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REPARAMETRIZATION INVARIANCE IN SOME NON-LOCAL 1-D FIELD THEORIES*

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Abstract

In this paper we consider 1-D non-local field theories with a particular $1/r^2$ interaction, a constant gauge field and an arbitrary scalar potential. We show that any such theory that is at a renormalization group fixed point also satisfies an infinite set of reparametrization invariance Ward identities. We also prove that, for special values of the gauge field, the value of the potential that satisfies the Ward identities to first order in the potential strength remains a solution to all orders in the potential strength, summed over all loops. These theories are of interest because they describe dissipative quantum mechanics with an arbitrary potential and a constant magnetic field. They also give solutions to open string theory in the presence of a uniform gauge field and an arbitrary tachyon field.

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1. Introduction

In 2-D statistical systems, scale invariance at a second order or higher phase transition and locality generally lead to conformal invariance. One dimensional systems can also have phase transitions, but only if they are non-local. Thus conformal invariance is not expected to be a symmetry of critical 1-D theories. However, it is possible that they could exhibit some other enhanced symmetry related to reparametrizations of the theory. One example of a non-local 1-D system that undergoes phase transitions is dissipative quantum mechanics, as described by the Caldeira-Leggett model [1]. The functional integral for this theory is also used to obtain the boundary state in open string theory [2]. This boundary state describes an open string where the field is free on the interior of the 2-D world sheet and all the interactions take place at the 1-D boundary.

Callan and Thorlacius have shown [3] that in order to be a solution to open string theory, the boundary state must satisfy a set of Ward identities that reflect the reparametrization invariance of the string theory. Even though the 1-D field theory is not manifestly reparametrization invariant, in order to correspond to a boundary state it must possess a "hidden" reparametrization symmetry, given by the Ward identities. The question arises as to whether we can use this symmetry to classify and construct 1-D critical dissipative theories, just as conformal symmetry was used to classify 2-D critical theories. Also, it is natural to ask whether, for 1-D theories, scale invariance of the theory implies that the theory satisfies the Ward identities. We will address this latter question in this paper for the case of an arbitrary tachyon potential and a uniform gauge field. For the dissipative quantum system, this corresponds to considering an arbitrary potential and a uniform magnetic field.

Callan and Thorlacius have shown to one loop that for an arbitrary gauge field the fixed point of the renormalization group also satisfies the reparametrization invariance Ward identities [3], and similarly for an arbitrary tachyon potential [2]. To lowest order in the potential strength, the critical tachyon potential is a cosine with period p_c , and the theory with this cosine potential looks very much like the Sine-Gordon model restricted to a line. In the Sine-Gordon model, the cosine potential that is at a zero of the β -function to lowest order in V_0 no longer remains so at higher orders in V_0 . However, in this paper we will prove that in the 1-D theory the cosine potential that satisfies the β -function to lowest order is at a renormalization group fixed point and has a vanishing reparametrization invariance anomaly to all orders in the potential.

The paper is organized as follows. In Section 2 we give a brief review of the open string boundary state and dissipative quantum mechanics. In Section 3 we present the reparametrization invariance Ward identities and our regularization scheme. In the following two sections, we will evaluate the reparametrization Ward identities for an arbitrary tachyon field and a constant, uniform gauge field, diagrammatically. The first part of Section 4 follows closely the appendix of [2]. The goal of these two sections is to reduce the formal expression for the Ward identity to an identity for the graphs in the perturbative expansion of the IP1 function, summed to all orders in the string tension, α' . This result enables us to show that if the tachyon is periodic with period p_c (where p_c depends on the strength of the gauge field), and if we could ignore the regulator, then the Ward identity would be satisfied to all orders in α' . Additionally, these calculations show that even when

the regulator is taken into account, the special choice of tachyon potential does satisfy the Ward identity at every *finite* order in the tachyon strength, V_0 , and α' . However, others have shown [4] that for closed string tachyons in 2D sigma models, even if the β -function is satisfied to all finite orders in α' , new divergences arise once it is summed to all orders. This can give anomalous contributions to the β -function. Therefore, in the remainder of the paper we evaluate the graphs, summed over all loops, more carefully.

In Section 6 we define what we mean by the flows generated by the Ward identity, and in Section 7 we calculate the flow for the coupling constant of $e^{i\phi}\phi(t)$. Readers who are only interested in the main results may wish to skip sections 4, 5 (except for 5.5) and 7. In Section 8 we find that, to leading order in the cutoff, the Ward identity is equivalent to the β -function. This means that the condition for the Ward identity to be satisfied is the same as for the system to lie at a zero of the β -function. Thus, once the system is scale invariant, it does have the infinite set of “hidden” symmetries corresponding to the reparametrization invariance of the underlying string theory. Next, in Section 9, we discuss the validity of our approximations, and in Section 10, we show that the flow for the coupling constant of $d\phi(t)/dt$ is always zero. In Section 11, we show that, for special values of the gauge field, the cosine potential with period p_c does satisfy the Ward identity to all orders in V_0 and α' . Consequently, the naive first order result is true to all orders, and we have found exact fixed points of dissipative quantum mechanics that correspond to solutions of string theory. In Section 12 we conclude the paper with a few remarks about the consequences of our results.

2. Background: Open String Theory and the Dissipative Hofstadter Model

In this section, we briefly review the open string boundary state and its connection to dissipative quantum mechanics. For more details, the reader is referred to references [1], [2], [5]. (The first part of this section closely follows the review in reference [6]). In the presence of open string background fields, interactions between a string and the background take place at the boundary of the string. Their effects can be represented by a boundary state $|B\rangle$, which lies in the closed string Hilbert space.

In [7] it is shown that this boundary state is given by

$$|B\rangle = \exp \left\{ \sum_{m=1}^{\infty} \frac{1}{m} \alpha_{-m} \cdot \tilde{\alpha}_{-m} \right\} \int [D\vec{X}(s)]' \exp(-S_{KE} - S_A - S_V - S_{LS})|0\rangle, \quad (2.1)$$

where

$$S_{KE} = \frac{1}{8\pi^2\alpha'} \int_0^T ds \int_{-\infty}^{\infty} ds' \frac{(\vec{X}(s) - \vec{X}(s'))^2}{(s-s')^2}; \quad (2.2)$$

$$S_A = i \int_0^T ds A_{\mu}(\vec{X}) \dot{\vec{X}}^{\mu}; \quad (2.3)$$

$$S_V = \int_0^T ds V(\vec{X}); \quad (2.4)$$

and

$$S_{LS} = \sqrt{\frac{2}{\alpha'}} \int ds \alpha(s) \cdot \vec{X}(s) \quad \text{with} \quad \alpha^{\mu}(s) = \sum_{m=1}^{\infty} i(\tilde{\alpha}_{-m}^{\mu} e^{-im\tau} + \alpha_{-m}^{\mu} e^{im\tau}). \quad (2.5)$$

