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Quantum Mechanics on a Closed Loop

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Abstract

Quantum mechanics on the loop of arbitrary shape is formulated, which is an extension of previous formulations of quantum mechanics on S^1 preserving its topology. It is shown that the representation spaces of the algebra do not change under the extension. We also derive, in the x -diagonal representation, the explicit expressions for the operators p_x and p_y that satisfy the constraints of the Dirac quantization on the closed loop.

From the previous formulations of quantum mechanics [1] ~ [5] on a sphere S^D , it can be anticipated that the argument does not depend crucially on the details of the shape of the manifold and that only its topology is playing an important role. By formulating quantum mechanics on a closed loop of arbitrary shape, we show explicitly that this in fact is the case. As an application of this result we explicitly determine, in the x -diagonal representation, the forms of the operators p_x and p_y in the Dirac formalism for the system constrained on the closed loop.

To start with we recapitulate our formulation of quantum mechanics [2], [4], [6], [7], [8] on S^1 together with the standard Dirac formalism applied to this problem. The fundamental algebra [2] in our formulation is

$$[G, W] = W \quad (1)$$

where G is a self-adjoint operator that gives displacement on S^1 and W is a unitary operator that plays the role of the position operator. We have seen that the operator G can be written as

$$G = \frac{1}{i} \frac{d}{d\theta} + \alpha \quad (2)$$

in the W -diagonal representation with

$$W = e^{i\theta} \quad \text{and} \quad \langle \theta | \psi \rangle = \psi(\theta), \quad (3)$$

where θ ($-\infty < \theta < \infty$) is the angle that fixes a point on S^1 and α ($0 \leq \alpha < 1$) is a continuous parameter characterizing an infinite number of inequivalent representations. We denote the representation space specified by α as $\mathcal{H}_\alpha$, where the wave function $\psi(\theta)$ undergoes the transformation

$$e^{iG\lambda} \psi(\theta) = e^{i\alpha\lambda} \psi(\theta + \lambda). \quad (4)$$

We can rewrite the unitary operator W as

$$W = r^{-1} (X + iY) \quad (X^2 + Y^2 = r^2)$$

then the relation (1) can be shown to be equivalent to

$$[X, G] = -iY, \quad [Y, G] = iX \quad \text{and} \quad X^2 + Y^2 = r^2. \quad (5)$$

On the other hand, in the Dirac formalism we consider the simplest Hamiltonian

$$H = \frac{1}{2} (p_x^2 + p_y^2), \quad (6)$$

$$[\phi_j, x] \Big|_{\phi_1=\phi_2=0} = [\phi_j, y] \Big|_{\phi_1=\phi_2=0}$$

$$= [\phi_j, p_x] \Big|_{\phi_1=\phi_2=0} = [\phi_j, p_y] \Big|_{\phi_1=\phi_2=0} = 0 \quad (j=1,2), \quad (16)$$

which proves the consistency of (13) with the constraints.

Next we try to formulate quantum mechanics on the closed loop (11) from the viewpoint stated in our previous paper [1], [2]. To this end we start with the following commutation relations:

$$[x, G] = -i(y + \frac{F'}{F}x), \quad [y, G] = i(x - \frac{F'}{F}y), \quad (17)$$

where x, y and G are self-adjoint operators. Since the operator G is commutable with ϕ_1 , any point on the closed loop can be moved to an arbitrary point on it by the unitary transformation $\exp[i\lambda G]$. Thus we see that the relations (17) provide us with the algebra that satisfies the requirement of quantization formulated in the configuration space.

