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Spin of Solitons in 2+1 Dimensional Models Coupled with the Chern-Simons Gauge Field

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ABSTRACT

The spin of solitons is considered in the models coupled with the Chern-Simons gauge field. In particular, we shall examine how the large gauge fixing of the Chern-Simons gauge field is related to the expression for spin of the solitons, in both the $O(3)$ σ -model coupled with the Chern-Simons gauge field and the Chern-Simons-Higgs model. Then, we shall show that, in order to obtain a plausible expression for the spin of the soliton, it is necessary to impose an appropriate large gauge condition on the Chern-Simons gauge field. We also comment on the relation between the spin of solitons and the extra angular momentum of static solitons.

1. Introduction

Particles in the $(2+1)$ -dimensional space-time are able to have fractional spins and obey fractional statistics. The possibility of fractional spin is based on the fact that the rotational group in 2-dimensional space, $SO(2)$, possesses multivalued representations. On the other hand, some field theoretical models in $(2+1)$ -dimensional space-time can admit topologically stable solitons.^[1,2] It has been pointed out, by use of topological argument for the soliton configuration, that the topological solitons may also have fractional spin and obey fractional statistics.^[2,3,4]

There are two independent methods to confirm that the solitons do have fractional spin: *The first method* is to show that the wave function of a soliton in the model becomes a multivalued function for adiabatic space rotation of the soliton. If the wave function is n -ply valued for the space rotation, then the value of spin for the soliton can be considered as $1/n$. It was shown along this line of thought that the baby skyrmion, which is a topological soliton in the $(2+1)$ dimensional $O(3)$ nonlinear σ -model, may have fractional spin.^[3,4] *The second method*, on the other hand, depends on rather classical argument. That is, it can be shown that the solitons in a model with the Chern-Simons (C-S) term can often be regarded as dyons, and the dyon in the $(2+1)$ dimensional system has an extra angular momentum, even if the soliton does not rotate. Then the value of the extra angular momentum is identified with that of the intrinsic fractional spin. The charged vortex, which is a topological soliton in the $(2+1)$ dimensional Chern-Simons-Higgs (C-S-H) model, was pointed out to have a fractional spin based on this consideration.^[2]

In the previous paper,^[5] however, it was pointed out that the two methods do not necessarily give the same values of spin in each model. In particular, it was found in the C-S-H model that unless we impose an appropriate large gauge fixing condition on the C-S gauge field (more precisely unless we fix soliton configuration at the space-time infinity, appropriately), the expression for the spin of vortex does not have a fixed form and depends on the gauge condition of the C-S gauge field. Therefore, it is necessary to fix the large gauge condition in order to specify the spin of vortex in the C-S-H model. The idea of fixing the large gauge condition is not very unfamiliar. When we quantize a model on the three dimensional manifold

The spin of baby skyrmion can be determined by the response of wave functional $\Psi[\vec{n}(\vec{x})]$ to the continuous deformation of configuration $\vec{n}(\vec{x})$ corresponding to 2π -rotation of space. The possibility that the baby skyrmions have fractional spins can be checked by observing topology of the homotopy class $\Pi_1(S^2 \rightarrow S^2)$. Since it is known that

$$\Pi_1(S^2 \rightarrow S^2) = \Pi_3(S^2) = \mathbb{Z}, \quad (2.6)$$

the baby skyrmions may have fractional spins, and associated topological charge is known as the Hopf term^[3]

$$H = \frac{1}{4\pi} \int d^3x d^3y \epsilon^{\mu\nu\lambda} j_\mu(x) \frac{(x-y)_\nu}{|x-y|^3} j_\lambda(y), \quad (2.7)$$

where j_μ is the topological current (2.4). We note that the homotopy class (2.6) classifies the time dependent configuration $\vec{n}(\vec{x}, t)$ with the periodic boundary condition $\vec{n}(\vec{x}, -\infty) = \vec{n}(\vec{x}, +\infty)$, while the homotopy class (2.3) classifies the static configuration $\vec{n}(\vec{x})$.

In order to actually confirm the baby skyrmion to have a fractional spin, it is necessary to find an effective interaction, by which, after the 2π rotation of space, the wave functional for the baby skyrmion gains an extra phase $e^{i\theta}$. There are two candidates for such an interaction: one is the nonlocal self interaction through the Hopf term (2.7) and another is the local interaction with the C-S gauge field. In both cases, it can be shown that the baby skyrmion does have a fractional spin.

First, let us consider the $O(3)$ σ -model with the Hopf term, effective action of which is given by

$$S^{O(3)H} = \frac{2}{f^2} \int d^3x \partial_\mu n^a \partial^\mu n^a + \frac{\theta}{4\pi} \int d^3x d^3y \epsilon^{\mu\nu\lambda} j_\mu(x) \frac{(x-y)_\nu}{|x-y|^3} j_\lambda(y). \quad (2.8)$$

In order to determine spin of the baby skyrmion in this action, we have to give both static and adiabatically rotating configurations of the baby skyrmions with

$Q = 1$. These configurations are generally expressed as

$$\frac{n_1 + in_2}{1 + n_3} = \frac{\sin 2F(\tau)}{1 + \cos 2F(\tau)} e^{i\Theta(\varphi)}, \quad (2.9)$$

and

$$\frac{n_1 + in_2}{1 + n_3} = \frac{\sin 2F(\tau)}{1 + \cos 2F(\tau)} e^{i\Theta(\varphi - g(\tau))}, \quad (2.10)$$

respectively, where $F(\tau)$, $\Theta(\varphi)$ and $g(\tau)$ are arbitrary functions that satisfy conditions $F(0) = 0$ and $F(\infty) = \pi/2$, $\Theta(\varphi + 2\pi) = \Theta(\varphi) + 2\pi$, and $g(-\infty) = 0$ and $g(\infty) = 2\pi$. It is known that in the case of exact solution of the field equation, $\sin 2F(\tau) = 2\tau/(\tau^2 + 1)$ and $\cos 2F(\tau) = (\tau^2 - 1)/(\tau^2 + 1)$, and $\Theta(\varphi) = \varphi$. Then, making use of the values of action for the static and the adiabatically rotating configurations which we denote as $S(st)$ and $S(ad)$, respectively, spin (J) of the baby skyrmion can be defined by

