta q + \sum_{j=1}^{r} \text{ad}^{-1}_J[H_j, q] \delta d_j \right\} \\
& \quad = -\text{Ad}(\hat{\Psi}_0) (\text{ad}^{-1}_J \delta q) + i \text{ad}^{-1}_J \left[ \delta T(0)T'(0), \text{ad}_3 \delta S_2 \right], \quad (69) \\
b) \quad & \hat{\Lambda}_- (\text{ad}^{-1}_J \delta S) = -\text{Ad}(\hat{\Psi}_0) \text{ad}^{-1}_J \delta q.
\end{align*}

**Proof.** Let us apply $\hat{\Lambda}_\pm$ to both sides of (60) and (61). Then taking into account (52) and (64), as well as the expansions for $\text{ad}^{-1}_J \delta q$ and $\text{ad}^{-1}_J[B, q]$ we
easily arrive to (68), (69b) and the first of the expressions (69a). The second one is obtained having in mind that:

\[ \frac{\partial S}{\partial x} = \frac{\partial}{\partial x}(\hat{\psi}_0 J \hat{\psi}_0) = -[S, \hat{\psi}_0 \hat{\psi}_0] \quad (70) \]

and since \( \text{Ad}(\hat{\psi}_0) \in \mathcal{H} \), we can also write

\[ \text{Ad}(\hat{\psi}_0)q = -i \text{ad}^{-1}_S \frac{\partial S}{\partial x} \quad (71) \]

Now we have all the necessary tools and can easily reformulate all the results I–IV from the previous section:

I The integrable equations which are analogs of (36) are:

\[ -i \text{ad}^{-1}_S \frac{\partial S}{\partial t} + \hat{\Lambda}_N \hat{\psi}_0(M_N) = 0, \quad M_N \in \mathcal{H}. \quad (72) \]

Since

\[ \hat{\Lambda}_N (\hat{\psi}_0 M_N) = \text{Ad}(\hat{\psi}_0) \text{ad}^{-1}_S [M_N, q] \]

\[ = -i \text{ad}^{-1}_S \left[ \text{Ad}(\hat{\psi}_0) M_N, \text{ad}^{-1}_S \frac{\partial S}{\partial x} \right] = -i \left( \text{ad}^{-1}_S \right)^2 \left[ M_N, \frac{\partial S}{\partial x} \right], \]

where \( M_N \equiv \text{Ad}(\hat{\psi}_0) M_N \) the above hierarchy can be written in the equivalent form:

\[ \text{ad}^{-1}_S \frac{\partial S}{\partial t} + \hat{\Lambda}_N^{-1} \left\{ \left( \text{ad}^{-1}_S \right)^2 \left[ M_N, \frac{\partial S}{\partial x} \right] \right\} = 0 \quad (73) \]

II In terms of the scattering data (72)–(73) are written in the same way as (37) but with tildes.

III The equations (72) have the following series of conservation laws:

\[ d_j^{(s)} = \frac{i}{s} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} (S_j, \hat{\Lambda}_N^{s+1} (\hat{\psi}_0 H_j^s)) dy \right] dx, \quad j = 1, 2, \ldots, r, \quad (74) \]

where \( s = 1, 2, \ldots \).

IV The equations (72) are Hamiltonian with respect to the hierarchy of symplectic forms:

\[ \tilde{\Omega}^{(m)}(\delta S_1, \delta S_2) = \int_{-\infty}^{\infty} (\delta S_1, \hat{\Lambda}^m \text{ad}^{-1}_S \delta S_2) dx, \quad (75) \]

\[ \hat{\Lambda} \equiv \frac{1}{2} (\hat{\Lambda}_+ + \hat{\Lambda}_-). \quad (76) \]
It is interesting to calculate $F^*\Omega^{(m)}$, that is the pull–back of $\Omega^{(m)}$ under the mapping $F : S(x) \rightarrow q[S(x)]$. It turns out that $F^*\Omega^{(m)}$ is equal to $\Omega^{(m+2)}$ up to terms that are exterior products of differentials of conservation laws. However, to perform this calculation we must calculate the Fréchet derivative of $F$ and we must go into details that are not needed for our task.