Defining the operators X and Y as

$$X = x F(\theta), \quad Y = y F(\theta), \quad (18)$$

we can show that they satisfy Eqs(5) with G given in Eq.(17). Accordingly the operators X, Y and G form the fundamental algebra on S^1 . Since $Y/X = y/x = \tan\theta$, we can write G in (17) as

$$G = -i \frac{d}{d\theta} + \alpha \quad (0 \leq \alpha < 1) \quad (19)$$

with

$$\theta = \tan^{-1}(y/x)$$

in the x -diagonal representation. The operator (19) gives us the transformation in (4) for the wavefunction $\psi(\theta)$ and hence can be regarded as an operator defined on the Hilbert space $\mathcal{H}_\alpha$, in which the inner product of the wavefunctions χ and ψ is given by

$$(\chi, \psi) = \int_0^{2\pi} d\theta \chi^*(\theta) \psi(\theta), \quad (20)$$

that is, the representation space remains unchanged under deformation of S^1 into the loop (11). Thus, we can argue about the algebra on the closed loop through the algebra on S^1 . The correspondence is shown in Fig. 1.

Fig. 1

It is remarked, in this connection, that there is a similar relation also in the case of the Dirac algebra. To see this, we define the operators P_x and P_y , using the fundamental operators x, y, p_x and p_y in the algebra (13), by

$$P_x = \frac{1}{2}(\{1/F, p_x\} + \{F'/F^2, p_y\}), \quad P_y = \frac{1}{2}(\{1/F, p_y\} - \{F'/F^2, p_x\}), \quad (21)$$

or with

$$\xi = 1/F,$$

$$P_y = \frac{1}{2}(\{\xi, p_x\} - \{\xi', p_y\}), \quad P_x = \frac{1}{2}(\{\xi', p_x\} + \{\xi, p_y\}). \quad (22)$$

from which P_x and P_y are conversely expressed as

$$P_x = \frac{1}{2} \left(\left\{ \frac{\xi}{\xi^2 + \xi'^2}, p_x \right\} + \left\{ \frac{\xi'}{\xi^2 + \xi'^2}, p_y \right\} \right),$$

$$P_y = \frac{1}{2} \left(\left\{ \frac{-\xi'}{\xi^2 + \xi'^2}, p_x \right\} + \left\{ \frac{\xi}{\xi^2 + \xi'^2}, p_y \right\} \right). \quad (23)$$

Then, using the first four relations of (7), validity of which will be shown in a minute, the constraint (15) can equivalently be rewritten as

then the quantum versions of the Dirac brackets are

$$\begin{aligned}
[X, Y] &= 0, \\
[X, P_x] &= i \frac{Y^2}{r^2}, \\
[Y, P_y] &= i \frac{X^2}{r^2}, \\
[X, P_y] &= [Y, P_x] = -i \frac{XY}{r^2}, \\
[P_x, P_y] &= i \frac{1}{r^2} (Y P_x - X P_y),
\end{aligned} \tag{7}$$

with the constraints

$$\begin{aligned}
X^2 + Y^2 &= r^2, \\
\{X, P_x\} + \{Y, P_y\} &= 0,
\end{aligned} \tag{8}$$

where X, Y, P_x, P_y are self-adjoint operators.

It can be shown that the operators in the above two formalisms on S^1 are transcribed to each other. The operators P_x and P_y in the Dirac algebra are expressed as

$$P_x = \frac{-1}{2r^2} \{Y, G\} \quad \text{and} \quad P_y = \frac{1}{2r^2} \{X, G\}, \tag{9}$$

while the generator G in the fundamental algebra is

$$G = X P_y - Y P_x. \tag{10}$$

Thus P_x and P_y written with G of (2) are operators in $\mathfrak{H}_\alpha$.

Now let us turn to quantum mechanics on a closed loop. To make our story concrete we examine a closed loop parametrized as follows:

$$(x^2 + y^2) F^2(\theta) = r^2 \tag{11}$$

where, $r = \text{const.}$, $\theta = \tan^{-1}(y/x)$ and $F(\theta) [= F(\theta + 2\pi)]$ is an arbitrary positive periodic function. Although a strongly deformed loop cannot be covered by this parametrization, we shall, for the present, be satisfied with it and start with the Dirac formalism of quantum mechanics on the loop (11). Generalization to the loops of a really arbitrary shape will be discussed in our forthcoming paper.