$$J = \frac{\Delta S}{2\pi} = \frac{S(ad) - S(st)}{2\pi}. \quad (2.11)$$

In the case of configurations (2.9) and (2.10), the spin of the baby skyrmion is expressed as $J = \Delta S/2\pi = -\theta/2\pi$ ^[3,5]. So, it can be said that the baby skyrmion in the effective action with the Hopf term (2.8) has a fractional spin ($J = -\theta/2\pi$). Note that the value of spin J is decided only by the value of the Hopf term (2.7) for the adiabatically rotating configuration, since $J \propto \Delta S \propto H(ad)$, where $H(ad)$ denotes the value of the Hopf term for adiabatically rotating configuration. Therefore, it can be said that the above argument for determination of the spin depends only on the topology associated with the homotopy class (2.6). Namely, the value of spin estimated by the above method is a topological quantity, since the spin J is directly related to the Hopf term. Indeed, the spin J is invariant under the continuous deformation of the shape ($F(\tau)$ and $\Theta(\varphi)$ in (2.9) and (2.10)) of the baby skyrmion.

Next, let us consider the model coupled with the C-S gauge field, the action of

with a boundary, the boundary condition of the fields should be considered at the boundary of the manifold. Especially in the C-S gauge theory, to impose an appropriate boundary condition is equivalent to restriction of the gauge symmetry in the theory.^[6] In the models under consideration, the boundary is not that of a finite region but the space-time infinity, and the boundary condition corresponds to the fixing of large gauge condition.

The purpose of this paper is to consider, more precisely, the spin of solitons in the models coupled with the C-S gauge field. In particular, we will examine how the large gauge fixing of the C-S gauge field is related to the expression for spin of the solitons, by use of *the first method*, in both the $O(3)$ σ -model coupled with the C-S gauge field and the C-S-H model. Then, we will show that the result for the vortex obtained in the previous paper seems to be a general consequence for solitons in the models coupled with the C-S gauge field; that is to say, in general, even though Lagrangian is given in a model coupled with the C-S gauge field, the model is not yet uniquely fixed, and so spin of the soliton in the model also can not be determined uniquely at this stage. In order to fix the model uniquely and to obtain a plausible expression for the spin of the soliton, it is necessary to impose an appropriate large gauge condition on the C-S gauge field in the model. Indeed, as we will see in Sect.2, spin for the baby skyrmion in the $O(3)$ σ -model coupled with the C-S gauge field becomes consistent with that of the model with the Hopf term, when we choose a gauge condition that $A_\mu(x)$ at space-time infinity satisfies $\partial_\mu A^\mu = 0$. But, if we adopt another gauge condition, spin for the baby skyrmion acquires an expression different from that in the model with the Hopf term.

Moreover, we shall find that value of the spin obtained by *the second method* depends on the shape of the soliton configuration. Thus, *the second method* does not generally give the true expression for spin of the soliton, but only *the first method* can give the plausible expression.

This paper is organized as follows. In Sect.2, spin of the baby skyrmion both in the $O(3)$ σ -model with the Hopf term and in the same model coupled with the C-S gauge field is studied, and we find the importance of the large gauge fixing for determination of the spin of solitons. Next, we discuss the spin of the vortex in the

C-S-H model in Sect.3. We consider the large gauge fixing condition, which leads to the plausible expression for the spin of vortex. In Sect.4, we briefly comment on *the second method* mentioned above, and show that this method does not generally give the true value of spin for the solitons. The final section is devoted to a summary and discussion.

2. Spin of the Baby Skyrmion in the $O(3)$ σ -Model Coupled with the Chern-Simons Gauge Field

Lagrangian of the $O(3)$ σ -model is given by

$$\mathcal{L} = \frac{1}{2f} \sum_{a=1}^3 (\partial_\mu n^a)^2 = \frac{1}{2f} (\partial_\mu \vec{n})^2, \quad (\mu = 0, 1, 2) \quad (2.1)$$

with a constraint

$$\sum_{a=1}^3 (n^a)^2 = (\vec{n})^2 = 1. \quad (2.2)$$

The static configuration $\vec{n}(\vec{x})$ gives a mapping from S^2 to S^2 , and the configuration space is divided into infinite sectors characterized by

$$\Pi_2(S^2) = \mathbb{Z}. \quad (2.3)$$

The associated topological current and the charge are given by

$$j^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} n^a(x) \partial_\nu n^b(x) \partial_\lambda n^c(x), \quad (2.4)$$

and

$$Q = \int d^2x j^0 = \frac{1}{8\pi} \int d^2x \epsilon^{ij} \epsilon_{abc} n^a(\vec{x}) \partial_i n^b(\vec{x}) \partial_j n^c(\vec{x}), \quad (2.5)$$

respectively. The baby skyrmions are the lowest energy configurations in each ($Q = \pm 1, \pm 2, \dots$) sector.^[1]

for the infinity of time direction should satisfy the relation (see Appendix A)

$$\Lambda(\vec{x}, \infty) = \Lambda(\vec{x}, -\infty). \quad (2.19)$$

Substituting the above static and rotating configurations into the action, and using the relation (2.19), we can find the value of spin (J) for the baby skyrmion in terms of the relation (2.11) (see Appendix B):

$$J = \frac{\Delta S}{2\pi} = \frac{S(ad) - S(st)}{2\pi} = -\frac{1}{16\pi\alpha}. \quad (2.20)$$

Note that the above result is independent of the choice of functions $F(\tau)$ and $\Theta(\varphi)$, if we fix boundary conditions of those functions. Namely, the spin of the baby skyrmion also becomes a topological quantity. This fact means that the above calculation is equivalent to that obtained by use of the action in which the C-S gauge field is integrated out under the Landau gauge condition, i.e. the action with the Hopf term which is the topological charge for the homotopy class $\Pi_1(S^2 \rightarrow S^2) = \mathbb{Z}$. Thus, the baby skyrmion in the effective action coupled with the C-S gauge field has also the fractional spin ($J = -1/16\pi\alpha = -\theta/2\pi$), consistently. Namely, the $O(3)$ σ -model coupled with the C-S gauge field with the gauge condition $\partial_\mu A^\mu = 0$ is equivalent to that with the Hopf term. On the other hand, spin of the baby skyrmion in the model with another gauge condition takes the expression different from that in the model with the above gauge condition. For example, if we take the following static and adiabatic configurations for the gauge field, instead of (2.16) and (2.17),

$$\begin{aligned} A_0 &= 0, \\ A_i &= -\frac{1}{8\pi\alpha} \frac{x_i}{r} \sin(2F(\tau)) F'(\tau) \Theta(\varphi), \end{aligned} \quad (2.21)$$

and

$$\begin{aligned} A_0 &= 0, \\ A_i &= -\frac{1}{8\pi\alpha} \frac{x_i}{r} \sin(2F(\tau)) F'(\tau) \Theta(\varphi - g(t)), \end{aligned} \quad (2.22)$$

then we obtain $J = 0$. Thus, under the gauge conditions different from that of $\partial_\mu A^\mu = 0$, the model does not become equivalent to that with the Hopf term.