4 The $sl(3, \mathbb{C})$ Case

Now we shall apply the abstract construction of the previous two sections to the case $G = sl(3, \mathbb{C})$. As we have mentioned the simplest case $G = sl(2, \mathbb{C})$ (rank 1) was studied in [11]. Let us consider the typical representation of $sl(3, \mathbb{C})$ and choose $J = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 > \lambda_2 > \lambda_3$, $\sum_{i=1}^{3} \lambda_i = 0$. Then the linear problem $L\psi = 0$ is:

$$\left[ i \frac{\partial}{\partial x} + \begin{pmatrix} 0 & q_{12}(x) & q_{13}(x) \\ q_{21}(x) & 0 & q_{23}(x) \\ q_{31}(x) & q_{32}(x) & 0 \end{pmatrix} - \lambda \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} \right] \psi = 0, \quad (77)$$

$q_{ij}(x)$ being Schwartz-type functions of $x$.

The Cartan subalgebra is spanned by all traceless diagonal matrices and is clearly two–dimensional. There are certain simple properties which, however will be very useful. We introduce them as propositions 4.1–4.3 below.

**Proposition 4.1.** The matrices $J$ and $J_1 = J^2 - \frac{1}{3} \text{tr}(J^2) I$ span the Cartan subalgebra $\mathcal{H} = \ker \text{ad} J$.

**Proposition 4.2.** The matrix $J$ satisfies the equation:

$$J^3 = \frac{1}{2} C_2 J + \frac{1}{3} C_3 I, \quad C_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad C_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3. \quad (78)$$

This equation is simply the Hamilton–Caley theorem for $J$.

**Proposition 4.3.** If $S \in O_J = \{ \tilde{X} : \tilde{X} = gJg^{-1}, g \in SL(3, \mathbb{C}) \}$ then $S$ satisfies (78). If in addition $\lambda_i \neq 0$ and $S \in sl(3, \mathbb{C})$ then the inverse is also true.

**Proof.** The fact that if $S \in O_J = \{ \tilde{X} : \tilde{X} = gJg^{-1}, g \in SL(3, \mathbb{C}) \}$ then $S$ satisfies (78) is obvious. The inverse is a bit more difficult to prove. First of all let us note that if $A$ satisfies (78) then in its canonical form there are no Jordan cells. Indeed, if for example:

$$A = \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} \quad (79)$$
and \( j \)-th column.

They form the Cartan–Weyl basis for \( \text{sl}(3, \mathbb{C}) \). Here as usual \( \xi \) means a matrix having a single non-zero entry equal to one on the intersection of the \( i \)-th row and \( j \)-th column.

\[
\begin{bmatrix}
\lambda_2 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]  

This completes the proof.

Below we assume that \( \lambda_j \neq 0 \).

The system of roots for \( \text{sl}(3, \mathbb{C}) \) is

\[
\Delta = \{\alpha_{i,j} = \varepsilon_i - \varepsilon_j, \quad i \neq j, \quad i, j = 1, 2, 3\}, \quad \varepsilon_i(\text{diag}(h_1, h_2, h_3)) = h_i, \quad (83)
\]

and the positive roots are

\[
\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \alpha_3 = \varepsilon_1 - \varepsilon_3 = \alpha_1 + \alpha_2. \quad (84)
\]

