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COHERENT STATES IN FINITE QUANTUM MECHANICS⁺)

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ABSTRACT

The "harmonic oscillator", introduced by Itzykson and Balian, is considered, on finite phase-space lattices with the structure of Galois fields $GF[p^2]$. We study this system at the classical and quantum mechanical level. Explicit construction of the quantum eigenstates provides us with a detailed understanding of the spectrum of scales relevant to this model. We construct also the coherent states appropriate to the finite Heisenberg-Weyl and Affine groups. Possible applications are discussed.

⁺) Work partially supported by the EEC grants, SCI-430C, CHRX-CT92-0063

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Introduction

Possible relevance of number theory in today's Physical theories, has been recently discussed in various contexts, ranging from discrete dynamical chaotic systems to field and string theories of fundamental particle interactions [1-5].

Also, introduction of quantum mechanical ideas, such as coherent states and representations of symmetries in the area of discrete algorithms for signal and image processing have recently attracted interest in the Physics Community [6]. So it seems to be an interesting problem the understanding of Quantum Mechanics and Quantum Field Theory on lattices over finite Galois fields [7].

Quantum Mechanics on a finite lattice, (FQM), was introduced by H. Weyl, [8] and later on it was discussed in detail by J. Schwinger [9]. Applications of (FQM) to the physics of the Quantum Hall (QH) effect started in the work of J. Zak and continued recently by J. Bellissard and collaborators [10], where the connection of QH effect with non-commutative geometry was emphasized [11].

In this paper we examine FQM of one degree of freedom over the simplest Galois field $GF[p]$, [7]. We consider in detail, the classical and quantum "harmonic oscillator" introduced by R. Balian and C. Itzykson (B-I) [12]. In chapter 1 we discuss aspects of Classical Mechanics of discrete maps on the Galois field $GF[p^2]$ regarded as a finite and discrete phase space, $GF[p] \times GF[p]$. In chapters 2 and 3 we describe the explicit construction of the eigenstates and eigenvalues of the B-I oscillator. In chapter 4 we present the coherent states of the finite Heisenberg-Weyl group (Gabor-states) and in chapter 5 we introduce the coherent states of the extended finite Affine group $SL(2,p)/O_p(2)$ and discuss their properties. We conclude in chapter 6 with the discussion of open problems and possible applications. Finally in the Appendix, we collect for completeness, some elementary facts needed from number theory and the representation theory of the finite group $SL(2, p)$ [13, 14].

transformations (1.2-1.3) forms a finite group over GF [p], the special linear (or symplectic) group SL (2, p) [12-14]. One choice for the generators of this group is :

$$g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (1.5)$$

where g, is a primitive element of the multiplicative group $GF^*[p] = GF[p] - \{0\}$, [7]. There is in the place of g_3 , another choice, which is a primitive or generating element, R_0 , for the abelian subgroup $O_p(2)$, of SL (2, p) of matrices commuting with g_3 [12].

$$\begin{aligned} O_p^{(2)} &\equiv \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \pmod{p} \right\} \\ &= \{R_0, R_0^2, \dots, R_0^{4k}\} \end{aligned} \quad (1.6)$$

Every primitive element R_0 , has period $4k$, for $p=4k\pm 1$, the order of $O_p(2)$. For some well known facts about Galois fields relevant to our discussion see the appendix. The way to determine either of the primitive elements g of $GF^*[p]$ or R_0 of $O_p(2)$ is by random search. For all prime numbers $p \leq 5.000$ the smallest primitive elements g are tabulated by Abramowitz and Stegun, [17], while all the primitive elements of all primes ≤ 5.000 are tabulated in ref. [18]. In ref. [19] a search was made for the generator R_0 , of the "rotation" matrices mod p , for primes of the form $p=4k-1$ in the range $p \leq 20.000$. In Table I we give g and R_0 for $GF[p]$, $p=4k-1 \leq 2003$. For primes p of the form $p=4k+1$ it is possible to determine R_0 once a primitive element g of $GF[4k+1]$ is known [see the appendix].

Coming back to the interpretation of the classical motion of the mappings g_1, g_2, g_3 , or R_0 (1.5, 1.6) the first represents free motion, the second hyperbolic and the third, R_0 , "Oscillatory" motion with periods $p, p-1$ and $4k$ where $p=4k\pm 1$, [4, 12]. The periods of elements in SL (2, p) must divide the order of the group, which is $p(p^2-1)$, so they must divide p or $p-1$ or $p+1$ [4].

Ch.2. The quantum B-I oscillator

In a famous paper A. Weil discovered the so called metaplectic representation of the symplectic group [20]. The basic idea, as far physics is concerned, is that since the representations of Heisenberg-Weyl, (HW), group are unique for a fixed value of the Planck constant $\hbar$, if we transform the phase-space (position, momentum) parameters of the H-W group by a symplectic transformation, we get another representation of this group for the same value of $\hbar$ and so this must be an equivalent representation to the original one. The similarity unitary matrix can then define an irrep of the symplectic group.

For the finite and discrete phase-space T_p^2 this representation has been discussed in ref. [12, 15].

We start by constructing first the representations of the discrete and finite Heisenberg-Weyl group [8-10, 12]. The configuration space of T_p is the discrete circle, $GF[p] = \{0, 1, \dots, p-1\}$, and so a basis for the position eigenfunctions is :

$$|q\rangle_k = \delta_{kq} \quad , q=1, 2, \dots, p \quad (2.1)$$

The translation operator P , which shifts by one position the states (2.1) is :

$$P = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & 1 & 0 \end{pmatrix} \quad (2.2)$$

and its finite Fourier transform

$$F P F^{-1} = Q, \quad (2.3)$$

where

$$J_{rs} J_{r's'} = \omega^{\frac{r's - rs'}{2}} J_{r+r', s+s'} \quad (2.8)$$

The metaplectic representation of $SL(2, p)$ is defined through matrices $U(A)$:

$$U^{-1}(A) J_{rs} U(A) = J_{r', s'} \quad (2.9)$$

$$(r' s') = (r, s) A, \quad \forall A \in SL(2, p)$$

In [12] R. Balian and C. Itzykson presented an explicit form of the metaplectic representation (2.9), [4, 13, 14] :

$$U(A) = \frac{\sigma(1) \sigma(\delta)}{p} \sum_{r,s=0}^{p-1} \omega^{\left[br^2 + (d-a)rs - cs^2 \right] / 2\delta} J_{rs} \quad (2.10)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\delta = 2 - a - d$ and

$$\sigma(a) = \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \omega^{ar^2} = (a/p) \begin{cases} 1 \\ i \end{cases} \quad \begin{matrix} p = 4k+1 \\ p = 4k-1 \end{matrix} \quad (2.11)$$

is the famous Gauss sum [7, 21] while (a/p) is ± 1 , depending on whether a is, (or is not), the square of an integer mod p (Jacobi symbol [7]).