Finally, we note the fact that the value of spin for the baby skyrmion depends on the large gauge fixing holds not only in the case of $Q = 1$ but also in the case of higher soliton number ($Q \geq 2$).

3. Spin of Vortex in the Chern-Simons-Higgs Model

In this section, we consider spin of soliton in the C-S-H model. Lagrangian of the C-S-H model is given by

$$\mathcal{L}^{CSH} = |(\partial_\mu + ieA_\mu)\phi|^2 + \frac{1}{4}\kappa\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} - V(|\phi|), \quad (3.1)$$

where ϕ is a complex scalar field and we assume that the potential $V(|\phi|)$ has a symmetry breaking minimum at $|\phi| = v$ as well as a symmetric minimum at $\phi = 0$.

By use of the polar coordinate (r, φ) , boundary condition of the static configurations with finite energy should be $\phi \rightarrow ve^{i\alpha(\varphi)}$ as $r \rightarrow \infty$. This fact means that static configurations are classified by

$$\Pi_1(U(1)) = \mathbb{Z}. \quad (3.2)$$

So, there exist topologically stable static soliton solutions (called vortices) in this model. By analogy to the case of the $O(3)$ σ -model, it is expected that the value of spin for the vortex is determined by nontriviality of the homotopy $\Pi_1(S^1 \rightarrow S^1) = \mathbb{Z}$,[#] and should be obtained as a topological invariant, which does not depend on the shape of vortex configuration.

The Lagrangian is invariant under local gauge transformation;

$$\begin{aligned} \phi(\vec{x}, t) &\rightarrow e^{i\Lambda(\vec{x}, t)} \phi(\vec{x}, t), \\ A_\mu(\vec{x}, t) &\rightarrow A_\mu(\vec{x}, t) - \frac{1}{e} \partial_\mu \Lambda(\vec{x}, t), \end{aligned} \quad (3.3)$$

[#] Private communication from Prof. Kazuyuki Fujii.

which is given by

$$S^{O(3)cs} = \int d^3x \left[\frac{2}{f^2} \partial_\mu n^a \partial^\mu n^a + A_\mu{}^\mu + \alpha \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} \right], \quad (2.12)$$

where j_μ is the topological current (2.4). The action is invariant under local gauge transformation;

$$A_\mu(\vec{x}, t) \rightarrow A_\mu(\vec{x}, t) - \partial_\mu \Lambda(\vec{x}, t), \quad (2.13)$$

when the gauge function $\Lambda(\vec{x}, t)$ satisfy the condition

$$\Lambda \rightarrow 0 \quad \text{as} \quad |\vec{x}| \rightarrow \infty, \quad |t| \rightarrow \infty. \quad (2.14)$$

Note, however, that it is not invariant under the large gauge transformation, where gauge function $\Lambda(\vec{x}, t)$ does not become a constant at space-time infinity. When we evaluate the spin of the baby skyrmion in the $O(3)$ σ -model coupled with the C-S gauge field, we have to impose a gauge fixing condition on the local symmetry on the finite region of the model. Usually under such a condition, however, the gauge is fixed not only on the finite region of the space-time, but also at the space-time infinity. This fact implies that there is no freedom to choose the gauge fixing condition, since the model is not invariant under the large gauge transformations. Namely, values of some physical quantities of the model which are related to the space-time infinity may be changed according as the choice of gauge condition. For example, spin of the baby skyrmion calculated by this method is just a quantity which is invariant under the gauge transformation on the finite region of the space-time, but is not independent of the choice of the large gauge. Then, the expression for the spin of baby skyrmion is changed, according as the choice of the gauge fixing conditions. Therefore, an appropriate large gauge condition (appropriate fixing of configuration at the space-time infinity) is needed to determine the spin of baby skyrmion uniquely. It is known that the $O(3)$ σ -model coupled with the C-S gauge field is equivalent to the $O(3)$ σ -model with Hopf term, when gauge condition,

$$\partial_\mu A^\mu = 0 \quad (2.15)$$

is satisfied. Indeed, when the gauge field A_μ is integrated out under the gauge condition (2.15), $O(3)$ σ -model coupled with the C-S gauge field is reduced to that

with the Hopf term with $\theta = \frac{1}{8\pi\alpha}$. Therefore, the baby skyrmion in this case has also a fractional spin; $J = -\frac{1}{16\pi\alpha}$.

We can also directly determine the spin of the baby skyrmion in the model coupled with the C-S gauge field. Let us consider, again, both the static and the adiabatically rotating configurations of the baby skyrmion in this model. These configurations are generally expressed as

$$\begin{aligned} \frac{n_1 + in_2}{1 + n_3} &= \frac{\sin 2F(r)}{1 + \cos 2F(r)} e^{i\Theta(\varphi)}, \\ A_0 &= 0, \\ A_i &= -\frac{1}{8\pi\alpha} \epsilon_{ij} \frac{x_j}{r^2} \sin^2 F(r) \Theta'(\varphi) - \partial_i \Lambda(\vec{x}), \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \frac{n_1 + in_2}{1 + n_3} &= \frac{\sin 2F(r)}{1 + \cos 2F(r)} e^{i\Theta(\varphi - g(t))}, \\ A_0 &= -\frac{1}{8\pi\alpha} \sin^2 F(r) \Theta'(\varphi - g(t))(-\dot{g}(t)) \\ &\quad + \frac{1}{8\pi\alpha} \Theta'(-g(t))(-\dot{g}(t)) - \partial_0 \Lambda(\vec{x}, t), \\ A_i &= -\frac{1}{8\pi\alpha} \epsilon_{ij} \frac{x_j}{r^2} \sin^2 F(r) \Theta'(\varphi - g(t)) - \partial_i \Lambda(\vec{x}, t), \end{aligned} \quad (2.17)$$