In these expressions, T is the parameter length of the boundary, α' is the string constant, and $A_{\mu}(\vec{X})$ and $V(\vec{X})$ are the gauge fields and tachyon fields, respectively. The creation operators of the left- and right-moving modes of the closed string, $\tilde{\alpha}_{-m}$ and α_{-m} , act on the closed string vacuum $|0\rangle$ to create some state in the closed string Hilbert space. The notation $[D\vec{X}(s)]'$ means that the zero-mode, $\vec{X}_0$, is not integrated out. The commuting objects $\tilde{\alpha}_{-m}$, α_{-m} and $\vec{X}_0$ together make up a set of coordinates which specify where the boundary lies in the target space, and the boundary state is just a functional of these coordinates. As an example of the usefulness of this construct, we note that the projection of $|B\rangle$ onto the graviton state is essentially the energy-momentum tensor of the open string object under study. This gives us a string-theoretic way to define such important notions as gravitational and inertial mass.

This path integral is the generating functional for a renormalizable “one-dimensional” field theory described by the underlying action $S_{KE} + S_A + S_V$. S_{LS} is the linear source term in the generating function. This theory is divergent, so it requires a regulator. One possible choice is to add a dimension-two operator to the action of the form

$$S_R = \int_0^T ds \frac{1}{2} M \dot{\vec{X}}^2(s). \quad (2.6)$$

To obtain the boundary state, in the end we must take the cut-off, M , to zero. In order for this limit to be meaningful, the field theory must lie at a renormalization group fixed point, which implies that the gauge and tachyon fields must satisfy some “vanishing beta function” equations of motion for open string background fields. This means that the associated 1-D field theory must lie at a phase transition.

This 1-D field theory also describes a quantum mechanical particle subject to a potential, a magnetic field, and a dissipative force. According to the Caldeira-Leggett model [1] for dissipation, the particle interacts linearly with a bath of harmonic oscillators. The interaction strengths and the frequencies of the oscillators are chosen so that in the classical limit the particle is subject to ohmic dissipation. Because the oscillators appear only linearly and quadratically in the action, they can be integrated out of the path integral. The resulting action for the particle is given by

$$S_{dqm} = S_R + S_{KE} + S_A + S_V, \quad (2.7)$$

where S_R , S_{KE} , S_A and S_V are defined in equations (2.6), (2.2), (2.3), and (2.4). However, now M is the mass of the particle, $A_{\mu}(\vec{X})$ is the vector potential and $V(\vec{X})$ is the scalar potential. The classical coefficient of friction, η , appears only in S_{KE} and is related to the string tension by $\eta = 1/(2\pi\alpha')$. In order to correspond to a solution of string theory, this system must lie at a phase transition.

One dissipative quantum system which exhibits phase transitions is the dissipative Wannier-Azbel-Hofstadter model (which is also known as the dissipative Hofstadter model). It is obtained by restricting the particle to lie in two dimensions and taking

$$A_\mu(\vec{X}) = \frac{eB}{2c} \epsilon_{\mu\nu} \dot{X}^\nu \quad (2.8)$$

and

$$V(\vec{X}) = -V_0 \cos\left(\frac{2\pi\dot{X}(t)}{a}\right) - V_0 \cos\left(\frac{2\pi\dot{Y}(t)}{a}\right). \quad (2.9)$$

This model depends on the dimensionless parameters defined by

$$2\pi\tilde{\alpha} = \frac{\eta a^2}{\hbar}, \quad 2\pi\tilde{\beta} = \frac{eB}{\hbar c} a^2. \quad (2.10)$$

In ref.[5] we have shown that the phase diagram of this model should have an infinite number of critical circles in the $\alpha - \beta$ plane, which suggests that we have found many solutions to open string theory.

Motivated by this connection, in this paper we will only consider the 1-D field theories where $\vec{X}$ is restricted to lie in two dimensions and the gauge field is taken to be a constant of the form given by eqn. (2.8). (Strictly speaking, to calculate the boundary state, one must still include the remaining 24 dimensions of $\vec{X}$, which, for convenience, will be assumed to be free fields. Although there is some relation, it is not clear that there is a direct connection between this model with $\vec{X}$ taken to be two-dimensional and open string theory in 1+1 dimensions as considered, for example, in reference [8].) Also in analogy with the dissipative quantum mechanics case, we would like to define the dimensionless parameters as in equation (2.10). However, for now we do not want to assume the tachyon field has a particular period, so instead we will define the dimensionful parameters

$$\alpha = \frac{1}{\alpha'}, \quad \beta = \frac{2\pi eB}{c}. \quad (2.11)$$

These differ from the usual choice of β and α for the dissipative Wannier-Azbel-Hofstadter model, given in eqn. (2.10), only in that we have left out the $\hbar$ and the factor of $a^2/(2\pi)^2$.

3. The Reparametrization Invariance Ward Identities

From the preceding discussion, it might appear that we have found many solutions to open string theory, since the dissipative Wannier-Azbel-Hofstadter model has many critical theories. However, the space-time physics of the string theory is not only invariant under changes in scale, but it is also invariant under reparametrizations of the world sheet. This means that the boundary state must be invariant under reparametrizations of the boundary of the world sheet. The condition for the reparametrization invariance of the boundary state is given by

$$(L_n - \tilde{L}_{-n})|B\rangle = 0 \quad \text{for} \quad -\infty \leq n \leq \infty. \quad (3.1)$$

Here, $|B\rangle$ is the boundary state defined in equation (2.1), and L_n and $\tilde{L}_{-n}$ are the closed-string Virasoro generators, given by

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m, \quad (3.2)$$

and similarly for $\tilde{L}_{-n}$. For $-m < 0$, the α_{-m} and $\tilde{\alpha}_{-m}$ are the closed string creation operators; and for $m \geq 0$, the α_m can be represented in terms of derivatives with respect to the creation operators, α_{-m} . When we apply the operator $L_n - \tilde{L}_{-n}$ to the expression for the boundary state in equation (2.1), then, as shown in [3], the condition for reparametrization invariance is reduced to a set of Ward identities. For $n \geq 0$, they are given by

$$0 = \delta_n \Gamma - \sum_{m=1}^{n-1} \frac{1}{\alpha'} m(n-m) \phi_m \cdot \phi_{n-m} - \sum_{m=1}^{n-1} \frac{1}{2} m(n-m) \frac{\partial^2 W}{\partial \alpha_{-m} \cdot \partial \alpha_{m-n}} + in \sqrt{\frac{\alpha'}{2}} \frac{\partial^2 W}{\partial \dot{X}_0 \cdot \partial \alpha_{-n}}. \quad (3.3)$$