The representation (2.10) is valid for $\delta \neq 0$ while when $\delta = 0$ there are the following

The characters of the $U_{L,R}$ representations are given in detail in [12]. The complete set of representations for $SL(2, GF[p^n])$ is described in [13], [14]. It is possible to find explicitly the matrix elements of $U(A)$, for every $A \in SL(2, p)$, using the fact that :

$$(J_{r,s})_{k,l} = \delta_{k-r,l} \omega^{\frac{k+l-2}{2} \cdot s} \quad (2.18)$$

and making repeated use of the Gauss sum (2.11) :

$$U_{k,l}(A) = \frac{1}{\sqrt{p}} (-2c|p) \begin{Bmatrix} 1 \\ -i \end{Bmatrix} \cdot \omega^{-\frac{1}{2c} [a(k-1)^2 + d(l-1)^2 - 2(k-1)(l-1)]} \quad (2.19)$$

where the bracket symbol distinguished between $p=4k+1$, $\{1\}$ and $p=4k-1$, $\{-i\}$, cases.

Our interest is in studying the eigenvalues and eigenvectors of the generator $U(R_0)$, where R_0 is the generator of $Op(2)$. For $p=4k+1$ the unitary matrix which represents the scaling map, see (1.9) :

$$U_{kl} \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} = \sigma(1) \sigma(\delta) \delta_{(k-1), g(l-1)} \quad (2.20)$$

where $\delta = 2 - g - g^{-1}$ and g a primitive element of $GF^*[p]$.

For $p=4k-1$ the generic form of $G=U(R_0)$ is :

$$G_{k,l} = \frac{1}{\sqrt{p}} i (2b_0|p) \cdot \omega^{\frac{1}{2b_0} [a_0(k-1)^2 + a_0(l-1)^2 - 2(k-1)(l-1)]}, R_0 = \begin{pmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{pmatrix}. \quad (2.21)$$

Ch.1. Galois Phase Spaces

Consider a particle moving on a discrete circle with finite number N of equidistant points. The classical phase space of this particle is a toroidal two dimensional finite lattice $N \times N$, which in appropriate units can be labelled by the pairs of integers, [15],

$$T_N^2 = \{(m, n) \mid m, n = 0, 1, 2, \dots, N-1\} \quad (1.1)$$

Classical linear canonical transformations of this phase-space lattice, are 2×2 integer matrices with matrix multiplication mod N . These matrices represent one-time step evolution maps, of Hamiltonians quadratic in phase-space variables [4, 5].

$$\begin{pmatrix} m_{t+1} \\ n_{t+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m_t \\ n_t \end{pmatrix} \quad t = 0, 1, 2, \dots \quad (1.2)$$

where m_t, n_t are the position and momentum of the particle at time step t and a, b, c, d are integers mod N . To get Hamiltonian evolution we must preserve phase-space area mod N :

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \mod N \quad (1.3)$$

The restriction we have from doing all the arithmetic operations between integers mod N in a unique way, implies that N must be a prime number. For example the inverse of an integer mod N is unique if and only if N is a prime number [7]. Thus, the set of all integers

$$GF[p] = \{0, 1, 2, \dots, p-1\} \quad (1.4)$$

is closed under the four arithmetic operations. This is the simplest of the finite Galois fields $GF[p^n]$.

When the number of lattice points is a prime number, the set of all linear canonical

Ch.3. Eigenstates and eigenvalues of the B-I Quantum oscillator

In [12] a method of construction was given, for the eigenstates of the Quantum B-I oscillator, through projector operators. Below we present a compact method, to construct the diagonalizing matrix of the Quantum B-I oscillator matrix G (ch.2). The period of matrix G is $4k$ for $p=4k\pm 1$. Since G is unitary the eigenvalues are:

$$\lambda_r = e^{\frac{2\pi i}{4k} r}, \quad -m \leq r \leq m \quad m = \begin{cases} 2k-1, & p=4k-1 \\ 2k, & p=4k+1 \end{cases} \quad (3.1)$$

The ground state of the B-I oscillator corresponds to the eigenvalue $r=0$ (no-nodes). We can construct the unitary matrix of the eigenvectors of G as the interwinner of G and D_G :

$$V = \frac{1}{4k} \sum_{m=0}^{4k-1} G^m D_G^{-m} \quad (3.3)$$

Indeed we easily check

$$G \cdot V = V \cdot D_G, \quad (3.4)$$

that is, the column r of the matrix V is the orthonormalized eigenvector of G , $|\Psi_{r-2k}\rangle$, corresponding to the eigenvalue λ_{r-2k} .

$$\langle q | \Psi_{r-2k} \rangle = \frac{1}{4k} \sum_{m=0}^{4k-1} \lambda_{r-2k}^{-m} (G^m)_{qr} \quad (3.5)$$

To construct the matrix power $G^n = U(R_0^n)$, we have to find the 2×2 matrix $R_0^m \bmod p$

In this work we mainly study the Quantum Mechanics of R_0 , the “harmonic oscillator” of Balian and Itzykson (B-I) [12]. According to the results in the appendix, in the case $p=4k+1$, R_0 is essentially the scaling generator g_2 , (1.5), while for $p=4k-1$ one has “true” rotations with “complex” eigenvalues $a_0 \pm ib_0$. These belong to the quadratic extension of $\text{GF}[p]$, [16],

$$\text{GF}[p^2] = \{a+ib \mid a, b \in \text{GF}[p], i^2+1=0\}. \quad (1.7)$$

on which $O_p(2)$ acts by multiplication

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow (a+ib)(x+iy). \quad (1.8)$$

The generator R_0 of $O_p(2)$, can be considered as the unit time step map in the phase-space T_p , (1.1) corresponding to a harmonic oscillator in an external magnetic field [4], which from now on we shall call the B-I oscillator.