The corresponding root vectors are written below:

\[
E_{\alpha_1} = \frac{1}{\sqrt{6}}\varepsilon_{12}, \quad E_{\alpha_2} = \frac{1}{\sqrt{6}}\varepsilon_{23}, \quad E_{\alpha_3} = \frac{1}{\sqrt{6}}\varepsilon_{13},
\]

\[
E_{-\alpha_1} = \frac{1}{\sqrt{6}}\varepsilon_{21}, \quad E_{-\alpha_2} = \frac{1}{\sqrt{6}}\varepsilon_{32}, \quad E_{-\alpha_3} = \frac{1}{\sqrt{6}}\varepsilon_{31}, \quad (85)
\]

and together with the matrices

\[
H_{\alpha_1} = \frac{1}{6}(\varepsilon_{11} - \varepsilon_{22}), \quad H_{\alpha_2} = \frac{1}{6}(\varepsilon_{22} - \varepsilon_{33}), \quad (86)
\]

satisfies (78) then the following equations hold:

\[
\mu_j^2 - \frac{1}{2}C_2\mu_j - \frac{1}{3}C_3 = 0, \quad 3\mu_j^2 - \frac{1}{2}C_2 = 0. \quad (80)
\]

They imply that \( \mu_j \) is a double root of the equation:

\[
\lambda^3 - \frac{1}{2}C_2\lambda - \frac{1}{3}C_3 = 0, \quad (81)
\]

which by assumption has three different roots \( \lambda_1, \lambda_2, \lambda_3 \).

The canonical form of \( S \in \text{sl}(3, \mathbb{C}) \) is then of the kind \( \text{diag}(\mu_1, \mu_2, \mu_3) \) where \( \mu_i \) is one of the numbers \( \lambda_1, \lambda_2, \lambda_3 \). If for example \( \mu_1 = \mu_2 \neq \mu_3 \), this assumption is in contradiction with the fact that \( \lambda_i \neq \lambda_j \) for \( i \neq j \). If \( \mu_1 = \mu_2 = \mu_3 \) then one of \( \lambda_i \) is zero, a possibility which we have excluded. Therefore \( (\mu_1, \mu_2, \mu_3) \) is a permutation of \( (\lambda_1, \lambda_2, \lambda_3) \). But all such matrices belong to \( \mathcal{O}_J \). For example:

\[
\begin{bmatrix}
\lambda_2 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]  

This completes the proof.

\[
E_{-\alpha_1} = \frac{1}{\sqrt{6}}\varepsilon_{21}, \quad E_{-\alpha_2} = \frac{1}{\sqrt{6}}\varepsilon_{32}, \quad E_{-\alpha_3} = \frac{1}{\sqrt{6}}\varepsilon_{31}, \quad (85)
\]

\[
H_{\alpha_1} = \frac{1}{6}(e_{11} - e_{22}), \quad H_{\alpha_2} = \frac{1}{6}(e_{22} - e_{33}), \quad (86)
\]

they form the Cartan–Weyl basis for \( \text{sl}(3, \mathbb{C}) \). Here as usual \( e_{ij} \) means a matrix having a single non-zero entry equal to one on the intersection of the \( i \)-th row and \( j \)-th column.
The Killing form is equal to $6 \text{tr} \ XY$. One has the following useful identities:

$$
(J, J) = 6 C_2, \quad (J_1, J_1) = C_2^2, \quad (J, J_1) = 6 C_3,
$$

(87)

It is not difficult to write the Gauss decompositions (15) in this particular case. If $T(\lambda) = (t)_{ij}$ is the scattering matrix, then for example:

$$
T^+ = \begin{pmatrix}
1 & T_{12}^+ & T_{13}^+
0 & 1 & T_{23}^+
0 & 0 & 1
\end{pmatrix}, \quad T_{12}^+ = \frac{1}{D_2} \begin{pmatrix}
t_{12} & t_{13}
t_{32} & t_{33}
\end{pmatrix}, \quad T_{13}^+ = \frac{t_{13}}{D_1^+},
$$

(88)

where $D_1^+, D_2$ are the lower principal minors:

$$
D_1^+ = t_{33}, \quad D_2 = \begin{pmatrix}
t_{22} & t_{23}
t_{32} & t_{33}
\end{pmatrix}.
$$

(91)