respectively. Here, Λ 's in (2.16) and (2.17) are chosen appropriately, so that A_μ 's in both cases satisfy the same gauge condition $\partial_\mu A^\mu = 0$. Inserting (2.16) or (2.17) into this condition, we find that the Λ 's should satisfy the following equations

$$\partial_i \partial^i \Lambda(\vec{x}) = \frac{1}{8\pi\alpha} \frac{\sin^2 F(r)}{r^2} \Theta''(\varphi), \quad (2.18a)$$

and

$$\begin{aligned} \partial_\mu \partial^\mu \Lambda(\vec{x}, t) &= \frac{1}{8\pi\alpha} \left[\frac{\sin^2 F(r)}{r^2} \Theta''(\varphi - g(t)) \right. \\ &\quad \left. + \{\sin^2 F(r) \Theta'(\varphi - g(t)) - \Theta'(-g(t))\}(\dot{g}(t)) \right. \\ &\quad \left. - \{\sin^2 F(r) \Theta''(\varphi - g(t)) - \Theta''(-g(t))\}(\dot{g}(t))^2 \right]. \end{aligned} \quad (2.18b)$$

Unfortunately, it is hard to obtain a solution of the equation (2.18) for any $F(r)$, $\Theta(\varphi)$ and $g(t)$. It can be found out, however, that the boundary value of $\Lambda(\vec{x}, t)$

different form at space-time infinity from that is satisfied by (3.9). Inserting (3.13) instead of (3.9) into (3.10), the action becomes^[5]

$$\Delta S = -\frac{\kappa}{2e^2}4\pi^2 n^2 + \frac{\kappa}{2e^2}4\pi^2 mn. \quad (3.14)$$

Thus, form of the expression for the spin is generally changed according to the choice of large gauge which corresponds to the choice of time dependent function $\Lambda(\vec{x}, t)$. The situation is similar to the case of $O(3)$ σ -model. Namely, so far, we have not considered problem of the gauge fixing. As seen in Sect. 2, however, the method is meaningless, unless an appropriate gauge condition is required. Therefore, what we have to do is to find out the consistent gauge condition such as (2.15) in case of the $O(3)$ σ -model.

Now, in order to find the appropriate gauge condition, let us consider the deformation of shape of vortex configurations. As mentioned in the case of $O(3)$ σ -model, the value of spin of soliton (vortex) should not depend on shape of the soliton configuration but depends only on the soliton number. The deformed static configuration is expressed by

$$\begin{aligned} \phi &= vG(r)e^{in\Theta(\varphi)}, \\ eA_i &= \epsilon_{ij}\frac{x_j}{r^2}[a(r) - n]\Theta'(\varphi), \\ eA_0 &= -\frac{\kappa n}{2e} \cdot \frac{a'(r)}{v^2 r G^2(r)} \Theta'(\varphi), \end{aligned} \quad (3.15)$$

with the same soliton number as (3.5), where $\Theta(\varphi)$ and $a'(r)$ are φ -derivative of $\Theta(\varphi)$ and r -derivative of $a(r)$, respectively. In (3.15), $\Theta(\varphi)$ is arbitrary function which satisfies a condition $\Theta(\varphi+2\pi) = \Theta(\varphi) + 2\pi$. At the first guess, it is natural to adopt the following Ansatz as the adiabatically rotating configuration of the deformed vortex (3.15),

$$\begin{aligned} \phi &= vG(r)e^{in\Theta(\varphi-g(t))}, \\ eA_i &= \epsilon_{ij}\frac{x_j}{r^2}[a(r) - n]\Theta'(\varphi - g(t)), \\ eA_0 &= -\frac{\kappa n}{2e} \cdot \frac{a'(r)}{v^2 r G^2(r)} \Theta'(\varphi - g(t)) + n\Theta'(\varphi - g(t))\dot{g}(t). \end{aligned} \quad (3.16)$$

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As a result, however, we obtain

$$\Delta S = S(ad) - S(st) = -\frac{\pi\kappa n^2}{e^2} \frac{1}{2\pi} \int |\Theta'(\varphi - g(t))|^2 \dot{g}(t) d\varphi dt. \quad (3.17)$$

Then, spin of the vortex is represented as

$$J = \frac{\Delta S}{2\pi} = -\frac{\kappa n^2}{4\pi e^2} \int |\Theta'(\varphi - g(t))|^2 \dot{g}(t) d\varphi dt. \quad (3.18)$$

Thus, it seems that the value of the spin of vortex depends on the shape of vortex. However, A_μ in (3.16) does not satisfy the same gauge condition as in (3.9). As mentioned before, it is necessary to define the spin by using the Ansatz satisfying a fixed gauge condition. For example, let us first adopt the Landau gauge $\partial_\mu A^\mu = 0$. In this case, one can find $\Delta S = 0$, and so the spin J also becomes zero. Moreover, this value is independent of the shape of configuration. Therefore, it can be recognized as topologically invariant quantity. However, the value is undesirable one. Then, let us next seek for the (large) gauge condition, in which the value of spin for the vortex becomes fractional in the topologically invariant way. We consider the following Ansatz,

$$\begin{aligned} \phi &= vG(r)e^{i\Lambda(\vec{x}, t)} e^{in\Theta(\varphi-g(t))}, \\ eA_i &= \epsilon_{ij}\frac{x_j}{r^2}[a(r) - n]\Theta'(\varphi - g(t)) - \partial_i \Lambda(\vec{x}, t), \\ eA_0 &= -\frac{\kappa n}{2e} \cdot \frac{a'(r)}{v^2 r G^2(r)} \Theta'(\varphi - g(t)) + n\Theta'(\varphi - g(t))\dot{g}(t) - \partial_0 \Lambda(\vec{x}, t), \end{aligned} \quad (3.19)$$

with

$$\Lambda(\vec{x}, t) = -f(r)[\Theta(\varphi - g(t)) - \Theta(\varphi) + g(t)], \quad (3.20)$$

where $f(r)$ is arbitrary function with the boundary conditions $f(0) = 0$ and $f(\infty) = n$. Unfortunately, we have not yet found the explicit form of the gauge condition which is written only in terms of a field A_μ , and solution of which is (3.20). It can be found, however, that using the configurations (3.15) and (3.19), expression of

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where the gauge function $\Lambda(\vec{x}, t)$ satisfies the condition

$$\Lambda(\vec{x}, t) \rightarrow 0 \text{ as } r \rightarrow \infty, |t| \rightarrow \infty. \quad (3.4)$$

Note, however, that it is not invariant under large gauge transformation, where the gauge function $\Lambda(\vec{x}, t)$ does not become a constant at space-time infinity.