$GF^*[4k+1]$. Thus,

$$R_0 = \begin{pmatrix} \frac{g+g^{-1}}{2} & \frac{g^{-1}-g}{2t} \\ \frac{g-g^{-1}}{2t} & \frac{g+g^{-1}}{2} \end{pmatrix} \quad (3.10)$$

and

$$G = U(R_0) = U(T) U \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} U^{-1}(T). \quad (3.11)$$

If the diagonalizing matrix of $U_g = U \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ is denoted by V_g and its diagonal form D_g :

$$U_g V_g = V_g D_g \quad (3.12)$$

then from (3.11) we get

$$G U(T) \cdot V_g = U(T) V_g \cdot D_g \quad (3.13)$$

or $U(T) V_g$ is the unitary matrix of eigenvectors of G and D_g is at the same time, the diagonal form of G . The scaling matrix U_g has matrix elements (rel. (2.20)) :

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$$F_{kl} = \frac{1}{\sqrt[p]{p}} \omega^{(k-1)(l-1)}, \quad k, l = 1, \dots, p \quad (2.4)$$

is the exponential position operator

$$Q = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \dots & \\ & & & \omega^{p-1} \end{pmatrix}, \quad \omega = e^{2\pi i/p} \quad (2.5)$$

The canonical Heisenberg commutation relations, in terms of Q and P, become,

$$QP = \omega PQ \quad (2.6)$$

The general element of the H-W group [10, 12, 15] is given by :

$$g_{t,r,s} = \omega^t J_{r,s} \equiv \omega^{t + \frac{rs}{2}} P^r Q^s \quad t, r, s = 0, 1, \dots, p-1 \quad (2.7)$$

The H-W group elements $J_{r,s}$ realize a projective representation of the translation group in the phase-space T_p^2 [10],

It is easy to check the action of the “circulant” scaling matrix U_g (3.14), on the extended characters Π_j , $j=1, \dots, p$:

$$U_g \Pi_j = - e^{-\frac{2\pi i}{p-1}j} \Pi_j \quad (3.19)$$

So the eigenvalue (-1) appears twice $j=p-1$, $j=p=0$, while the ground state $j=\frac{p-1}{2}$ is the quadratic character

$$10 > = \Pi_{\frac{p-1}{2}}(a) = \begin{pmatrix} 0 \\ (a|p) \end{pmatrix} \quad a=1, \dots, p-1, \quad (3.20)$$

where $(a|p)$ is the Jacobi symbol [7].

The matrix C_g is the eigenvector matrix V_g of U_g we were looking for. From rel. (3.13) we check that the matrix

$$V_G = U(T) \cdot C_g \quad (3.21)$$

is the eigenvector matrix of the B-I oscillator G for $p=4k+1$. The eigenvalue matrix D_g is at the same time the eigenvalue matrix of G (3.2), but with some different order for the eigenvalues.

Since G commutes with F , the finite Fourier transform (FFT) (2.4), the eigenvectors of G are eigenvectors of F with eigenvalues (2.22) :

$$f_j = i^n (-1)^k \cdot \lambda_j^k, \quad j = 1, 2, \dots, p \quad (3.22)$$

$n = 0, 1$ for $p = 4k \pm 1$

These eigenvectors split the degeneracies of the FFT eigenvalues [12, 21].

particular cases [12] :

$$\delta=0, b \neq 0 \quad U(A) = \frac{\sigma(-2b)}{\sqrt[p]{p}} \sum_{r=0}^{p-1} \omega^{r^2/2b} J_{\frac{r(a-1)}{b}, r} \quad (2.12)$$

$$\delta=b=0, c \neq 0 \quad U(A) = \frac{\sigma(-2c)}{\sqrt[p]{p}} \sum_{r=0}^{p-1} \omega^{-r^2/2c} P^r \quad (2.13)$$

$$U(I) = I_{p \times p} \quad (2.14)$$

The representation of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has as the finite Fourier transform (FFT):

$$U \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (-1)^{k+1} i^n F, \quad (2.15)$$

where $n=0, 1$ according to if, $p=4k+1$ or $p=4k-1$.

The matrix $U(A)$ commutes with the square of FFT which is the parity operator,

$$S = F^2, \quad S|q\rangle = |-q\rangle, \quad q = 1, 2, \dots, p \quad (2.16)$$

So we have the left-mover and the right-mover representations of the H-W group, [12].

$$U_L = U(I + S) / 2, \quad U_R = U(I - S) / 2 \quad (2.17)$$

$$\frac{1}{N} \sum_{\vec{m}} |\vec{m}\rangle \langle \vec{m}| = I \quad (4.4)$$

because the left hand side (lhs), commutes with all the $J_{\vec{n}}$'s and the irreducibility of the H-W group representation implies that the lhs is a scalar number, times the identity matrix. We normalize the sum by

$$N = \sum_{\vec{m}} |\langle \vec{n} | \vec{m} \rangle|^2 \quad (4.5)$$

The coherent states, eventhough normalized, are not orthogonal :

$$\langle \vec{n} | \vec{m} \rangle = \omega^{\frac{\vec{m} \times \vec{n}}{2}} \langle 0 | J_{\vec{m}-\vec{n}} | 0 \rangle, \quad (4.6)$$

where $\vec{m} \times \vec{n} = m_1 n_2 - m_2 n_1$.

The overlap constants $\langle \vec{n} | \vec{m} \rangle$ are constant on the R_0 orbits :

$$\langle \vec{n} | \vec{m} \rangle = \langle \vec{n} R_0^{-1} | \vec{m} R_0^{-1} \rangle \quad (4.7)$$

Finally the matrices $J_{\vec{m}}$ on the coherent states realize a projective representation [23] of the T_2 translation group of $GF[p^2]$.

$$J_{\vec{m}} |\vec{n}\rangle = \omega^{\frac{\vec{n} \times \vec{m}}{2}} |\vec{n} + \vec{m}\rangle \quad (4.8)$$

The p^2 coherent states are linearly dependent and the system is overcomplete.