For $n < 0$, we just replace α_{-k} with $\tilde{\alpha}_{-k}$ and ϕ_k with ϕ_{-k} . In equation (3.3), $W(\alpha, \tilde{\alpha}, \dot{X}_0)$ is the connected generating functional given by

$$W(\alpha, \tilde{\alpha}, \dot{X}_0) = \int [D\vec{X}(s)]' \exp(-S_R - S_{KE} - S_A - S_V - S_{LS})|0\rangle, \quad (3.4)$$

and $\Gamma(\phi)$ is the 1PI generating functional. It is obtained by Legendre transforming W , as follows:

$$\Gamma(\phi^{cl}) = W(\alpha, \tilde{\alpha}, \dot{X}_0) - i \sqrt{\frac{2}{\alpha'}} \sum_{m=1}^{\infty} (\alpha_{-m} \cdot \phi_m^{cl} + \tilde{\alpha}_{-m} \cdot \phi_{-m}^{cl}), \quad (3.5)$$

where ϕ^{cl} and α are related by

$$\begin{aligned} \phi_m^{cl} &= -i \sqrt{\frac{\alpha'}{2}} \frac{\partial W}{\partial \alpha_{-m}} & \phi_{-m}^{cl} &= -i \sqrt{\frac{\alpha'}{2}} \frac{\partial W}{\partial \tilde{\alpha}_{-m}} \\ \alpha_{-m} &= i \sqrt{\frac{\alpha'}{2}} \frac{\partial \Gamma}{\partial \phi_m^{cl}} & \tilde{\alpha}_{-m} &= i \sqrt{\frac{\alpha'}{2}} \frac{\partial \Gamma}{\partial \phi_{-m}^{cl}} \end{aligned} \quad (3.6)$$

$$\phi_0^{cl} = \dot{X}_0.$$

The first term in equation (3.3), $\partial_n \Gamma$, is the variation of Γ under an infinitesimal reparametrization $s \rightarrow s + f$, with $f_n(s) = i\epsilon e^{ins}$. The remaining terms reflect the fact that the effective action is not reparametrization invariant. In fact, the non-local kinetic term, S_{KE} , in the classical action is not reparametrization invariant, so the second term in the Ward identity subtracts off its variation, and the third term corrects for the effect that this non-invariance has on higher order diagrams. The last term makes up for

the fact that we have selected out a particular parametrization in order to define the zero mode of $\vec{X}$ so that we could treat it separately.

To evaluate the Ward identity, we use a perturbative background-field expansion. To find $\Gamma(\phi^{cl}(t))$, we begin by expanding $V(\vec{X})$ around $\vec{X}(t) = \phi^{cl}(t)$, where ϕ_m^{cl} is defined in equation (3.6). We will assume that $\phi^{cl}(t)$ is a smooth function of t , and we will use the notation $\vec{\phi}^{cl}(t)$ and $\vec{\phi}(t)$ interchangeably. If we set $\vec{X} = \vec{\phi} + \vec{x}$, then S_V becomes

$$S_V = \int \frac{d^2}{2\pi} V(\vec{X}) = \int \frac{d^2}{2\pi} V(\vec{\phi}) + \int \frac{d^2}{2\pi} \nabla_\mu V(\vec{\phi}) x^\mu + \frac{1}{2} \int \frac{d^2}{2\pi} \nabla_\mu \nabla_\nu V(\vec{\phi}) x^\mu x^\nu + \dots \quad (3.7)$$

Similarly, to find $W(\alpha, \vec{\alpha}, \vec{X}_0)$, we expand $V(\vec{X})$ around the zero mode, $\vec{X}_0$. Next, we treat S_V as a perturbation and Taylor expand $\exp(-\int S_V)$. The propagator is then determined by the remaining quadratic terms in the action, $S_{KE} + S_A$ and S_R . However, as explained above, S_R acts only as an ultra-violet regulator, so we will set M to zero and choose a more convenient cutoff. The quadratic part of the action can then be written in momentum space as

$$S_{KE} + S_A = \frac{1}{2} \int \frac{d\omega}{2\pi} \vec{X}_\mu(\omega) \vec{X}_\nu(\omega) \tilde{S}^{\mu\nu}(\omega), \quad (3.8)$$

where the inverse propagator, $\tilde{S}(\omega)$, is given by

$$\tilde{S}^{\mu\nu}(\omega) = \frac{\alpha}{2\pi} |\omega| \delta^{\mu\nu} + \frac{\beta}{2\pi} \omega \epsilon^{\mu\nu}. \quad (3.9)$$

For the ultra-violet regulator, we will multiply the propagator by an $\exp(-\epsilon|\omega|)$ cutoff. In addition, we will need an infra-red cutoff, which we will define by putting the time, t , on a circle of circumference T . (For simplicity, we will take $T = 2\pi$, but this should not significantly affect our results.) The diagonal part of the propagator is then given by

$$\begin{aligned} G^{\mu\nu}(t) &= \frac{\alpha}{\alpha^2 + \beta^2} \sum_{m \neq 0} \frac{1}{|m|} e^{imt} e^{-\epsilon|m|} \delta^{\mu\nu} \\ &= -\frac{\alpha}{\alpha^2 + \beta^2} \ln(1 + e^{-2\epsilon} - 2e^{-\epsilon} \cos t) \delta^{\mu\nu}. \end{aligned} \quad (3.10)$$

The off-diagonal part is

$$\begin{aligned} G^{\mu\nu}(t) &= -\epsilon^{\mu\nu} \frac{\beta}{\alpha^2 + \beta^2} \sum_{m \neq 0} \frac{1}{m} e^{imt} e^{-\epsilon|m|} \\ &= \epsilon^{\mu\nu} \frac{\beta}{\alpha^2 + \beta^2} \ln \left[\frac{\sin\{(t + i\epsilon)/2\}}{\sin\{(-t + i\epsilon)/2\}} e^{-i\epsilon} \right]. \end{aligned} \quad (3.11)$$

Note that when $\beta = 0$, we have $\frac{\alpha}{\alpha^2 + \beta^2} = \alpha'$.

We will find that all diagrams of interest can be written in terms of the following sum over $G^{\mu\nu}(t)$:

$$E(t; \mu, \nu) = \sum_n \frac{1}{n!} (G(t)^{\mu\nu})^n = \epsilon^{G^{\mu\nu}}. \quad (3.12)$$

The propagator $G^{\mu\nu}(t)$, and not the propagator $E(t; \mu, \nu)$, determines the connectedness and irreducibility of the diagrams. That means that if we express the graphs in terms of $E(t; \mu, \nu)$, it is more difficult to keep track of whether they are connected or 1PI. However, by using the propagator $E(t; \mu, \nu)$, we will have automatically summed over all orders in α' or $1/\alpha$.