The other decomposition can also be written without difficulties. It is clear that one can construct explicitly the adjoint solutions but this would be superfluous here as they are not needed to write down the hierarchies. All we need now is to calculate the generating operators $A_{\pm}$ in terms of $S$, as $A_{\pm}$ are already expressed through $q(x)$. First of all the gauge equivalent system is:

$$
S^- = \begin{pmatrix}
1 & 0 & 0
S_{21}^- & 1 & 0
S_{31}^- & S_{32}^- & 1
\end{pmatrix}, \quad S_{21}^- = -\frac{1}{D_2} \begin{pmatrix}
t_{21} & t_{23}
t_{31} & t_{33}
\end{pmatrix},
$$

(90)

$$
S_{31}^- = \frac{1}{D_2} \begin{pmatrix}
t_{21} & t_{22}
t_{31} & t_{32}
\end{pmatrix}, \quad S_{32}^- = -\frac{t_{32}}{D_1^-},
$$

where $D_{1,2}^-$ are the lower principal minors.

$$
T(\lambda) = (t)_{ij},
$$

(92)

To calculate $\tilde{A}_{\pm}$ we shall use the equation

$$
i \frac{\partial}{\partial x} \tilde{W} - \lambda [S, \tilde{W}] = 0,
$$

(93)

which is satisfied by every function of the kind $\tilde{W} = \tilde{x} A \tilde{x}^{-1}$ where $A$ is a constant matrix and $\tilde{x}$ is a fundamental solution of (92).
We have
\[ \hat{W} = \hat{W}^h + \hat{W}^a, \quad \hat{W}^h \in \mathcal{H}, \quad \hat{W}^a \in \mathcal{H}^\perp. \] (94)
Since \( J \) and \( J_1 \) span \( \mathcal{H} \) then \( S = \text{Ad}(\hat{\theta}_0)J \) and \( S_1 = \text{Ad}(\hat{\theta}_0)J_1 = S^2 - \frac{1}{2} \mathcal{C}_2 \mathcal{H} \) span \( \mathcal{H} \). Therefore:
\[ \hat{W}^h = a(W)S + b(W)S_1, \] (95)
where \( a(W) \) and \( b(W) \) are coefficients. We have:
\[ i\frac{\partial}{\partial x} \left[ \hat{W}^a + aS + bS_1 \right] - \lambda [S, \hat{W}^a] = 0. \] (96)

In order to find the coefficients \( a \) and \( b \) let us calculate the scalar product of the left hand side of (96) with \( S \) and \( S_1 \). Taking into account (87) we arrive to the following system:
\begin{align*}
6C_2a_x + 6C_3b_x &= -\left( \frac{\partial \hat{W}^a}{\partial x}, S \right) = \left( \hat{W}^a, \frac{\partial S}{\partial x} \right), \\
6C_2a_x + C_3b_x &= -\left( \frac{\partial \hat{W}^a}{\partial x}, S_1 \right) = \left( \hat{W}^a, \frac{\partial S_1}{\partial x} \right) = \left( \hat{W}^a, \frac{\partial S^2}{\partial x} \right). \quad (97)
\end{align*}

The determinant of this system is:
\[ m_1 = 6(C_2^2 - 6C_3^2). \] (98)

A cumbersome calculation shows that
\[ m_1 = 12(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_3)^2 \equiv 12m, \] (99)
and therefore \( m_1 \neq 0 \).