Notice also the relations [12] :

$$F = i^n (-G)^k, \quad S = (-1)^n G^{2k} \quad (2.22)$$

$n=0, 1$ if $p=4k\pm 1$ respectively.

$$\vec{lm} > = J_{\vec{m}} \vec{U}(T) \prod_{\frac{p-1}{2}} \quad (4.13)$$

or using the definition of the metaplectic representation :

$$\vec{lm} > = U(T) J_{\vec{m} \cdot T} \prod_{\frac{p-1}{2}} \quad (4.14)$$

where $\vec{m} = (m_1, m_2)$,

$$\vec{m} \cdot T = \frac{1}{1+t} (m_1 - t m_2, m_1 + t m_2) \equiv (m_1^T, m_2^T) \quad (4.15)$$

The position representation of $\vec{lm} >$ can be expressed in terms of hybrid Gauss sums with Jacobi symbols as coefficients. To get an easier representation we pass over the modified coherent states

$$\vec{lm} >_T = J_{\vec{m} \cdot T} \prod_{\frac{p-1}{2}} \quad (4.16)$$

In this basis where G is represented by a circulant matrix, rel. (3.14), we obtain :

$$\langle \vec{r} | \vec{lm} \rangle_T = \frac{1}{\sqrt{p-1}} (r - m_1^T - 1)p \omega^{\frac{2(r-1) - m_1^T \cdot m_2^T}{2}} \quad (4.17)$$

and m_1^T, m_2^T are given in rel. (4.15). The overlap constants $\langle \vec{n} | \vec{m} \rangle$ are the same :

$${}_T \langle \vec{n} | \vec{m} \rangle_T = \langle \vec{n} | \vec{m} \rangle. \quad (4.18)$$

$$R_0^n = \begin{pmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{pmatrix}^n \equiv \begin{pmatrix} a_{n-1} & b_{n-1} \\ -b_{n-1} & a_{n-1} \end{pmatrix}, \quad n=1, \dots, 4k-1 \quad (3.6)$$

$$R_0^0 = \begin{pmatrix} a_{-1} & b_{-1} \\ -b_{-1} & a_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and then to use rel. (2.19). We find that the eigenvectors (3.5) are $(a_{-1}=b_{-1}=0)$;

$$\langle q | \Psi_{r-2k} \rangle = \frac{1}{\sqrt{p(p+1)}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \sum_{m=0}^{4k-1} (2b_{m-1} | p) \cdot \lambda_{r-2k}^{-im} \cdot \omega^{1/2 b_{m-1} [a_{m-1}(q-1)^2 + a_{m-1}(r-1)^2 - 2(q-1)(r-1)]} \quad (3.7)$$

These sums are the so called Kloosterman generalized sums in number theory and only very specific cases can be expressed in closed form [16].

In the case $p=4k+1$ it is possible to find explicit closed expressions for the eigenstates of the B-I oscillator in a particular basis. As we discuss in the appendix, in this case, the $O_p(2)$ generator R_0 is similar to the scaling map, ($t^2=-1 \bmod p$) :

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = T \begin{pmatrix} a-tb & 0 \\ 0 & a+tb \end{pmatrix} T^{-1}, \quad a^2 + b^2 = 1 \bmod p \quad (3.8)$$

where the matrix

$$T = \frac{1}{\sqrt{2t}} \begin{pmatrix} 1 & 1 \\ -t & t \end{pmatrix} \quad (3.9)$$

belongs to $SL(2,p)$ since $2t = (1+t)^2$ and $\det T = 1$.

To find R_0 we choose $a_0-tb_0=g$ and so $a_0+tb_0=g^{-1}$ where g a primitive element of

$$(A \cdot B)_{\vec{m},\vec{n}} = \frac{1}{N} \sum_{\vec{k}} A_{\vec{m},\vec{k}} B_{\vec{k},\vec{n}} \quad (4.24)$$

and

$$\langle \vec{m} | A | \psi \rangle = \frac{1}{N} \sum_{\vec{n}} A_{\vec{m},\vec{n}} \langle \vec{n} | \psi \rangle \quad (4.25)$$

$$\left(U^{-1}(R_0) A U(R_0) \right)_{\vec{m},\vec{n}} = A_{\vec{m}R_0, \vec{n}R_0} \quad (4.26)$$

We close this section by noticing that, we can get a representation with classical behaviour, for the transformation properties of any matrix A , under any element of $SL(2,p)$ (ch. 2), if we first expand A on the basis $J_{\vec{m}}$:

$$A = \frac{1}{p} \sum_{\vec{m}} A_{\vec{m}} J_{\vec{m}} \quad (4.27)$$

where

$$A_{\vec{m}} = \text{Tr} (A J_{-\vec{m}}) \quad (4.28)$$

Indeed from ch.2 we get

$$U^{-1}(R) A U(R) = \frac{1}{p} \sum_{\vec{m}} A_{\vec{m}} J_{\vec{m} \cdot R} = \frac{1}{p} \sum_{\vec{m}} A_{\vec{m} \cdot R^{-1}} J_{\vec{m}} \quad (4.29)$$

So the representaion of A by the matrix

$$U_{kl} \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} = \delta_{k-1, g(l-1)} \quad (3.14)$$

We find explicit forms of the eigenvectors of U_g by using the multiplicative characters of $GF^*[p]$, [16].

Relatively to the primitive element g of $GF^*[p]$, the multiplicative characters are defined as the $p-1$ functions on $GF^*[p]$:

$$\Pi_j(g^n) = e^{\frac{2\pi i}{p-1} j n} / \sqrt{p-1} \quad \begin{matrix} j=1,2,\dots, p-1 \\ n=1,2,\dots, p-1 \end{matrix} \quad (3.15)$$

and the p -th function is defined as :

$$\Pi_0(g^n) = 0, \quad n = 1, 2, \dots, p-1 \quad (3.16)$$

We extend these characters also on the zero-element of $GF[p]$:

$$\Pi_0(p) = 1, \Pi_j(0) = 0 \quad j=1, \dots, p-1 \quad (3.17)$$

Thus we construct the extended $p \times p$, character matrix C_g

$$C_g = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ \Pi_1 & \Pi_2 & & \Pi_{p-1} & 0 \\ & & & & \vdots \\ & & & & 0 \end{pmatrix} \quad (3.18)$$

In the case of the discrete field $F=GF[p]$ [29] we can use our elementary matrices P (2.2), and the circulant matrix, $D = D_g^{-1}$ (2.20), to construct the general element $U(\lambda, \alpha)$ (5.4),

$$[Pf](k) = f(k-1) \quad k=0, 1, \dots, p-1 \quad (5.5)$$

$$[Df](k) = f(gk) \quad k=0, 1, \dots, p-1 \quad (5.6)$$

We check that

$$U(\lambda, \alpha) = D^{1/\lambda} \cdot p^{-\alpha} \quad (5.7)$$

The representation defined above is reducible and there is an invariant subspace of dimension one (the constant function) [30].