4. Ward Identity with $\beta = 0$

We will begin by considering the case when $\beta = 0$ and restrict $\vec{X} = x$ to lie in one dimension. In that case, according to standard perturbation theory for path integrals, the general $O(V_0^N)$ graph will have N vertices with a factor of $V(x(i))$ associated to each vertex. The i^{th} vertex has M_i legs coming out of it, and for each leg there is a derivative $\frac{\partial}{\partial x(i)}$ acting on $V(x(i))$. Each leg is joined to one other leg by the propagator $G(t)$, which is obtained from $G^{xx}(t)$ by setting $\beta = 0$. We will define a petal to be a propagator that joins two legs from the same vertex, and we will let P_i equal the number of petals coming out of the i^{th} vertex. Also, we will define a link to be a propagator that joins legs coming from two different vertices, and we will let l_{ij} be the number of links between two different vertices i and j . The total number of legs coming from a vertex is related to the number of petals and links coming from the vertex by the equation

$$M_i = 2P_i + \sum_j l_{ij}. \quad (4.1)$$

For example, the following graph with three vertices has $P_1 = 0$, $P_2 = 3$, $P_3 = 2$, $l_{12} = 1$, $l_{23} = 1$ and $l_{31} = 2$.

The value of a general graph is given by

$$\begin{aligned} \Gamma(l_{ij}, P_i) &= -\frac{1}{N!} \frac{1}{\prod_{i,j} l_{ij}! \prod_i P_i! 2^{P_i}} \\ &\times \int \prod_i \frac{ds_i}{2\pi} [G(0)]^{\sum_i P_i} \prod_{i < j} [G(s_i - s_j)]^{l_{ij}} \prod_i \left[\frac{\partial}{\partial x(s_i)} \right]^{M_i} V(x(s_i)). \end{aligned} \quad (4.3)$$

The symmetry factor is exactly what one would expect. $1/l_{ij}!$ takes into account that there are l_{ij} identical links joining the i^{th} and j^{th} vertex, and $1/(P_i! 2^{P_i})$ takes into account that there are P_i identical petals coming from each vertex, each with two identical end points. To make a graph for the 1PI vacuum function, we just replace $x(s_i)$ with $\phi^{cl}(s_i)$ (we will write $\phi(s_i)$ for short), and keep only those graphs that remain connected when any one link is cut. For the connected generating-functional, we must remember that the action includes the linear source term $i\sqrt{2/\alpha'} \sum_{m \neq 0} \alpha_m x_m$, which generates extra vertices weighted by $\sqrt{2/\alpha'} \alpha(t)$ and having only one leg. Then the $O(V_0^N)$ graph for W is given

by replacing $x(s_i)$ by X_0 , the zero mode, and keeping only the connected graphs of the form

$$W_{l_{ij}, P_i}(\alpha) = -\frac{1}{N!} \frac{1}{\prod_{i,j} l_{ij}! \prod_i P_i! 2^{P_i}} \prod_{i \leq N} \left[\frac{\partial}{\partial X_0} \right]^{M_i} V(X_0) \times \int \prod_i \frac{ds_i}{2\pi} [G(0)]^{\sum_{i=1}^N P_i} \prod_{i < j} [G(s_i - s_j)]^{l_{ij}} \prod_{i > N} \sqrt{\frac{2}{\alpha}} \alpha(s_i). \quad (4.4)$$

An example of such a graph is

4.1. Variation of the 1PI Vacuum Function

First, we will calculate $\delta_n \Gamma(\phi(s))$. Strictly speaking, this is the change in $\Gamma(\phi(s))$ when $s \rightarrow s + f_n$, where f_n is given by $f_n(s) = i e^{i n s}$. Under this reparametrization, the change in ϕ is $\delta\phi(s) = \left[\frac{\partial\phi}{\partial s} \right] f_n(s)$, which implies that the change in $V(\phi)$ is given by

$$\delta V(\phi(s_i)) = \left[\frac{d}{d\phi(s_i)} V(\phi(s_i)) \right] \delta\phi = \left[\frac{d}{ds_i} V(\phi(s_i)) \right] f_n(s_i). \quad (4.6)$$

Then $\delta_n \Gamma(\phi(s))$ is just the sum over graphs for Γ with one of the $V(\phi(s_i))$'s replaced by $f_n(s_i) \frac{d}{ds_i} V(\phi(s_i))$. Now we can integrate by parts. For each vertex, we obtain two types of terms, one with the derivative acting on $f_n(s_i)$, and the other with the derivative acting on $G(s_i - s_j)$. The latter is just the change in G under a reparametrization of s_i . The resulting graphs are minus the change of Γ under reparametrizations where $V(\phi(s_i))$ behaves like a scalar. The first term, when $f_n(s_k)$ is varied, is given by

$$\begin{aligned} \delta_n \Gamma_P(l_{ij}, P_i) &= \frac{1}{N!} \frac{1}{\prod_{i,j} l_{ij}! \prod_i P_i! 2^{P_i}} [G(0)]^{\sum_i P_i} \\ &\times \int \prod_i \frac{ds_i}{2\pi} f_n(s_k) \prod_{i < j} [G(s_i - s_j)]^{l_{ij}} \prod_i \left[\frac{\partial}{\partial\phi(s_i)} \right]^{M_i} V(\phi(s_i)) \\ &= \int \frac{ds_k}{2\pi} f_n(s_k) h_{l_{ij}, P_i}(s_k). \end{aligned} \quad (4.7)$$

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In this equation, $h_{l_{ij}, P_i}(s_k)$ is defined to be a graph that is integrated over all vertices except for s_k , with P_i petals at the i th vertex and l_{ij} links joining the i th and j th vertex. The second term, when $G(s_k - s_m)$ is varied, is given by

$$\begin{aligned} \delta_n \Gamma_L(l_{ij}, P_i) &= \frac{1}{N!} \frac{1}{\prod_{i,j} l_{ij}! \prod_i P_i! 2^{P_i}} \int \prod_i \frac{ds_i}{2\pi} f_n(s_k) [G(0)]^{\sum_i P_i} \\ &\times \prod_{i < j} [G(s_i - s_j)]^{l_{ij}} l_{km} \frac{d}{ds_k} \frac{G(s_k - s_m)}{G(s_k - s_m)} \prod_i \left[\frac{\partial}{\partial\phi(s_i)} \right]^{M_i} V(\phi(s_i)) \\ &= \int \frac{ds_k}{2\pi} \frac{ds_m}{2\pi} f_n(s_k) \left[\frac{d}{ds_k} G(s_k - s_m) \right] \frac{\partial}{\partial\phi(s_k)} \frac{\partial}{\partial\phi(s_m)} h_{l_{ij}, P_i}(s_k, s_m). \end{aligned} \quad (4.8)$$