As before we consider the Hamiltonian

$$H = \frac{1}{2} (p_x^2 + p_y^2). \tag{12}$$

In classical mechanics the primary constraint

$$\phi_1^{cl} = \frac{1}{2} [(x^2 + y^2) F^2(\theta) - r^2] \approx 0$$

leads to the following secondary constraint

$$\phi_2^{cl} = (xF - yF') P_x + (yF + xF') P_y \approx 0.$$

The classical Poisson bracket between these is

$$\{\phi_1^{cl}, \phi_2^{cl}\}_{cl} \approx r^2 (F'^2 + F'^2)/F.$$

Thus using the classical Dirac brackets obtained with this relation we may write the quantum version as

$$\begin{aligned}
[x, y] &= 0, \\
[x, p_x] &= i \left(1 - \frac{F^2(xF - yF')^2}{r^2(F^2 + F'^2)} \right), \\
[y, p_y] &= i \left(1 - \frac{F^2(yF + xF')^2}{r^2(F^2 + F'^2)} \right), \\
[x, p_y] &= [y, p_x] = -i \frac{F^2(xF - yF')(yF + xF')}{r^2(F^2 + F'^2)}, \\
[p_x, p_y] &= i \frac{1}{2} \left\{ \frac{F^3(F + F'')}{r^2(F^2 + F'^2)}, (yP_x - xP_y) \right\}.
\end{aligned} \tag{13}$$

with the constraints

$$\phi_1 = \frac{1}{2} [(x^2 + y^2) F^2(\theta) - r^2] = 0, \tag{14}$$

$$\phi_2 = \{ (xF - yF'), p_x \} + \{ (yF + xF'), p_y \} = 0, \tag{15}$$

where x, y, p_x, p_y are self-adjoint operators. Applying the above commutation relations we can explicitly show that

$$\begin{aligned}
P_Y &= \frac{1}{2} \left(\{-\eta, P_x\} + \{\xi, P_y\} \right) \\
&= \frac{1}{2r^2} \{ \xi X + \eta Y, G \},
\end{aligned} \tag{33}$$

where we write

$$\xi = \frac{\xi}{\xi^2 + \xi'^2} \quad \text{and} \quad \eta = \frac{\xi'}{\xi^2 + \xi'^2}. \tag{34}$$

Then the explicit form of the Hamiltonian becomes

$$H = \frac{1}{2r^2} \left(G \nabla G + \frac{1}{4} (Y - Y') + \frac{1}{2} (\xi' \eta - \eta' \xi) + \frac{1}{4} (\xi'^2 + \eta'^2) \right) \tag{35}$$

with

$$Y = \frac{1}{\xi^2 + \xi'^2}. \tag{36}$$

Needless to say, the operators P_x and P_y and the Hamiltonian (35) with G of (19) are those defined in the Hilbert space $\mathcal{H}_\alpha$ [9].

We have seen that quantum mechanics on the loop (11) can be reduced to that on the S^1 and that although the algebra of the observables gets deformed, the essential structure of the representation space remains unchanged, which implies that only the topological structure of the manifold is playing an important role in formulating quantum mechanics on it.

Finally, we conclude the note with a few remarks.

1) The extension of our arguments to the higher dimensional surfaces is straightforward and can be based on the quantum mechanics on S^D , deforming the latter by an arbitrary function $F(n_1, n_2, \dots, n_{D+1})$, where n_α ($\alpha = 1, 2, \dots, D+1$) are the direction cosines in the flat space R^{D+1} where S^D is embedded.

2) As has already been referred we are unable to exhaust all the closed loops by deforming the circle with an arbitrary function on the rays from the origin. It is noted, however, that we can extend our approach so as to include an arbitrary closed loop by introducing a suitable homeomorphic mapping between the algebras on the loop and S^1 . Then it can be shown that the main result in this note remains unchanged. Details will be published elsewhere together with related topics.

3) Since we have the explicit expression for the Hamiltonian on the closed loop, we can carry out the path integration on it [10]. This problem is under study now.