It is known that, taking the Ansatz for static soliton solutions as

$$\begin{aligned} \phi &= vG(r)e^{in\varphi}, \\ eA_i &= \epsilon_{ij} \frac{x_j}{r^2} [a(r) - n], \end{aligned} \quad (3.5)$$

the vortex solutions with boundary conditions

$$\begin{aligned} G(0) &= 0, \quad a(0) = n, \\ G(r) &\rightarrow 1, \quad a(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty, \end{aligned} \quad (3.6)$$

can be obtained.[2] The time component of the C-S gauge field is the auxiliary field with no degrees of freedom, and is solved as

$$A_0 = \frac{1}{2e^2} \frac{\kappa B}{|\phi|^2}, \quad (3.7)$$

where B denotes the magnetic field for C-S gauge field. Topological charge of the vortices is represented by the quantized magnetic flux number

$$\Phi = \int B d^2x = \frac{2\pi}{e} [a(0) - a(\infty)] = \frac{2\pi n}{e}. \quad (3.8)$$

Now, we will examine spin of the vortex, applying the method which was illustrated in the Sect.2, to the action (3.1) and the vortex configuration (3.5). The static Ansatz (3.5) suggests that the following Ansatz is valid for adiabatically

rotating configuration with finite energy;

$$\begin{aligned} \phi &= vG(r)e^{in(\varphi - g(t))}, \\ eA_i &= \epsilon_{ij} \frac{x_j}{r^2} [a(r) - n]. \end{aligned} \quad (3.9a)$$

In this case the time component of the C-S gauge field is solved as

$$A_0 = \frac{1}{2e^2} \frac{\kappa B + e j_0}{|\phi|^2} = -\frac{\kappa n}{2e^2} \cdot \frac{a'(r)}{v^2 r G^2(r)} + \frac{n}{e} g'(t). \quad (3.9b)$$

We get $S(st)$ and $S(ad)$ by substituting (3.5) and (3.9) into the action

$$S^{CSH} = \int d^3x \left[|(\partial_\mu + ieA_\mu)\phi|^2 + \frac{1}{4} \kappa \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} - V(|\phi|) \right]. \quad (3.10)$$

After some careful calculations, we find that the action becomes

$$\Delta S \equiv S(ad) - S(st) = -\frac{\kappa}{2e^2} 4\pi^2 n^2. \quad (3.11)$$

This fact means that the spin of the charged vortex is expressed as

$$J = -\frac{\pi\kappa}{e^2} n^2. \quad (3.12)$$

At first glance, it seems that this result is consistent with that obtained in terms of classical argument on the extra angular momentum of vortex (see Sect.4).[2]

However, the expression for the spin (3.12) is changed with the choice of gauge in general, since the action (3.10) is not invariant under the large gauge transformations. For example, we may consider the following Ansatz as the rotating configuration

$$\begin{aligned} \tilde{\phi} &= e^{i\Lambda} \phi, \\ \tilde{A}_\mu &= A_\mu - \frac{1}{e} \partial_\mu \Lambda, \end{aligned} \quad (3.13)$$

where ϕ and A_μ are the ordinary rotating configurations (3.9), and $\Lambda(\vec{x}, t) = \alpha(t)$ with $\alpha(\infty) - \alpha(-\infty) = 2\pi n$. The gauge condition which (3.13) satisfies has

invariant quantity under the gauge transformation with any gauge function $\Lambda(\vec{x}, t)$, and so we can not change the value obtained in (4.4) by means of appropriate choice of gauge, in contrast to the method discussed in the previous section. Then, this method does not correspond to the topological argument of the previous section.

Next, let us examine the extra angular momentum of the baby skyrmion. There are two types of possible system; that is, the $O(3)$ σ -model with the Hopf term, or that coupled with the C-S gauge field. In the following argument, we may consider whichever system, and the same result can be obtained. For example, by using either (2.8) and (2.12) we obtain the energy-momentum tensor $T_{\mu\nu}$

$$T_{\mu\nu} = \frac{2}{f^2}(2\partial_\mu n^a \partial_\nu n^a - g_{\mu\nu} \partial_\alpha n^a \partial^\alpha n^a). \quad (4.5)$$

Since the $(0, j)$ -component of $T_{\mu\nu}$ explicitly involves a time derivative of n^a , there is no extra angular momentum for the static baby skyrmion:

$$J = \int d^2x \epsilon_{ij} x_i T_{0j} = \frac{4}{f^2} \int d^2x \epsilon_{ij} x_i n^a \partial_j n^a = 0. \quad (4.6)$$

This result does not depend on the shape of configuration of the baby skyrmion. The intuitive reason for the result is that the baby skyrmion can not be regarded as dyon, since electric field around the baby skyrmion is zero, in contrast with the case of the charged vortex in the C-S-H model. Thus, within the framework of this argument, the baby skyrmion would not have fractional spin, and this fact is inconsistent with the results obtained in the previous section.

Thus, J which is calculated from classical energy-momentum tensor should not be considered as the spin of soliton. It was pointed out, however, in ref. [7] that J for the baby skyrmion can be identified with the fractional spin, if we take account of canonical quantization of the system, and consider J as the operator. In this case, the spin of the baby skyrmion becomes a topologically invariant quantity. In the case of the vortex in the C-S-H model, classical J is not topological invariant. However, J for the vortex may become topologically invariant spin, if we quantize the system of the C-S-H model. This is an open problem.