Here, we have defined $h_{l_{ij}, P_i}(s_k, s_m)$ to be a graph that is integrated over all vertices except for s_k and s_m . It has P_i petals at the i th vertex; l_{ij} links joining the i th and j th vertex for $\{i, j\} \neq \{k, m\}$; and $l_{km} - 1$ links joining the k th and m th vertex. The two types of graphs we obtain from varying $\Gamma(l_{ij}, P_i)$ can be summarized by

$$\delta_n \Gamma(l_{ij}, P_i) = \begin{array}{c} \text{Diagram 1: Circle with } s_k \text{ and } s_m \text{ vertices, } l_{km} \text{ links between them.} \\ \text{Diagram 2: Circle with } s_k \text{ vertex and } l_{km} \text{ links forming a loop.} \end{array} + \frac{\partial}{\partial\phi(s_k)} \frac{\partial}{\partial\phi(s_m)} f_n(s_k) h_{l_{ij}, P_i}(s_k, s_m), \quad (4.9)$$

where

$$\begin{array}{c} \text{Diagram 1: Circle with } s_k \text{ vertex.} \\ \text{Diagram 2: Circle with } s_k \text{ and } s_m \text{ vertices, } l_{km} \text{ links between them.} \end{array} = h_{l_{ij}, P_i}(s_k), \quad (4.10)$$

$$\begin{array}{c} \text{Diagram 1: Circle with } s_k \text{ and } s_m \text{ vertices, } l_{km} \text{ links between them.} \\ \text{Diagram 2: Circle with } s_k \text{ and } s_m \text{ vertices, } l_{km} \text{ links between them.} \end{array} = h_{l_{ij}, P_i}(s_k, s_m), \quad (4.11)$$

and

$$s_k \longrightarrow s_m = \frac{d}{ds_k} G(s_k, s_m). \quad (4.12)$$

The second term in the Ward identity explicitly subtracts off the variation of the part of Γ due to the classical non-local kinetic term.

4.2. Evaluation of the "Kinetic Term" in the Ward Identity

Now we turn our attention to the third term in the Ward identity. We will need the identity (used in [2])

$$\frac{1}{\alpha} \sum_{m=1}^{n-1} m(n-m) \tilde{F}_{m, n-m} = \frac{1}{2} \int \frac{ds}{2\pi} \frac{ds'}{2\pi} \delta_f H_0(s-s') F(s, s'), \quad (4.13)$$

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where s and t can be equal. Using the definition of $\delta_f H_0$ and integrating by parts, we have

$$I = - \int \frac{ds' dt'}{2\pi 2\pi} \left(\frac{d}{ds'} G(s-s') \right) f_n(s') [H_0(s'-t') G(t'-t)] - \int \frac{ds' dt'}{2\pi 2\pi} [G(s-s') H_0(s'-t')] f_n(t') \frac{d}{dt'} G(t'-t). \quad (4.26)$$

When we perform the integral over $H_0 G$, we obtain the function $\Delta - 1$, an approximate delta-function minus a constant term due to the zero mode. The part of I resulting from the constant term is

$$I_c = \int \frac{ds'}{2\pi} f_n(s') \left[\frac{d}{ds'} G(s-s') + \frac{d}{ds'} G(t-s') \right]. \quad (4.27)$$

Upon integrating by parts, it becomes

$$I_c = - \int \frac{ds'}{2\pi} f'_n(s') [G(s-s') + G(t-s')]. \quad (4.28)$$

The terms in I coming from the approximate delta-function have the form

$$I_\Delta = - \int \frac{ds'}{2\pi} f'_n(s') \left[\Delta_\epsilon(t-s') \frac{d}{ds'} G(s-s') + \Delta_\epsilon(s-s') \frac{d}{ds'} G(t-s') \right]. \quad (4.29)$$

Returning to the integrals for the full diagrams, when δH is inserted into a petal, we have two integrals. The first is due to ignoring the zero mode in G , and it is obtained from I_c when $s = t$. It is given by

$$D_{P,0}(l_{ij}, P_k) = \int \frac{ds ds'}{2\pi 2\pi} \frac{\partial^2}{\partial \phi^2(s)} h_{l_{ij}, P_k}(s) f'_n(s') G(s-s'). \quad (4.30)$$

The second term comes from the approximate delta function and is obtained from I_Δ with $s = t$. It has the form

$$D_{P,\Delta}(l_{ij}, P_k) = \int \frac{ds ds'}{2\pi 2\pi} \frac{\partial^2}{\partial \phi^2(s)} h_{l_{ij}, P_k}(s) f_n(s') \Delta_\epsilon(s-s') \frac{d}{ds'} G(s-s') = \frac{\partial^2}{\partial \phi(s)} \text{Diagram} \quad (4.31)$$

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We are interested in the limit as $\epsilon \rightarrow 0$. In that limit, Δ_ϵ becomes a delta function, so, for small ϵ , we should be able to evaluate the integral for small values of the argument of Δ_ϵ . For small ϵ and t , Δ_ϵ and G become

$$\Delta_\epsilon(t) \approx \frac{2\epsilon}{\epsilon^2 + t^2} (1 + O(\epsilon^2) + O(t^2)) \quad (4.32)$$

and

$$\frac{d}{dt} G(t) \approx -\frac{2\alpha' t}{t^2 + \epsilon^2} ((1 + O(\epsilon^2) + O(t^2))) \quad (4.33)$$

To perform the integral, we will also make the change of variables

$$s_+ = s \quad s_- = s'; \quad (4.34)$$

and we Taylor expand f around s_+ , keeping the lowest order term that gives a non-vanishing contribution to $D_{P,\Delta}$. Then we obtain the following expression for $D_{P,\Delta}$:

$$D_{P,\Delta}(l_{ij}, P_k) = - \int \frac{ds_+ ds_-}{2\pi 2\pi} \frac{\partial^2}{\partial \phi^2(s_+)} h_{l_{ij}, P_k}(s_+) f'_n(s_+) \frac{4\alpha' \epsilon s_-^2}{(\epsilon^2 + s_-^2)^2}. \quad (4.35)$$