From (97) we get:
\begin{align*}
a_x &= \frac{C_2}{12m} \left( \hat{W}^a, \frac{\partial S}{\partial x} \right) - \frac{C_3}{2m} \left( \hat{W}^a, \frac{\partial S^2}{\partial x} \right), \\
b_x &= -\frac{C_3}{2m} \left( \hat{W}^a, \frac{\partial S}{\partial x} \right) + \frac{C_2}{2m} \left( \hat{W}^a, \frac{\partial S^2}{\partial x} \right). \quad (100)
\end{align*}

As for the eigenfunctions of \( \tilde{\Lambda}_+ \) we have \( \lim_{x \to +\infty} a = \lim_{x \to +\infty} b = 0 \) and for the eigenfunctions of \( \tilde{\Lambda}_- \) we have \( \lim_{x \to -\infty} a = \lim_{x \to -\infty} b = 0 \) it is not difficult to find that:
\[ \tilde{\Lambda}_\pm (\tilde{Z}) \]
\begin{align*}
&= i\text{Ad}^{-1}_b S_0 \left\{ \frac{\partial \tilde{Z}}{\partial x} + \left[ \frac{C_2}{12m} \int_{\pm \infty} \left( \tilde{Z}, \frac{\partial S}{\partial y} \right) dy - \frac{C_3}{2m} \int_{\pm \infty} \left( \tilde{Z}, \frac{\partial S^2}{\partial y} \right) dy \right] S_x \right. \\
&\quad + \left[ -\frac{C_3}{2m} \int_{\pm \infty} \left( \tilde{Z}, \frac{\partial S}{\partial y} \right) dy + \frac{C_2}{2m} \int_{\pm \infty} \left( \tilde{Z}, \frac{\partial S^2}{\partial y} \right) dy \right] (S^2)_x \right\} \quad (101)
\end{align*}
The last thing that remains to be done is to express the operator $\text{ad}_S^{-1}$ through $S$. For this let us note that the operators $\text{ad}_J$ and $\text{ad}_J^{-1}$ ($\text{ad}_S$ and $\text{ad}_S^{-1}$ respectively) are simple and have common eigenvectors. Then we can apply the following theorem which is just the special theorem for the simple matrix $A$.

**Proposition 4.4.** Let $A$ be a simple matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. Let $\mu_1, \mu_2, \ldots, \mu_N$ be arbitrary numbers. Then the matrix $f(A)$ defined by:

$$f(\lambda) = \sum_{k=1}^{N} \mu_k l_k(\lambda),$$

(102)

where $l_k$ are the Lagrange interpolation polynomials

$$l_k(\lambda) = \prod_{i \neq k} \frac{(\lambda - \lambda_i)}{\lambda_k - \lambda_i}$$

(103)

has as eigenvalues $\mu_1, \mu_2, \ldots, \mu_N$ and the same eigenvectors as $A$. Among the polynomials with the above properties the polynomial $f(\lambda)$ is of minimal degree.

**Remark 4.5.** It is not difficult to see that $l_k(A)$ is the projector onto the subspace corresponding to the eigenvalue $\lambda_k$ (since $A$ is simple this subspace is one dimensional).

The operator $\text{ad}_J^{-1}$ has as eigenvalues $1/\alpha(J)$, $\alpha \in \Delta$ and eigenvectors $E_\alpha$, $\alpha \in \Delta$. The operator $\text{ad}_J$ has the same set of eigenvalues and eigenvalues $\alpha(J)$. Then $\text{ad}_J^{-1}$ is equal to $P(\text{ad}_J)$, where $P(\lambda)$ is the polynomial:

$$P(\lambda) = \sum_{\alpha \in \Delta^+} \frac{\lambda r_\alpha(\lambda)}{\alpha(J)^2},$$

(104)

where $r_\alpha(\lambda) = \prod_{\beta \in \Delta^+, \beta \neq \alpha} \frac{\lambda^2 - \beta^2(J)}{\alpha(J)^2 - \beta(J)^2}.$

(105)

In our case $\alpha_{ij} = e_i - e_j$ are the roots and $\alpha_{ij}(J) = \lambda_i - \lambda_j$ the corresponding eigenvalues. A long calculation shows that:

$$P(\lambda) = \frac{\lambda (\lambda^2 - \frac{3}{2}C_2)^2}{(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_3)^2} = \frac{\lambda (\lambda^2 - \frac{3}{2}C_2)^2}{m}.$$

(106)