5. Summary and Discussion

In Sect. 2 and 3, we have studied whether the solitons in the models coupled with the C-S gauge field have fractional spins or not, by seeing the multi-valuedness of the wave function of the solitons. As a result, we have shown that, in general, the expression for the spin of solitons depends on the large gauge fixing condition for the C-S gauge field. For example, in case of the $O(3)$ σ -model coupled with C-S gauge field, the baby skyrmion has the fractional spin in the Landau gauge $\partial_\mu A_\mu = 0$. This result is consistent with the fact that $O(3)$ σ -model with C-S gauge field is equivalent to the $O(3)$ σ -model with the Hopf term, in which the baby skyrmion actually has fractional spin. Under another gauge condition, the expression for the spin of the baby skyrmion does not become consistent with that of the model with the Hopf term. Next, in case of the C-S-H model, expression for the spin of vortex also depends on the large gauge condition. In this case, under the Landau gauge condition, the spin of vortex does not become fractional, but necessarily becomes zero. Alternatively, if we choose the gauge condition given by (3.20), the expression for the spin of vortex becomes fractional one.

When we evaluate the spin of soliton in a model coupled with the C-S gauge field, we impose a gauge condition on the model. Usually under such a condition, the gauge is fixed not only on the finite region of the space-time, but also on the space-time infinity. However, the model is not invariant under the large gauge transformations which are gauge transformations at the space-time infinity. This fact means that there is no freedom to choose the gauge fixing condition. Namely, values of some physical quantities of the model which are related to the space-time infinity may be changed according as the choice of gauge condition, since the large gauge at the space-time infinity may also be transformed according as the condition, which is not allowed in the model. For example, spin of the soliton obtained by this method is just a quantity which is invariant under the gauge transformation on the finite region of the space-time, but is not independent of the choice of the large gauge. Therefore, expression for the spin of soliton is changed, according as the (large) gauge fixing conditions.

Furthermore, in Sect. 4, we have examined another method to show solitons to

the spin becomes

$$J = \frac{\Delta S}{2\pi} = \frac{S(ad) - S(st)}{2\pi} = -\frac{\pi\kappa^2}{e^2} n^2. \quad (3.21)$$

This has a desirable form and is the same as that obtained before, (3.12). Moreover, it is obvious that, under the gauge condition which is satisfied by A_μ in (3.19) with (3.20), J is independent of the shape of configuration, and so it can be recognized as topological invariant. This fact implies that J in (3.21) is proportional to the topological charge of $\Pi_1(S^1 \rightarrow S^1)$. Note that the above result can be obtained, when the (large) gauge condition, which is satisfied by A_μ in (3.19), is realized.

4. Comments on the Extra Angular Momentum of Static Solitons

It can be shown that the solitons in some models with the C-S term can often be regarded as dyons, and dyon in the (2+1)-dimensional system has classical extra angular momentum, even if the dyon does not rotate. Then, it has been believed that the value of the extra angular momentum could be identified with that of the intrinsic fractional spin. Indeed, some authors pointed out that the vortex in the C-S-H model has fractional spin, based on this consideration. So, in this section, it is instructive to compare this method with those explained in the previous two sections.

Procedure of the method is as follows: First, we derive expression of the energy-momentum tensor $T_{\mu\nu}$ from the Lagrangian. Next, by means of the (0, j)-component of $T_{\mu\nu}$, the extra angular momentum of soliton, which could be regarded as the spin of soliton, is expressed as

$$J = \int d^2x \epsilon_{ij} x_i T_{0j}. \quad (4.1)$$

Then, the value of angular momentum for the soliton is estimated by substituting the static soliton solution into (4.1).

First, let us estimate J of the vortex in the C-S-H model, using the most general form of the configuration, (3.15). From the Lagrangian of the C-S-H model (3.1), (0, j)-component of the energy-momentum tensor is calculated as

$$T_{0j} = \frac{1}{2} i e A_0 (\phi^* \partial_j \phi - \partial_j \phi^* \phi) + e^2 A_0 A_j |\phi|^2. \quad (4.2)$$

Substituting the Ansatz of the general vortex solution (3.15) with soliton number n ;

$$\begin{aligned} \phi &= v G(r) e^{in\Theta(\varphi)}, \\ e A_i &= \epsilon_{ij} \frac{x_j}{r^2} [a(r) - n] \Theta'(\varphi), \\ e A_0 &= \frac{1}{2e} \frac{\kappa B}{|\phi|^2} = -\frac{\kappa n}{2e} \frac{a'(r)}{v^2 r G^2(r)} \Theta'(\varphi), \end{aligned} \quad (3.15)$$

with the boundary conditions

$$\begin{aligned} G(0) &= 0, \quad a(0) = n, \\ G(r) &\rightarrow 1, \quad a(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty, \\ \Theta(\varphi + 2\pi) &= \Theta(\varphi) + 2\pi, \end{aligned} \quad (4.3)$$

into (4.1), the angular momentum of the static vortex is obtained as

$$\begin{aligned} J &= \int d^2x \epsilon_{ij} x_i T_{0j} \\ &= \int d^2x B \epsilon_{ij} x_i \{ \partial_j \text{Arg}(\phi) - e A_j \} \\ &= -\frac{\pi\kappa}{e^2} \times \frac{n^2}{2\pi} \int_0^{2\pi} \{ \Theta'(\varphi) \}^2 d\varphi. \end{aligned} \quad (4.4)$$

Thus, it is obvious that in general the extra angular momentum depends on the shape of soliton configuration, and is of the same form as that of (3.18). The result coincides with (3.12) or (3.21), when $\Theta(\varphi) = \varphi$, by which the exact soliton solution can be obtained.^[2] Since the extra angular momentum of classical vortex can have various values, it can not be regarded as the topologically invariant quantity in this method. Moreover, one can find that the value of extra angular momentum J is an

rotation of the baby skyrmion, $A_\mu(\vec{x})$ has to be the same forms at $t = -\infty$ and $t = \infty$. This requirement leads to the relation

$$\partial_\mu \Lambda|_{t=\infty} = \partial_\mu \Lambda|_{t=-\infty}, \quad \text{for } \mu = 0, 1, 2. \quad (A.6)$$

Then, the boundary conditions for $\Lambda(\vec{x}, t)$ at $t = \pm\infty$ satisfy the relation

$$\Lambda(\vec{x}, \infty) - \Lambda(\vec{x}, -\infty) = \text{constant}. \quad (A.7)$$