After integrating over s_- , we get the final form for $D_{P,\Delta}$,

$$D_{P,\Delta}(l_{ij}, P_k) = -\alpha' \int \frac{ds_+}{2\pi} \frac{\partial^2}{\partial \phi^2(s_+)} h_{l_{ij}, P_k}(s_+) f'_n(s_+). \quad (4.36)$$

As a result, the two types of graphs involved in the calculation of D_P are

$$D_P(l_{ij}, P_k) = -\frac{\partial^2}{\partial \phi(s)} \text{Diagram} - \alpha' \frac{\partial^2}{\partial \phi(s_+)} \text{Diagram} \quad (4.37)$$

Now we must calculate the effect of inserting $\delta_f H_0$ into a link. Again we have two integrals. The first is

$$D_{L,0}(l_{ij}, P_k) = \frac{1}{2} \int \frac{ds dt ds'}{2\pi 2\pi 2\pi} \frac{\partial}{\partial \phi(s)} h_{l_{ij}, P_k}(s, t) f'_n(s') [G(s-s') + G(t-s')]; \quad (4.38)$$

and the second is

$$D_{L,\Delta}(l_{ij}, P_k) = \frac{1}{2} \int \frac{ds dt ds'}{2\pi 2\pi 2\pi} \frac{\partial}{\partial \phi(s)} \frac{\partial}{\partial \phi(t)} h_{l_{ij}, P_k}(s, t) f'_n(s') \times \left[\Delta_\epsilon(t-s') \frac{d}{ds'} G(s-s') + \Delta_\epsilon(s-s') \frac{d}{ds'} G(t-s') \right]. \quad (4.39)$$

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$$\begin{aligned}
 & \frac{1}{2} \frac{\partial}{\partial \phi(s)} \frac{\partial}{\partial \phi(t)} \\
 & \left[\begin{array}{c} \text{Diagram 1: A shaded circle with points } s \text{ and } t \text{ on its boundary. A solid line segment connects } s' \text{ (outside) to } t \text{ (on the boundary).} \end{array} \right. \\
 & + \\
 & \left. \begin{array}{c} \text{Diagram 2: A shaded circle with points } s \text{ and } t \text{ on its boundary. A solid line segment connects } s' \text{ (outside) to } s \text{ (on the boundary).} \end{array} \right] \\
 & (4.40)
 \end{aligned}$$

Finally, we must evaluate the last term in the Ward Identity. It is given by

where $\frac{\partial W}{\partial \alpha(C)}$ is the sum of all the connected graphs with one α truncated, leaving a free leg; and where $\frac{\partial^2 X}{\partial \alpha^2}$ acts on the $V(X)_0$'s. Again we have two different situations. The first is when the $\frac{\partial}{\partial \alpha}$ acts on the same vertex that the free leg is connected to, and the second is when the $\frac{\partial}{\partial \alpha}$ acts on a different vertex. If we calculate the contribution to C from the graph in (4.5), then in the first case we will get graphs like

$$\frac{\partial}{\partial X_0} \left(\frac{f_n(s)}{\alpha(s')} \right) \alpha(t) \quad (4.42)$$

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$$\begin{array}{ccc} f_n(s) & & \alpha(s') \quad \alpha(r) \\ & \searrow \quad \nearrow & \\ & \frac{\partial}{\partial X_0} & \\ & \nearrow \quad \searrow & \\ t_1 & & t_2 \end{array} \quad (4.43)$$
$$C(l_{ij}, P_k) = C_1(l_{ij}, P_k) + C_2(l_{ij}, P_k)$$

(4.44)

$$C_1(l_{ij}, P_k) = \int \frac{dt}{2\pi} \frac{ds}{2\pi} f'_n(t) G(t-s) \tilde{h}_{l_{ij}, P_k}(s, \alpha) \quad (4.45)$$
$$C_2(l_j, P_k) = \int \frac{dt}{2\pi} \frac{ds}{2\pi} \frac{ds'}{2\pi} f_n'(s') G(s' - s) \tilde{h}_{l_{ij}, P_k}(s, t, \alpha).$$

$\hat{h}_{t_1, P_k}(s, \alpha)$ is a connected graph integrated over all its vertices, except s , and it has two extra derivatives, $\frac{\partial^2}{\partial X_0^2}$ acting on the vertex at s . For example, for the graph in (4.42), $\hat{h}_{t_1, P_k}(t_1, \alpha)$ is given by

$$\tilde{h}_{li..P_2}(t_1, \alpha) = \left[\frac{\partial}{\partial v_0} \right]^4 v(x_0) \bigvee \bigwedge \left[\frac{\partial}{\partial v_0} \right]^5 v(x_0) \alpha(s') \alpha(t) \quad (4.47)$$

Similarly, $\tilde{h}_{ij}, P_\alpha(s, t, \alpha)$ is a connected graph integrated over all vertices except s and t , and it has two extra $\frac{\partial}{\partial \lambda_s}$'s, one acting on the $V(X_0)$ at vertex s and one on the $V(X_0)$ at vertex t .

In order to compare C_1 with $D_{P,0}$ and C_2 with $D_{L,0}$ we must perform the Legendre transformation on D that is described after equation (4.17). From this transformation, we obtain all the connected graphs $\tilde{h}_{l_{ij}, P_k}(s, \alpha)$ from $\frac{\partial^2}{\partial \phi(s) \partial \phi(t)} h_{l_{ij}, P_k}(s, t)$ and all the connected graphs $\tilde{h}_{l_{ij}, P_k}(s, t, \alpha)$ from $\frac{\partial^2}{\partial \phi(s) \partial \phi(t)} h_{l_{ij}, P_k}(s, t)$. In particular, for a connected graph with petals labeled by P_k and links labeled by l_{ij} , we will obtain a contribution to D of the form

$$D_{P,0}(l_{ij}, P_k) = \int \frac{ds}{2\pi} \frac{dt}{2\pi} f_n'(t) G(s-t) \tilde{h}_{l_{ij}, P_k}(s, \alpha) \quad (4.48)$$

and

$$D_{L,0}(l_{ij}, P_k) = \int \frac{ds}{2\pi} \frac{dt}{2\pi} f_n'(s') G(s-s') \tilde{h}_{l_{ij}, P_k}(s, t, \alpha). \quad (4.49)$$

As a result, when we subtract the third term, D , from the last term, C , in the Ward identity, $D_{P,0} + D_{L,0}$ cancels with $C_1 + C_2$, (after appropriately symmetrizing in s and t). As we expected, the terms in the Ward identity arising from the separate treatment of the zero mode are accounted for by the last term in the Ward identity.