Let us make an important observation. As $P(\lambda)$ is of the form $\lambda P_1(\lambda)$, $P_1$ - polynomial, then $P(\text{ad}_J) = P_1(\text{ad}_J) \text{ad}_J$. Therefore we have $P(\text{ad}_J) \pi_0 = P(\text{ad}_J)$. Quite in the same way

$$P(\text{ad}_S) = \text{ad}_S^{-1},$$

$$P(\text{ad}_S) \pi_0 = P(\text{ad}_S),$$

(107)
The formula (107) is true irrespective of the choice of the representation. However, we are working in a fixed representation and can simplify (107) using (92). Each time we have $S^3$ we can substitute it with $\frac{1}{2}C_2 S + \frac{1}{3}C_3 \mathbb{1}$ and thus get expressions whose degree in $S$ is at most 2. Thus finally we obtain:

$$\text{ad}^{-1}_S X = \frac{1}{m} \left\{ C_2^2 [J, X] + \frac{3}{2} C_2 J [J, X] J - 3 C_3 J^2 [J, X] \right\}, \quad X \in \mathcal{H}^4,$$

(108)

$$\text{ad}^{-1}_S \tilde{X} = \frac{1}{m} \left\{ C_2^3 [S, \tilde{X}] + \frac{3}{2} C_2 \tilde{S} [S, \tilde{X}] S - 3 C_3 \tilde{S}^2 [S, \tilde{X}] \right\}, \quad \tilde{X} \in \tilde{\mathcal{H}}^4,$$

(109)

For (101) we need to calculate $\text{ad}^{-1}_S \partial S/\partial x$ and $\text{ad}^{-1}_S \partial \tilde{S}^2/\partial x$. We have:

$$m \text{ad}^{-1}_S \frac{\partial S}{\partial x} = -3 C_3 [S^2, S_x] + \frac{3}{2} C_2 S [S, S_x] S,$$

$$m \text{ad}^{-1}_S \frac{\partial \tilde{S}^2}{\partial x} = C_2^3 [S^2, S_x] - 2 C_2 C_3 [S, S_x] S - 3 C_3 \tilde{S} S [S, S_x] S,$$

(110)

Since $m = C_2^2 / 2 - 3 C_3^2$ from this system we get two very simple formulae:

$$\text{ad}^{-1}_S \left[ \frac{C_2^2 (S^2)_x + C_3 S_x}{2} \right] = [S^2, S_x],$$

(111)

$$\text{ad}^{-1}_S \left[ \frac{C_2^3 S_x + C_3 (S^2)_x}{3} \right] = \frac{2 C_2}{3} [S, S_x] + S [S, S_x] S,$$

(112)

These formulae were obtained in [3] in a different way. We must note however that in [3] in the right hand side of (112) instead of the expression $(2/3)C_2 [S, S_x] + S [S, S_x] S$ there stands the expression:

$$[S, \{S, S_x\}] + \frac{1}{6} C_2 [S, S_x],$$

(113)

where $\{,\}$ means anticommutator. It is not difficult to check that taking into account the relation $S^3 = (1/2)C_2 S - (1/3) C_3 \mathbb{1}$, these expressions are equal.

Let us now list some of the equations of the two-parameter hierarchy related to (77) and their counterparts related to (92).

The system $L$:

1. $i q_t + [A J + B J^2, q_t] = 0$, \quad $A, B$—constants,

(114)

2. $q_t + \left\{ \frac{A}{2} \mathbb{1} + B J, q_x \right\} + i B [J, q_x^2] = 0$,

(115)

Special cases:

1. $B = 0$, \quad $q_t + A q_x = 0,$

(116)
\[ II^b \quad A = 0, \quad q_1 + B(J, q_2) + iB[J, q^2] = 0, \quad (117) \]

\[ III \quad i\frac{\partial}{\partial x}(\pi_0 q^2) - \frac{1}{2} \text{ad}_J(\pi_0) \text{ad}_J^{-1} \left\{ BJ + \frac{A}{2} \mathbb{1}, q_2 \right\} = 0, \quad (118) \]