If we take $\Lambda(\vec{x}, t) = 0$ in (A.1), then

$$\begin{aligned} \partial^\mu A_\mu &\equiv \Omega = \frac{H(r)}{r^2} \Theta''(\varphi - g(t)) \\ &+ \{H(r) \Theta'(\varphi - g(t)) - \frac{1}{8\pi\alpha} \Theta'(-g(t))\}(\ddot{g}(t)) \\ &- \{H(r) \Theta''(\varphi - g(t)) - \frac{1}{8\pi\alpha} \Theta''(-g(t))\}(\dot{g}(t))^2. \end{aligned} \quad (A.8)$$

So, it is found that the gauge field configuration in (A.1) satisfies the Landau gauge condition, $\partial^\mu A_\mu = 0$, when the following relation holds:

$$\partial^\mu A_\mu = \Omega - (\partial_0^2 - \partial_1^2 - \partial_2^2) \Lambda(\vec{x}, t) = 0. \quad (A.9)$$

In particular, at $r = \infty$ and $\varphi = 0$, where $\Omega = 0$ (see (A.8)), $\Lambda(r = \infty, \varphi = 0, t) = 0$ satisfies the condition (A.9) at any time. (in general $\Lambda(r = \infty, \varphi = 0, t) = \text{constant}$). Therefore, it is found from (A.7) at $r = \infty$ and $\varphi = 0$ that the constant value on the right hand side in (A.7) becomes zero as follows;

$$\begin{aligned} \text{constant} &= \Lambda(\vec{x}, \infty) - \Lambda(\vec{x}, -\infty) \\ &= \Lambda(r = \infty, \varphi = 0, t = \infty) - \Lambda(r = \infty, \varphi = 0, t = -\infty) \\ &= 0. \end{aligned} \quad (A.10)$$

This leads to the boundary condition

$$\Lambda(\vec{x}, \infty) = \Lambda(\vec{x}, -\infty). \quad (2.19)$$

Appendix B

In this appendix, we derive the result (2.20). As seen in the Appendix A, Ansatz of the gauge field and the topological current for the baby skyrmion are expressed as

$$\begin{aligned} A_1 &= \frac{\sin \varphi}{r} H(r) \Theta'(\varphi - g(t)) - \partial_1 \Lambda(\vec{x}, t), \\ A_2 &= -\frac{\cos \varphi}{r} H(r) \Theta'(\varphi - g(t)) - \partial_2 \Lambda(\vec{x}, t), \\ A_0 &= -H(r) \Theta'(\varphi - g(t))(-\dot{g}(t)) + \frac{1}{8\pi\alpha} \Theta'(-g(t))(-\dot{g}(t)) - \partial_0 \Lambda(\vec{x}, t), \end{aligned} \quad (A.11)$$

and

$$\begin{aligned} j^0 &= \frac{1}{2\pi} (\sin^2 F(r))' \Theta'(\varphi - g(t)), \\ j^1 &= \frac{1}{2\pi} \sin \varphi (\sin^2 F(r))' \Theta'(\varphi - g(t))(-\dot{g}(t)), \\ j^2 &= -\frac{1}{2\pi} \cos \varphi (\sin^2 F(r))' \Theta'(\varphi - g(t))(-\dot{g}(t)), \end{aligned} \quad (A.12)$$

respectively. Here, $g(t) \equiv 0$ for static configurations, and $g(-\infty) = 0$ and $g(\infty) = 2\pi$ for adiabatically rotating configurations in both (A.1) and (A.2). $H(r)$ is associated with $F(r)$ as

$$H(r) = \frac{1}{8\pi\alpha} \sin^2 F(r), \quad (A.13)$$

and satisfies the boundary condition

$$\begin{aligned} H(0) &= 0, \\ H(\infty) &= \frac{1}{8\pi\alpha}. \end{aligned} \quad (A.14)$$

By use of the field equations (A.3), the action (2.12) is rewritten as

$$\begin{aligned} S^{O(3)CS} &= \int d^3x \left[\frac{2}{f^2} \partial_\mu n^a \partial^\mu n^a + A_\mu j^\mu + \alpha \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} \right] \\ &= \int d^3x \left[\frac{2}{f^2} \partial_\mu n^a \partial^\mu n^a + \frac{1}{2} A_\mu j^\mu \right]. \end{aligned} \quad (B.1)$$

Now, let us define $\tilde{S}$ as

$$\tilde{S} = \int \frac{1}{2} A_\mu j^\mu d^3x, \quad (B.2)$$

have fractional spins, which is based on a rather classical argument. It has been found that, in both models, the spins calculated by the method explained in Sect. 2 and 3 can be regarded as topological invariants, since the spins are independent of the shape of static soliton configurations. On the other hand, the value of spin calculated by the method explained in Sect. 4 depends on the shape of static soliton configuration. Therefore, it can be expected that only the former method gives the true values of spin for solitons and is consistent with the topological argument for the possibility that the solitons have fractional spins.

Finally, as mentioned in Sect. 3, we have not yet found the explicit form of the gauge condition expressed only in terms of field variables, by means of which the expression for the spin of vortex becomes plausible one. So, we have to look for the explicit form of the gauge condition, which A_μ in (3.19) with (3.20) satisfies. So far, we have examined only spin of solitons, and have not considered statistics of the solitons. It is not manifest whether the statistics of soliton is also consistently determined by the same large gauge fixing condition, or not. So, further investigation is needed to confirm that solitons can actually be considered as anyons.

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Appendix A

In this Appendix, we derive the relation (2.19). The Ansatz of gauge fields for the $O(3)$ σ -model is given by

$$\begin{aligned} A_1 &= \frac{\sin \varphi}{r} H(\tau) \Theta'(\varphi - g(t)) - \partial_1 \Lambda(\vec{x}, t), \\ A_2 &= -\frac{\cos \varphi}{r} H(\tau) \Theta'(\varphi - g(t)) - \partial_2 \Lambda(\vec{x}, t), \\ A_0 &= -H(\tau) \Theta'(\varphi - g(t))(-\dot{g}(t)) + \frac{1}{8\pi\alpha} \Theta'(-g(t))(-\dot{g}(t)) - \partial_0 \Lambda(\vec{x}, t). \end{aligned} \quad (A.1)$$

The explicit form of the topological current $j^\mu(x)$ for the Ansatz (2.10) are expressed as

$$\begin{aligned} j^0 &= \frac{1}{2\pi r} (\sin^2 F(\tau))' \Theta'(\varphi - g(t)), \\ j^1 &= \frac{1}{2\pi} \sin \varphi (\sin^2 F(\tau))' \Theta'(\varphi - g(t))(-\dot{g}(t)), \\ j^2 &= -\frac{1}{2\pi} \cos \varphi (\sin^2 F(\tau))' \Theta'(\varphi - g(t))(-\dot{g}(t)). \end{aligned} \quad (A.2)$$

If $g(t) \equiv 0$ in both (A.1) and (A.2), these configurations stand for the static configurations. On the other hand, these become the adiabatically rotating configurations by 2π , when $g(t)$ is a function with boundary conditions $g(-\infty) = 0$, $g(\infty) = 2\pi$.