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$$\{X, P_x\} + \{Y, P_y\} = 0, \quad (24)$$

thus the operators P_x and P_y satisfy the constraints of the Dirac quantization on S^1 together with the operators X and Y . Furthermore using

$$x = \frac{\cos\theta}{F}r, \quad y = \frac{\sin\theta}{F}r, \quad (25)$$

and

$$[A(\theta), P_x] = -i \frac{A'(\theta)F^3}{r^2(F^2 + F'^2)}(yF + xF'),$$

$$[A(\theta), P_y] = i \frac{A'(\theta)F^3}{r^2(F^2 + F'^2)}(xF - yF'), \quad (26)$$

which can also be written as

$$\begin{aligned} [A(\theta), P_x] &= -i \frac{A'(\theta)}{r^2} Y, \\ [A(\theta), P_y] &= i \frac{A'(\theta)}{r^2} X, \end{aligned} \quad (27)$$

with

$$A(\theta) = A(\theta + 2\pi).$$

we easily obtain, from (13), the first four relations in the Dirac algebra (7) for quantum mechanics on S^1 . However, the direct derivation of the last relation of (7) from (13) using these relations is a tedious and complicated task, so we shall proceed in the following way: Applying the constraint (24) together with the second and the fourth relations in (7) we get

$$\begin{aligned} 0 &= \{X, P_x\} + \{Y, P_y\}, P_x\} \\ &= \{X, P_x\}, P_x\} + \{Y, P_x\}, P_y\} - \{Y, [P_x, P_y]\} \\ &= \frac{i}{r^2} \{Y^2, P_x\} - \{XY, P_y\} - \{Y, [P_x, P_y]\} \\ &= \left\{ Y, \frac{i}{r^2} (YP_x - XP_y) - [P_x, P_y] \right\}. \end{aligned} \quad (28)$$

On the other hand we obtain from the second and third relations of (7)

$$\begin{aligned} &\left[Y, \frac{i}{r^2} (YP_x - XP_y) - [P_x, P_y] \right] \\ &= \frac{i}{r^2} (Y[Y, P_x] - X[Y, P_y]) - [Y, P_x], P_y] - [P_x, [Y, P_y]] = 0, \end{aligned} \quad (29)$$

and combining (28) and (29) we have

$$Y \left(\frac{i}{r^2} (YP_x - XP_y) - [P_x, P_y] \right) = 0,$$

and consequently

$$Y^2 \left(\frac{i}{r^2} (YP_x - XP_y) - [P_x, P_y] \right) = 0, \quad (30)$$

similarly

$$X^2 \left(\frac{i}{r^2} (YP_x - XP_y) - [P_x, P_y] \right) = 0. \quad (31)$$

Summing up these and with the use of the first relation of (8) we are lead to the last relation of (7).

Thus starting with the Dirac algebra on the closed loop we have arrived at the Dirac algebra on S^1 . It is also evident from the above argument that the former can be derived from the latter. Related to this we note that the non-canonical transformation as seen in (17) and (20) plays a crucial role in establishing a connection between these two types of algebras.

With the aid of (9) and (19) we can write P_x and P_y , in $\mathfrak{H}_\alpha$, as

$$\begin{aligned} P_x &= \frac{1}{r} \left(i \sin\theta \frac{d}{d\theta} + \frac{i}{2} \cos\theta - \alpha \sin\theta \right), \\ P_y &= \frac{-1}{r} \left(i \cos\theta \frac{d}{d\theta} - \frac{i}{2} \sin\theta - \alpha \cos\theta \right), \end{aligned} \quad (32)$$

from which using (23) and (9) we obtain the explicit expressions for p_x and p_y in the x -diagonal representation;

$$\begin{aligned} p_x &= \frac{1}{2} \left(\{ \xi, P_x \} + \{ \eta, P_y \} \right) \\ &= \frac{1}{2r^2} \{ \eta X - \xi Y, G \}, \end{aligned}$$