The system \( \dot{L} \):

\[
\begin{align*}
\dot{\mathbf{I}} & \quad \frac{\partial S}{\partial t} = \frac{1}{i} [S, AJ + BJ^2], \quad (119) \\
\dot{\mathbf{II}} & \quad \frac{\partial S}{\partial t} = -AS_\varepsilon - B(S^2)_\varepsilon, \quad (120) \\
\dot{\mathbf{III}} & \quad \frac{\partial S}{\partial t} = \frac{1}{i} \text{ad}^{-1}_S (AS_\varepsilon + (S^2)_\varepsilon). \quad (121)
\end{align*}
\]

Special cases:

a) \( A = \mu C_3, \quad B = \mu C_3/2, \quad \mu = \text{const.} \)

\[
\dot{\mathbf{III}}^a \quad \frac{\partial S}{\partial t} = \frac{\mu}{i} \left( [S^2, S_{xx}] + [S, S^2_\varepsilon] \right). \quad (122)
\]

b) \( A = \mu C_3/3, \quad B = \mu C_3, \quad \mu = \text{const.} \)

\[
\dot{\mathbf{III}}^b \quad \frac{\partial S}{\partial t} = \frac{\mu}{i} \left( \frac{2}{3} C_2 [S, S_{xx}] + S_x[S, S_\varepsilon]S + S[S, S_{xx}]S \right). \quad (123)
\]

If one prefers the expression (113) then (123) can be cast in an equivalent form:

\[
\dot{\mathbf{III}}^b \quad \frac{\partial S}{\partial t} = \frac{\mu}{i} \left( \frac{1}{6} C_2 [S, S_{xx}] + [S, \{S, (S^2)_{xx}\}] \right). \quad (124)
\]

According to [3], when \( iS \) belongs to the compact real form \( \text{su}(3) \) of \( \text{sl}(3, \mathbb{C}) \) the equations (122) - (124) describe the dynamics of spin systems with spin 1. Imposing \( iS \in \text{su}(3) \) does not affect in the slightest way all that has been said. Indeed, if \( J = J, \quad q^\dagger = q \) (that is, if \( iJ \in \text{su}(3), \quad iq \in \text{su}(3) \)) then \( \Psi_0 \) belongs to the group \( SU(3) \) and therefore \( iS = i\Psi_0 J \Psi_0 \in \text{su}(3) \). In other words the requirements \( iJ \in \text{su}(3), \quad iq \in \text{su}(3) \) impose an algebraical restriction which is compatible with all our constructions.

Appendix

Let \( \mathcal{G} \) be a simple Lie algebra over \( \mathbb{C} \). If \( \mathcal{H} \) is the Cartan subalgebra (the maximal abelian subalgebra), then \( \mathcal{H}^\perp \) splits into one-dimensional subspaces - root subspaces:

\[
\mathcal{H}^\perp = \bigoplus_{\alpha \in \Delta} \mathbb{C} E_\alpha, \quad (125)
\]
$E_{\alpha}$ are called root vectors, $\alpha \in \Delta \subseteq \mathcal{H}^*$ — roots, $\Delta$ — root system. $CE_{\alpha}$ are $\mathcal{H}$-invariant and one has:

$$\text{ad}_HE_{\alpha} = [H,E_{\alpha}] = \alpha(H)E_{\alpha}, \quad \alpha \in \Delta, H \in \mathcal{H}. \quad (126)$$

It is well known that if $\alpha \in \Delta$, then $-\alpha \in \Delta$ also and one has the following properties:

$$[E_{\alpha},E_{-\alpha}] = H_{\alpha} \in \mathcal{H}, \quad (127)$$

$$[E_{\alpha},E_{\beta}] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta, \alpha + \beta \neq 0, \\ 0 & \text{if } \alpha + \beta \in \Delta, \alpha + \beta = 0. \end{cases} \quad (128)$$