In terms of the field equation for $A_\mu(x)$

$$\epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda(x) = -\frac{1}{4\alpha} j^\mu(x), \quad (A.3)$$

$H(\tau)$ is related to $F(\tau)$ through the expression

$$H(\tau) = \frac{1}{8\pi\alpha} \sin^2 F(\tau), \quad (A.4)$$

and satisfies the boundary conditions

$$\begin{aligned} H(0) &= 0, \\ H(\infty) &= \frac{1}{8\pi\alpha}. \end{aligned} \quad (A.5)$$

(Configurations for the gauge fields in (2.16) and (2.17) are those of (A.1), where the relation (A.4) holds.) In order to make the configurations (A.1) represent 2π

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ΔS can be expressed, by means of $\tilde{S}$, as

$$\begin{aligned}\Delta S &= S(ad) - S(st) \\ &= \tilde{S}(ad) - \tilde{S}(st),\end{aligned}\quad (B.3)$$

where $\tilde{S}(ad)$ and $\tilde{S}(st)$ represent the action $\tilde{S}$ for adiabatically rotating configuration and static configuration, respectively. Now, we estimate the value of ΔS substituting (A.1) and (A.2) into (B.3). First of all, it is easily found from (A.1) and (A.2) that $A^\mu j_\mu \equiv 0$ in the case of static configuration, and so

$$\tilde{S}(st) = 0. \quad (B.4)$$

Next, $\tilde{S}(ad)$ is calculated as follows;

$$\begin{aligned}\tilde{S}(ad) &= \frac{1}{2} \int \left[\frac{\sin \varphi}{r} H(r) \Theta'(\varphi - g(t)) \cdot j^1 + \frac{-\cos \varphi}{r} H(r) \Theta'(\varphi - g(t)) \cdot j^2 \right. \\ &\quad \left. + \left\{ -H(r) \Theta'(\varphi - g(t)) + \frac{1}{8\pi\alpha} \Theta'(-g(t)) \right\} (-\dot{g}(t)) \cdot j^0 \right] d^3x \\ &\quad - \frac{1}{2} \int \partial_\mu \Lambda \cdot j^\mu d^3x \\ &\equiv S_1 + S_2.\end{aligned}\quad (B.5)$$

In (B.5), S_1 denotes combination of the terms independent of Λ in $\tilde{S}(ad)$, while S_2 denotes the last term which includes Λ . S_1 can be calculated as follows;

$$\begin{aligned}S_1 &= \frac{1}{2} \int \left[\frac{\sin \varphi}{r} H(r) \Theta'(\varphi - g(t)) \cdot \frac{1}{2\pi} \sin \varphi (\sin^2 F(r))' \Theta'(\varphi - g(t)) (-\dot{g}(t)) \right. \\ &\quad \left. + \frac{-\cos \varphi}{r} H(r) \Theta'(\varphi - g(t)) \cdot \frac{-1}{2\pi} \cos \varphi (\sin^2 F(r))' \Theta'(\varphi - g(t)) (-\dot{g}(t)) \right. \\ &\quad \left. + \left\{ -H(r) \Theta'(\varphi - g(t)) + \frac{1}{8\pi\alpha} \Theta'(-g(t)) \right\} (-\dot{g}(t)) \right. \\ &\quad \left. \times \left\{ \frac{1}{2\pi r} (\sin^2 F(r))' \Theta'(\varphi - g(t)) \right\} \right] d^3x \\ &= -\frac{1}{32\pi^2\alpha} \int_0^\infty (\sin^2 F(r))' dr \times \int_0^{2\pi} d\varphi \int_{-\infty}^\infty dt \Theta'(-g(t)) \Theta'(\varphi - g(t)) \dot{g}(t) \\ &= -\frac{1}{8\alpha},\end{aligned}\quad (B.6)$$

where we used boundary conditions for $F(r)$, $\Theta(\varphi)$ and $g(t)$ to derive the third line

of the right hand side. Next, S_2 is expressed as

$$\begin{aligned}S_2 &= -\frac{1}{2} \int \partial_\mu \Lambda \cdot j^\mu d^3x \\ &= -\frac{1}{2} \left\{ \int \Lambda \cdot j^0 d^2x \right\}_{t=-\infty}^{t=\infty} + \int dt \int \vec{C} \cdot d\vec{x},\end{aligned}\quad (B.7)$$

where

$$\begin{aligned}\vec{C} &\equiv (-\Lambda \cdot j^2, \Lambda \cdot j^1), \\ d\vec{x} &= (-r \sin \varphi d\varphi, r \cos \varphi d\varphi).\end{aligned}\quad (B.8)$$

From the explicit form of $\vec{C}$ in (B.8) and (A.2), it is easy to see that $\vec{C} \cdot d\vec{x} = 0$, and then the second term of (B.7) becomes zero. Furthermore, in terms of the relation

$$\Lambda(\vec{x}, \infty) = \Lambda(\vec{x}, -\infty), \quad (2.19)$$

the first term of (B.7) also becomes zero. Then, we obtain $S_2 = 0$. Therefore, it is found that

$$\begin{aligned}\Delta S &= \tilde{S}(ad) - \tilde{S}(st) \\ &= S_1 + S_2 \\ &= -\frac{1}{8\alpha}.\end{aligned}\quad (B.9)$$

Thus, we derive (2.20). Note that the field equation for $\vec{n}$ (or for $F(r)$) was not used in the above calculations, while we used the field equation (A.3).