4.4. Summing the Graphs

The remaining terms in the Ward identity come from $\delta_n \Gamma$ and $D_{P,\Delta} + D_{L,\Delta}$. For a graph described by $\{l_{ij}, P_k\}$, the terms in the Ward identity due to varying a petal at the vertex labeled by s are given by

$$\begin{aligned} A(l_{ij}, P_k; s) &= \delta_n \Gamma_P(l_{ij}, P_k) - D_{P,\Delta}(l_{ij}, P_k) \\ &= \int \frac{ds}{2\pi} f_n'(s) (1 + \alpha' \square_s) h_{l_{ij}, P_k}(s) \\ &= (1 + \alpha' \square_s) f_n'(s) \end{aligned} \quad (4.50)$$

where $\square_s = \frac{\partial^2}{\partial \phi(s) \partial \phi(s)}$. Similarly, the contribution to the Ward identity from the same graph when a link between vertices labeled by s and t is varied is

$$\begin{aligned} A(l_{ij}, P_k; s, t) &= \delta_n \Gamma_L(l_{ij}, P_k) - D_{L,\Delta}(l_{ij}, P_k) \\ &= \int \frac{ds}{2\pi} \frac{dt}{2\pi} \frac{\partial}{\partial \phi(s)} h_{l_{ij}, P_k}(s, t) \\ &\quad \left\{ f_n(s) \frac{d}{ds} G(s-t) + f_n(t) \frac{d}{dt} G(s-t) \right. \\ &\quad \left. - \int \frac{ds'}{2\pi} f_n(s') \left[\Delta_s(t-s') \frac{d}{ds'} G(s-s') + \Delta_t(s-s') \frac{d}{ds'} G(t-s') \right] \right\}. \end{aligned} \quad (4.51)$$

(To obtain this, we have symmetrized in s and t .) The contribution due to varying a link can be summarized diagrammatically as:

$$\delta_n \Gamma_L(l_{ij}, P_k) - D_{L,\Delta}(l_{ij}, P_k)$$

$$= \frac{\partial}{\partial \phi(s)} \frac{\partial}{\partial \phi(t)} \left[\text{Diagram 1} + \text{Diagram 2} \right] \quad (4.52)$$

These equations were only for the contribution of δH inserted into a particular petal or link of a particular graph. We must sum over all locations of the insertion of δH and then over all 1PI graphs. First we define

$$W_P(l_{ij}, P_k) = \sum_s A(l_{ij}, P_k; s), \quad (4.53)$$

where s runs over all the vertices in the graph, and

$$W_L(l_{ij}, P_k) = \sum_{(s,t)} A(l_{ij}, P_k; s, t), \quad (4.54)$$

where the sum is over all pairs of vertices, $\{s, t\}$, in the graph. Next, to perform the sum over all 1PI graphs with N vertices, we find it much more convenient to extend the sum to one over all graphs with N vertices and then later subtract off the disconnected and one-particle reducible graphs. In that case, the l_{ij} 's and P_k 's are allowed to range from zero to infinity, and we are interested in the expressions

$$\sum_{\{l_{ij}\}=0}^{\infty} \sum_{\{P_k\}=0}^{\infty} h_{l_{ij}, P_k}(s_m) \quad \text{and} \quad \sum_{\substack{l_{ij}=0 \\ l_{mn} \neq l_{mn}}}^{\infty} \sum_{\substack{l_{mn}=1 \\ l_{mn} \neq l_{mn}}}^{\infty} h_{l_{ij}, P_k}(s_m, s_n). \quad (4.55)$$

$h_{l_{ij}, P_k}(s_m)$ and $h_{l_{ij}, P_k}(s_m, s_n)$ are both integrals over

$$h_{l_{ij}, P_k}(\phi) = \frac{1}{N!} \frac{1}{\prod_{i,j} l_{ij}!} \prod_{i,j} P_i! 2^{P_i} [G(0)]^{\sum_i P_i} \prod_{i < j} [G(s_i - s_j)]^{l_{ij}} \prod_i \left[\frac{\partial}{\partial \phi(s_i)} \right]^{M_i} V(\phi(s_i)), \quad (4.56)$$

where $M_i = \sum_j (2P_j + l_{ij})$, and $\tilde{l}_{ij} = l_{ij}$ except for $\tilde{l}_{mn}$, which equals $l_{mn} - 1$. We can easily sum this expression over the P_i 's and l_{ij} 's to obtain

$$h(\phi) = \frac{1}{N!} \prod_{i < j} e^{G(s_i - s_j)} \frac{\partial^{P_i}}{\partial \phi(s_i)} \frac{\partial^{P_j}}{\partial \phi(s_j)} \prod_i e^{\frac{1}{2} G(0) \square_i} (V(\phi(s_i))). \quad (4.57)$$

Therefore, when we sum $W_P(l_{ij}, P_k)$ and $W_L(l_{ij}, P_k)$ over the l_{ij} 's and P_k 's, the Ward identity at order V^N becomes

$$W_P + W_L = 0, \quad (4.58)$$

where

$$W_P = \sum_{\{l_{ij}\}=0}^{\infty} \sum_{\{P_k\}=0}^{\infty} W_P(l_{ij}, P_k) = \frac{1}{N!} \sum_k \int \prod_i \frac{ds_i}{2\pi} f'(s_k) [1 + \alpha' \square_k] h(\phi) \quad (4.59)$$

and

$$W_L = \sum_{\substack{\{l_{ij}\}=0 \\ l_{ij} \neq l_{mn}}}^{\infty} \sum_{\substack{l_{n,m}=1 \\ \{P_k\}=0}}^{\infty} W_L(l_{ij}, P_k) = \frac{1}{N!} \sum_{k \neq m} \int \prod_i \frac{ds_i}{2\pi} \frac{\partial}{\partial \phi(s_k)} \frac{\partial}{\partial \phi(s_m)} h(\phi) \times \left[f(s_k) \frac{d}{ds_k} G(s_k - s_m) - \int dt f(t) \left[\Delta_c(t - s_k) \frac{d}{dt} G(s_m - t) \right] \right]. \quad (4.60)$$

5. The Ward Identity with Uniform Magnetic Field

When a non-zero magnetic field is applied, there are two changes to our system. The first is that $\tilde{\phi}(s)$ and $\tilde{X}_0$ now have two components. We will call $\phi^x(s) = x(s)$ and $\phi^y(s) = y(s)$. The second change is that $G(s)$ is replaced by $G^{\mu\nu}(s)$. The diagonal term in $G^{\mu\nu}$ is just $G(s)$ rescaled:

$$G^{\mu\nu}(s) = \frac{\alpha^2}{\alpha^2 + \beta^2} G(s). \quad (5.1)$$

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The off-diagonal term of $G^{\mu\nu}$ is given by equation (3.11). A sample 1PI graph is

where

$$x(s) \text{ ————— } x(t) = G_{xx}(s - t) \quad \text{and} \quad x(s) \text{ ————— } y(t) = G_{xy}(s - t). \quad (5.3)$$

For simplicity, we will assume that at each vertex all the legs are x 's or all are y 's. This is the case for the cosine potential of the dissipative Hofstadter model in equation (2.9). The results of this section are not changed if we take a more general potential instead.