It can be proved that one can normalize the root vectors in such a way that:

$$\langle E_{\alpha},E_{-\beta} \rangle = \delta_{\alpha,\beta}, \quad \alpha, \beta \in \Delta, \quad (129)$$

and at the same time fix $N_{\alpha,\beta}$ to satisfy:

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}. \quad (130)$$

see for example [14]. If we fix a basis $\{H_j\}_{j=1}^r$, $r = \text{dim} \mathcal{H} = \text{rank} G$ then $\{H_j, E_{\alpha}, j = 1, 2, \ldots, r, \alpha \in \Delta\}$ form a basis in $G$ — the Cartan—Weyl basis.

Usually the set of roots is ordered, that is one must introduce ordering in $\Delta$. The following ordering is appropriate for our purposes:

$$\begin{align*}
\alpha > \beta & \text{ if } \alpha(J) > \beta(J), \\
\alpha < \beta & \text{ if } \alpha(J) < \beta(J),
\end{align*} \quad (131)$$

with $J$ as in (5).

Then the set of roots is divided into two subsets: the set of positive roots $\Delta_+$ and the set of negative roots $\Delta_-:

$$\Delta = \Delta_+ \cup \Delta_- \quad (132)$$

The simple roots are introduced in the usual way.

There is an one-to-one correspondence between the irreducible representations of the so-called fundamental weights $\omega_j$, $j = 1, 2, \ldots, r = \text{dim} \mathcal{H}$, $\omega_j \in \mathcal{H}^*$. Let the representation corresponding to $\omega_j$ be $(V_j, F_j)$, where $V_j$ is the vector space with $\text{dim} V_j < \infty$ and $F_j : G \to \text{End}(V_j, V_j)$. $V_j$ splits into a direct sum of subspaces — the so-called weight subspaces invariant under the action of $\mathcal{H}$:

$$V_j = \bigoplus_{\beta \in \Delta_j} V_j(\beta), \quad \Delta_j \subseteq \mathcal{H}^*, \quad (133)$$

Here $\Delta_j$ is the system of weights and $\beta$ is a weight.

For every $H \in \mathcal{H}$ and $|\alpha\rangle \in V_j(\beta)$ one has:

$$\begin{align*}
F_j(H) | \alpha \rangle = \beta(H) | \alpha \rangle, \\
F_j(E_\alpha) | \beta \rangle = 0 & \text{ if } \alpha + \beta \notin \Delta_j, \\
F_j(E_\alpha) | \beta \rangle \in V_j(\alpha + \beta) & \text{ if } \alpha + \beta \in \Delta_j.
\end{align*} \quad (134)$$

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In the set of weights a fundamental role is played by the lowest and the highest weights (with respect to a fixed ordering). In fact $\omega_j$ is the highest weight of $(V_j,F_j)$, $\dim V_j(\omega_j) = 1$ and the action of $G$ on $V_j(\omega_j)$ generates the whole space $V_j$. There are also lowest weights $\omega_j$ with analogous properties and a simple construction allows one to pass from a lowest to a highest weight. Indeed, if $(V_j,F_j)$ is a representation, then $(V_j^*, F_j^*)$ is also a representation. It is not difficult to see that if $\omega_j$ is the highest weight for $(V_j,F_j)$ it is the lowest for $(V_j^*, F_j^*)$ and vice-versa. If there is a basis $\{ | a \}$ in $V_j$ we introduce the dual basis $\{ \langle a | \}$ of $V_j^*$ with the usual properties:

$$\langle a | b \rangle = \delta_{ab}, \quad (135)$$

and also the "matrix elements":

$$\langle a | F_j(x) | b \rangle = \langle a | V_j^*, x \rangle | b \rangle, \quad (136)$$

For the sake of brevity we shall not write the symbol $F_j$ for the homeomorphism $F_j : G \to \text{End}(V_j,V_j)$.

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