Here we shall construct generalized wavelets (we call them p -wavelets), on the discrete field $GF[p]$, as coherent states parameterized by the set $SL(2, p) / O_p(2)$. Every element $g \in SL(2, p)$ can be factorized as a product of an $O_p(2)$ element $r(g)$, times another unique element $h(g)$

$$g = h(g) r(g) \quad (5.8)$$

Indeed let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, p)$. If the vector (c, d) has norm which is a square modulo p ,

(We choose here $p=4k-1$. If $p=4k+1$ we have an analogous argument),

$$c^2 + d^2 = l^2 \pmod{p}, \quad l \in GF^*[p] \quad (5.9)$$

then we can find a unique relation $r(g)$ such that

$$g \cdot r^{-1}(g) = \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} \quad (5.10)$$

Ch.4. Heisenberg - Weyl coherent states

Coherent states are usually defined as quantum mechanical states, which evolve in time following as closely as possible the motion of a classical system, that is with the smallest possible dispersion. There is an extensive literature for coherent states in physics, [22, 23], but also recent applications to signal and image analysis are developing fast [6, 24]. Coherent states in discrete rational phase-space lattices, (or Von-Neumann lattices), have been discussed in papers by J. Zak and collaborators, which deal with the problem of quantum Hall effect [10, 25].

In ch. 3 we have determined the eigenstates of the B-I oscillator so we can proceed to define the corresponding coherent states over the Galois Field GF [p] :

$$|\vec{m}\rangle = J_{\vec{m}} |0\rangle, \quad \vec{m} = (m_1, m_2) \in \text{GF}[p] \times \text{GF}[p] \approx \text{GF}[p^2], \quad (4.1)$$

where the coset elements of the H-W group over its U(1) center are [rel. (2.7)] :

$$J_{\vec{m}} = \omega^{\frac{m_1 m_2}{2}} P^{m_1} Q^{m_2} \quad m_1, m_2 = 0, 1, \dots, p-1 \quad (4.2)$$

and $|0\rangle$ is the normalized ground state of the B-I oscillator. From the definition (2.21), we see that :

$$U(R_0) |\vec{m}\rangle = |\vec{m} \cdot R_0^{-1}\rangle \quad (4.3)$$

which implies that the state $|\vec{m}\rangle$ "moves" classically under the action of $U(R_0)$. The coherent states just defined resolve the identity operator :

In ref. [19] we show that always exists primitive element of $GF[p^2]$, $w=w_1+iw_2$ which has norm:

$$w_1^2 + w_2^2 = g \quad (5.18)$$

From (5.17), (5.18) it follows that we can find a rotation $r(g)$ such that

$$(c, d) = (l w_1, l w_2) r(g) \quad (5.19)$$

The decomposition of g in this case becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = h(g) \cdot r(g) \quad (5.20)$$

where,

$$h(g) = \begin{pmatrix} (lg)^{-1} & 0 \\ 0 & lg \end{pmatrix} \begin{pmatrix} 1 & (ac+bd)g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_2 & -w_1 \\ \frac{w_1}{g} & \frac{w_1}{g} \end{pmatrix} \quad (5.21)$$

$$r(g) = (lg)^{-1} \begin{pmatrix} cw_1+dw_2 & dw_1-cw_2 \\ -dw_1+cw_2 & cw_1+dw_2 \end{pmatrix} \quad (5.22)$$

In both cases the non-rotational part of an element g , in $SL(2, p)$, or its coset $(SL(2, p)/O_p)$ representative, is labeled by the elements of $GF[p^2]$:

$$z \equiv g \cdot i = \frac{ai+b}{ci+d}, \quad (5.24)$$

The transformation of cosets under a group multiplication g' is given as :

$$z' = g' \cdot z = (g' \cdot g) \cdot i \quad (5.25)$$

$$\vec{m} > = \frac{1}{N} \sum_{\vec{m}} \vec{m} > < \vec{m} | \vec{n} > \quad (4.9)$$

for all states $|\vec{n}\rangle$.

Although the standard quantum mechanical coherent states, [22], are easily visualized as frequency modulated gaussians, which have been shifted in position and momentum space, their discrete and finite analogs, (4.1), have non-gaussian patterns. Below we shall discuss the case of Galois fields $GF[p]$, where $p=4k+1$. In this case, we found in ch.3 that the B-I oscillator is similar to the scaling operator and its eigenstates are the multiplicative characters of $GF[p]$ in an appropriate basis. From (3.9-3.20) we see that the eigenstates of B-I oscillator for $p=4k+1$ are :

$$|j\rangle = U(T) \Pi_{\frac{p-1}{2}, j}, \quad j=0, 1, 2, \dots, p-1 \quad (4.10)$$

and the generator G is similar to the scaling operator :

$$G = U(T) U \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} U^{-1}(T), \quad (4.11)$$

where g is a primitive element of $GF[p]$ and Π_j are the multiplicative characters (rel. 3.15-3.17). The matrix $T \in SL(2, p)$, is given by rel. (3.9),

$$T = \frac{1}{1+t} \begin{pmatrix} 1 & 1 \\ -t & t \end{pmatrix}, \quad t^2 = -1 \text{ mod } p \quad (4.12)$$

The coherent states for $p=4k+1$ are defined as :

Ch. 6 Conclusions

Although in the literature there is a lot of discussion, about p-adic Quantum Mechanics (QM), we think that examining in detail QM, over Galois fields is a separate issue which is worth studying from the point of view of mathematical completeness and structure as well as from the physics point of view.

Classical linear dynamical systems on toroidal phase spaces, exhibit “chaotic” behaviour, which can be controlled and studied also at the Quantum Mechanical level with smaller computational effort than the non-linear ones [4a, 5, 31, 32]. We consider our work as one more step in the discretization and “decomposition” of the linear dynamical systems on toroidal phase spaces.

We hope, we have showed that, the classical and quantum mechanical behaviour of discrete linear systems over Galois phase-spaces, (especially of the BI oscillator), can be studied in great detail with the help of elementary Number theory.

Tractable problems in our approach are extensions to other Galois fields, $GF[p^n]$, increasing the number of degrees of freedom or to formulate field and string theories over Galois fields. More fundamental problem seems to be the approach to the continuum and its stability .

The relation of classical and quantum behaviour also is of great interest and should be studied along the lines of reference [5].

Finally but not lastly the coherent state analysis of signals and images over Galois fields is an intriguing and challenging problem due to the intrinsic randomization introduced by the algebraic operations of these number fields.

Acknowledgments

We thank C. Itzykson, A. Voros and G. Savvidy, for useful discussions.

We discuss finally some general properties of the coherent states $|\vec{m}\rangle$. It is easy to see that the overlapping constants $\langle \vec{m} | \vec{n} \rangle$ define a reproducing Kernel

$$K(\vec{m} | \vec{n}) = \langle \vec{m} | \vec{n} \rangle, \quad (4.19)$$

$$\frac{1}{N} \sum_{\vec{m}} K(\vec{l} | \vec{m}) K(\vec{m} | \vec{n}) = K(\vec{l} | \vec{n}) \quad (4.20)$$

which acts on functions like a δ -function :

$$\frac{1}{N} \sum_{\vec{n}} K(\vec{m} | \vec{n}) \psi(\vec{n}) = \psi(\vec{m}), \quad (4.21)$$

where

$$|\psi\rangle = \frac{1}{N} \sum_{\vec{n}} |\vec{n}\rangle \psi(\vec{n}). \quad (4.22)$$

Also an arbitrary $p \times p$ matrix A , can be represented through its symbol [22] :

$$\langle \vec{m} | A | \vec{n} \rangle = A_{\vec{m}, \vec{n}}, \quad (4.23)$$

which has the convolution property,

$$c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}, \quad (\text{A.7})$$

and b finally is the generator R_0 , of the $O_p(2)$ subgroup of $SL(2,p)$ (rel. 1.6) of order $4k=p\pm 1$.

The representations are divided in two classes [13, 14]. Those of the first class are constructed on the space of homogeneous functions on $G \equiv GF[p] \times GF[p]$

for some multiplicative character π of $GF[p]$ and those of the second class on the space of factorizable function on $GF[p^2] = \{x + \sqrt{g}y \mid x, y, \in GF[p]\}$,

$$f(tz) = \pi(t) f(z), \quad \forall z \in GF[p^2], \quad (\text{A.9})$$

$$t \in \{x + \sqrt{g}y \mid x^2 - gy^2 = 1\} \quad (\text{A.10})$$

and π a character of the $O_p(2)$. One finally has in total, eight types of representations and the non trivial ones have dimensions, $q, q+1, \frac{1}{2}(q+1), q-1, \frac{1}{2}(q-1)$. The metaplectic representation which we studied is the unique of dimension q .

The elements of $GF^*[p]$ are either quadratic residues (squares) mod p or not (both $(p-1)/2$ in number). The function which separates these two, is the Legendre symbol $(n|p)=1, -1, 0$ for q_{res}, n_{qres} , and $n=0$, respectively.

The Legendre symbol satisfies the famous quadratic reciprocity law

$$(p|q) (q|p) = (-1)^{(p-1)(q-1)/4}. \quad (\text{A.11})$$

For example the numbers $-1, 2$ are characterized as follows :

$$(-1|p) = (-1)^{(p-1)/2} \quad (\text{A.12})$$

$$(2|p) = (-1)^{(p^2-1)/8} \quad (\text{A.13})$$

$$J(A) = \{A_m\}_{m, m \neq 1, \dots, p} \quad (4.30)$$

follows the classical motion of its indices.

Ch.5 Coherent states for the affine group $SL(2,p)/O_p(2)$

The affine group, $Aff(F)$, over a number field F , (which is taken to be here $GF[p]$), is defined as the group generated by translations and scalings on F :

$$g(\lambda, \alpha) x = \lambda x + \alpha \quad \forall x \in F, \quad \alpha \in F, \quad \lambda \in F - \{0\} \quad (5.1)$$

The composition law follows to be,

$$g(\lambda, \alpha) g(\lambda', \alpha') = g(\lambda\lambda', \lambda\alpha' + \alpha) \quad (5.2)$$

and the inverse is,

$$g^{-1}(\lambda, \alpha) = g(\lambda^{-1}, -\alpha\lambda^{-1}) \quad (5.3)$$

The representation theory of this $Aff(R)$ group over the field of real number is known [26] and coherent states corresponding to this group have been studied in [27, 28] (see also [22]).

In signal and image processing, a new basis of functions (called wavelets) have been constructed which probe the spectrum of scales of signals at every position and the method is called multiresolution analysis (MRA) [6, 29]. Quite recently the method has been applied also to discrete fields [30]. In principle one uses the following representation of $Aff[F]$ on the space of complex functions on F , $C(F)$,

$$[U(\lambda, -\alpha)f](x) = f(g^{-1}(\lambda, \alpha)x), \quad \forall x \in F. \quad (5.4)$$

Choosing appropriate discrete set of values of λ_n , (usually $\lambda_n = \lambda_0 \cdot 2^n$), α_m and an initial function f , a set of functions f_{nm} is constructed through (5.4), which is at least complete in $C(F)$. The MRA method describe the conditions under which, we can analyze on the wavelets a signal and how we can reconstruct it from its wavelet components.

in the case $p=4k+1$.

In this case the diagonalizing matrix of any element in $O_p(2)$ is :

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = T \begin{pmatrix} a-tb & 0 \\ 0 & a+tb \end{pmatrix} T^{-1} \quad (\text{A.21})$$

where

$$T = \frac{1}{\sqrt{2t}} \begin{pmatrix} 1 & 1 \\ t & -t \end{pmatrix} \quad (\text{A.22})$$

and $t^2 = -1 \pmod{p}$. We notice that $t = \left(\frac{p-1}{2}\right)!$ is a solution in $\text{GF}[p]$ and that $2t = (1+t)^2$, so $\sqrt{2t} = 1+t \in \text{GF}[p]$.

We choose

$$a-tb=g, \quad a+tb=g^{-1} \quad (\text{A.23})$$

where g is a primitive element of $\text{GF}[p]$ then

$$a_0 = \frac{g+g^{-1}}{2}, \quad b_0 = \frac{g^{-1}-g}{2t} \quad (\text{A.24})$$

form the elements of R_0

$$R_0 = \begin{pmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{pmatrix} \quad (\text{A.25})$$

with minimum period $p-1=4k$.

We check easily that this geometrical construction gives $r(g)$ and $h(g)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} l^{-1} & (ac+bd)l^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d \cdot l^{-1} & c \cdot l^{-1} \\ -c \cdot l^{-1} & d \cdot l^{-1} \end{pmatrix} = h(g) \cdot r(g) \quad (5.11)$$

On the other hand $h(g)$, is a product of scaling and translation operations

$$h(g) = \begin{pmatrix} l^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & ac+bd \\ 0 & 1 \end{pmatrix}. \quad (5.12)$$

Acting on $GF[p]$ as, (see ref. [28] for the continuum case, $SL(2R)/O(2)$),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}, \quad x \neq -d/c \quad x \in GF[p], \quad (5.13)$$

We find

$$h(g) \cdot x = l^{-2} (x+ac+bd) \quad (5.14)$$

If we denote by i the square root of $-1 \in GF[p]$ and we extend in $GF[p^2]$ the action (5.13),

$$g \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}, \quad z=x+iy \in GF[p^2], \quad (5.15)$$

we can check that the real and imaginary parts of

$$g \cdot i = h(g) \cdot i \quad (5.16)$$

contain the two parameters of $h(g)$, l and $ac+bd$. If on the other hand c^2+d^2 is not the square of an integer mod p , then we can find a primitive element g of $GF[p]$, such that

$$c^2 + d^2 = l^2 \cdot g, \quad l \in GF^*[p] \quad (5.17)$$

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We define now the p-wavelets as coherent states based on the B-I oscillator ground state of ch.3.

$$|z(h)\rangle = U(h(g)) |0\rangle \quad \forall g \in SL(2,p) \quad (5.26)$$

Any other element, g' , of $SL(2,p)$ acts classically on the states $|z\rangle$,

$$U(g') |z(h)\rangle = |z(h(g'))\rangle \quad (5.27)$$

Because the metaplectic representation of $SL(2,p)$ is irreducible, from (5.27) we find that the matrix,

$$I = \frac{1}{N} \cdot \sum_{h \in SL(2,p)/O_p(2)} |z(h)\rangle \langle z(h)| \quad (5.28)$$

is the identity matrix, $I_{p \times p}$, adjusting the normalization constant N (see chapter 4).

Any complex function f on the Galois field $GF[p]$, can be represented on the coherent states $|z(h)\rangle$ using its components. The set of states $|z(h)\rangle$ acts like an "orthonormal" system, although it is not:

$$|f\rangle = \frac{1}{N} \cdot \sum_h |z(h)\rangle \langle z(h)| f \rangle \quad (5.29)$$

Matrices or functions on $GF[p^2]$ can also be represented on the coherent states $|z(h)\rangle$ in an analogous manner with the HW coherent states (ch.4).

Table Capture

In tables I_{a-c} , we give the primitive elements of $GF^*[p]$, $O_p(2) \cong C_p$ and $GF[p^2]$ for primes of the form $p=4k-1 \leq 2003$.

Appendix

In this appendix we summarize for completeness, some elementary facts about Galois fields [7,16]. The simplest Galois field is the set of numbers, (p prime number),

$$GF[p] = \{0, 1, 2, \dots, p-1\} \quad (A.1)$$

where all arithmetic operation are done modulo p .

The little Fermat theorem says that

$$a^p = a \mod p \quad \forall a \in GF[p] \quad (A.2)$$

and it follows that

$$(a+b)^p = a^p + b^p \mod p \quad \forall a, b \in GF[p] \quad (A.3)$$

The multiplicative group of $GF[p]$, $GF^*[p]$:

$$GF^*[p] = \{1, 2, \dots, p-1\} \quad (A.4)$$

is cyclic and there are primitive elements of period $p-1$, [18].

Now we pass over the representation theory of the group $SL(2,p)$ the 2×2 integer matrices of unit determinat mod p [13, 14].

The class representatives are given as

$$1, z, a^l, b^m, c, d, zc, zd \quad (A.5)$$

where

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z = -1, \quad a = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad (A.6)$$

g , a primitive element of $GF[p]$

TABLE Ia

4k-1 Prime	GF[p]	C _p	GF[p ²]	4k-1 Prime	GF[p]	C _p	GF[p ²]
3	2	0+I	1+I	347	2	205+225I	30+181I
7	3	2+5I	2+3I	359	7	154+112I	2+196I
11	2	3+5I	2+3I	367	6	74+221I	15+191I
19	2	16+12I	6+2I	379	2	304+228I	175+68I
23	5	19+10I	2+I	383	5	307+154I	2+I
31	3	10+26I	1+8I	419	2	272+173I	86+231I
43	3	35+18I	21+11I	431	7	153+254I	216+54I
47	5	10+29I	2+I	439	15	183+118I	56+55I
59	2	48+36I	7+22I	443	2	355+178I	26+395I
67	2	14+41I	22+11I	463	3	371+186I	133+298I
71	7	51+13I	18+31I	467	2	94+281I	409+29I
79	3	65+11I	39+33I	479	13	339+85I	343+445I
83	2	3+65I	2+9I	487	3	98+293I	187+337I
103	5	83+42I	2+I	491	2	152+227I	251+443I
107	2	22+65I	68+73I	499	7	400+300I	374+208I
127	3	26+77I	72+36I	503	5	368+313I	258+43I
131	2	27+53I	40+57I	523	2	419+210I	233+145I
139	2	99+25I	10+72I	547	2	110+329I	320+387I
151	6	59+12I	28+66I	563	2	451+226I	511+26I
163	2	131+66I	28+14I9	571	3	115+229I	60+551I
167	5	34+101I	2+I	587	2	112+508I	88+181I
179	2	144+108I	18+6I	599	7	109+10I	76+102I
191	19	130+181I	24+4I	607	3	122+365I	54+27I
199	3	87+81I	27+95I	619	2	496+372I	328+97I
211	2	43+85I	39+198I	631	3	98+146I	298+312I
223	3	179+90I	50+25I	643	11	515+258I	59+292I
227	2	46+137I	71+78I	647	5	130+389I	2+I
239	7	12+151I	72+9I	659	2	138+607I	46+259I
251	6	51+101I	77+139I	683	5	547+274I	2+682I
263	5	211+106I	2+I	691	3	139+277I	66+669I
271	6	126+188I	129+80I	719	11	593+508I	329+262I
283	3	227+114I	25+129I	727	5	146+437I	2+I
307	5	293+135I	11+240I	739	3	592+444I	93+708I
311	17	257+220I	4+I	743	5	595+298I	2+I
331	3	67+133I	226+35I	751	3	531+133I	188+47I

For $p=4k+1$ we check that -1 , is a qres with square root,

$$x_+ = \left(\frac{p-1}{2}\right)! \quad (\text{A.14})$$

The Gauss quadratic sum is very useful and it is known in closed form, [7, 12, 16, 21]

$$\frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} e^{\frac{2\pi i}{p} a n^2} = (a|p) \begin{cases} 1 \\ n \end{cases} \begin{matrix} p=4k+1 \\ p=4k-1 \end{matrix}, \quad \forall a \in \text{GF}^*[p] \quad (\text{A.15})$$

The basic additive and multiplicative functions on $\text{GF}[p]$ are the characters of the corresponding groups of $\text{GF}[p]$, $\text{GF}^+[p]$ and $\text{GF}^*[p]$, ($\omega=e^{2\pi i/p}$), [16]:

$$\chi_j(a) = \omega_p^{ja} \quad j=0, 1, \dots, p-1, \quad a \in \text{GF}[p] \quad (\text{A.16})$$

$$\pi_j(a) = \omega_{p-1}^{jn} \quad j=1, \dots, p-2, \quad a = g^n \in \text{GF}^*[p] \quad (\text{A.17})$$

We can extend π_j on $\text{GF}[p]$

$$\pi_j(0) = 0 \quad (\text{A.18})$$

$$\pi_0 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix} \quad (\text{A.19})$$

These are orthogonal and complete systems of functions on $\text{GF}[p]$.

We close the appendix by giving details of how construct the generator R_0 , of the $O_p(2)$ subgroup of $SL(2,p)$:

$$O_p(2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \bmod p \right\} \quad (\text{A.20})$$

TABLE Ib

4k-1 Prime	GF[p]	C _p	GF[p ²]	4k-1 Prime	GF[p]	C _p	GF[p ²]
787	2	158+473I	305+241I	1291	2	259+517I	538+251I
811	3	163+325I	559+84I	1303	6	1043+522I	167+735I
823	3	659+330I	195+509I	1307	2	262+785I	704+955I
827	2	166+497I	267+280I	1319	13	544+466I	317+409I
839	11	297+494I	259+694I	1327	3	266+797I	844+422I
859	2	688+516I	160+233I	1367	5	274+821I	2+I
863	5	532+797I	42+682I	1399	13	1153+988I	712+178I
883	2	707+354I	83+400I	1423	3	1139+570I	986+493I
887	5	178+533I	2+I	1427	2	286+857I	487+470I
907	2	182+545I	75+416I	1439	7	130+1330I	472+519I
911	17	376+322I	4+I	1447	3	290+869I	1184+592I
919	7	299+50I	764+179I	1451	2	1099+667I	946+326I
947	2	692+589I	96+931I	1459	3	1168+876I	32+962I
967	5	194+581I	2+I	1471	6	793+453I	445+89I
971	6	195+389I	215+719I	1483	2	1187+594I	1173+155I
983	5	787+394I	2+I	1487	5	298+893I	2+I
991	6	817+700I	268+67I	1499	2	1059+265I	84+21I
1019	2	717+799I	471+431I	1511	2	491+82I	847+393I
1031	14	196+892I	506+428I	1523	2	1030+1441I	736+385I
1039	3	77+355I	34+581I	1531	2	307+613I	105+1496I
1051	7	211+421I	101+667I	1543	5	1235+618I	2+I
1063	3	851+426I	224+112I	1559	19	276+459I	942+1015I
1087	3	218+653I	568+284I	1567	3	1107+277I	901+617I
1091	2	219+437I	619+570I	1571	2	315+629I	681+227I
1103	5	274+579I	364+846I	1579	3	1264+948I	1101+1212I
1123	2	899+450I	30+1108I	1583	5	1267+634I	2+I
1151	17	178+266I	349+300I	1607	5	322+965I	2+I
1163	5	931+466I	2+1162I	1619	2	1072+1146I	872+1510I
1171	2	235+469I	98+748I	1627	3	326+977I	1323+152I
1187	2	238+713I	837+175I	1663	3	1331+666I	488+244I
1223	5	34+1025I	1085+1200I	1667	2	1155+898I	34+660I
1231	3	725+797I	1198+915I	1699	3	1360+1020I	1558+47I
1259	2	1008+756I	461+993I	1723	3	1379+690I	718+1364I
1279	3	788+1181I	1001+456I	1747	2	350+1049I	1261+243I
1283	2	1027+514I	574+996I	1759	6	1237+1379I	1745+1175I

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TABLE Ic

4k-1 Prime	GF[p]	C _p	GF[p ²]	4k-1 Prime	GF[p]	C _p	GF[p ²]
1783	10	1427+714I	1467+1625I	1879	6	290+434I	1510+302I
1787	2	1489+1210I	202+236I	1907	2	1347+337I	742+1139I
1811	6	363+725I	955+922I	1931	2	387+773I	746+1536I
1823	5	1459+730I	2+I	1951	3	572+714I	838+1177I
1831	3	324+539I	1016+2541I	1979	2	1371+1066I	453+1287I
1847	5	370+1109I	2+I	1987	2	398+1193I	111+938I
1867	2	374+1121I	1583+142I	1999	3	824+706I	168+42I
1871	14	1370+776I	392+1083I	2003	5	1603+802I	2001+I

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