The calculation of the Ward identity proceeds in much the same way as when $\beta = 0$. Apart from the rescaling of the diagonal propagator, the only changes occur when an off-diagonal propagator is involved in some way. It turns out that the derivative of the off-diagonal propagator and its convolution with H_0 are related to similar functions of the original propagator, $G(s)$. In fact, they satisfy the following identities.

$$\begin{aligned} \frac{d}{ds} G_{xy}(s - s') &= i \frac{\beta}{\alpha^2 + \beta^2} [\Delta_c(s - s') - 1] \\ &= i \frac{\beta}{\alpha^2 + \beta^2} \int \frac{ds''}{2\pi} H_0(s' - s'') G(s'' - s) \end{aligned} \quad (5.4)$$

and

$$\int \frac{ds''}{2\pi} H_0(s' - s'') G_{xy}(s - s'') = -i \frac{\alpha^2 \beta}{\alpha^2 + \beta^2} \frac{d}{ds'} G(s - s'). \quad (5.5)$$

5.1. Variation of the 1PI Vacuum Function

We will now proceed to evaluate the diagrams, and again begin with $\delta_n \Gamma$. This time, when we reparametrize the 1PI diagrams, we obtain a term due to the reparametrization of the ds ,

$$\delta_n \Gamma_P(l_{ij}, P_k) = \int \frac{ds}{2\pi} f'_n(s) h_{l_{ij}, P_k}(s); \quad (5.6)$$

and a term due to the reparametrization of $G^{\mu\nu}(s - t)$,

$$\delta_n \Gamma_L(l_{ij}, P_k) = \int \frac{ds}{2\pi} \frac{dt}{2\pi} f_n(s) \left[\frac{d}{ds} G^{\mu\nu}(s - t) \right] \frac{\partial}{\partial \phi^{\mu}(s)} \frac{\partial}{\partial \phi^{\nu}(t)} h_{l_{ij}, P_k}(s, t). \quad (5.7)$$

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When $\mu = \nu$, this is the same expression we had when $\beta = 0$, except that here we have $G^{\mu\mu}$ instead of G , and here h_{ij} , P can be a graph with both x vertices and y vertices. When $\mu \neq \nu$, we obtain a new contribution to the Ward identity. Using eqn. (5.4) for $\frac{d}{ds} G_{xy}(s)$, we can write this new term as

$$\delta_n \Gamma(l_{ij}, P_k) = -\epsilon^{\mu\nu} i \frac{\beta}{\alpha^2 + \beta^2} \int \frac{ds}{2\pi} \frac{dt}{2\pi} f_n(s) [\Delta_\epsilon(s-t) - 1] \times \frac{\partial}{\partial \phi^\mu(s)} \frac{\partial}{\partial \phi^\nu(t)} h_{ij}, P_k(s, t). \quad (5.8)$$

Thus, once we add the uniform magnetic field, there are four distinct types of graphs for $\delta \Gamma$. They can be represented by

$$\delta_n \Gamma = \left[\begin{array}{c} \text{Diagram 1: } f_n(s) \text{ with } x(s) \text{ and } x(0) \text{ vertices} \\ \text{Diagram 2: } f_n(s) \text{ with } x(s) \text{ and } x(0) \text{ vertices, loop} \\ \text{Diagram 3: } f_n(s) \text{ with } x(s) \text{ and } y(0) \text{ vertices, loop} \\ \text{Diagram 4: } f_n(s) \text{ with } x(s) \text{ and } y(0) \text{ vertices, loop} \end{array} \right] \times \frac{\partial}{\partial x(s)} \frac{\partial}{\partial y(t)} \frac{\beta}{\alpha^2 + \beta^2} \quad (5.9)$$

(and similarly with x and y interchanged.)

5.2. The Kinetic Term

The changes to the third term of the Ward identity are more complicated. As we have seen, the third term of the Ward identity can be described as inserting a $\delta_n H_0$ into 1PI diagrams and it arises from truncating an α_{-m} and α_{m-n} from connected diagrams. However, now we have two different ways of inserting a δH , coming from the two terms of

$$\frac{\partial^2 W}{\partial \alpha_{-m} \cdot \partial \alpha_{m-n}} = \frac{\partial^2 W}{\partial \alpha_{-m}^x \partial \alpha_{m-n}^x} + \frac{\partial^2 W}{\partial \alpha_{-m}^y \partial \alpha_{m-n}^y}. \quad (5.10)$$

For the first term, we must join δH_0 to two vertices from which α^x 's have been truncated. In terms of the 1PI diagrams, this can be described as an insertion of $\delta H_x(s-s')$, where $\delta H_x(s-s')$ is defined to be $\delta H_0(s-s')$ with x vertices at s and s' . (This is depicted in the first diagram in equations (5.11) and (5.21).) Similarly, the second term is given by an insertion of $\delta H_y(s-s')$ into 1PI diagrams, with $\delta H_y(s-s')$ defined in the same way.

First we will calculate the effect of inserting $\delta H_x(s'-t') + \delta H_y(s'-t')$ into a $G^{\mu\nu}(s-t)$ propagator when $\mu = \nu$. If we put $\delta H_x(s'-t')$ and $\delta H_y(s'-t')$ between two like vertices, say two x -vertices, then for a particular graph we obtain the following contribution to the third term in the Ward identity.

$$D_1(l_{ij}, P_k) = - \int \frac{ds}{2\pi} \frac{ds'}{2\pi} \frac{dt}{2\pi} \frac{dt'}{2\pi} \frac{\partial}{\partial x(s)} \frac{\partial}{\partial x(t)} h_{ij}, P_k(s, t) \times [G_{xx}(s-s') \delta H_x(s'-t') G_{xx}(t'-t) + G_{xy}(s-s') \delta H_y(s'-t') G_{yx}(t'-t)] \quad (5.11)$$

To include the case when $s = t$, we define $h_{ij}, P_k(s, s) = \frac{1}{2} h_{ij}, P_k(s, s)$, and when $s = t$ we omit the integration over t . Since G_{xx} is just a rescaling of G and $\delta H_x = \delta H_0$, we can write the first term in the square brackets as

$$\frac{\alpha^4}{(\alpha^2 + \beta^2)^2} G(s-s') \delta H_0(s'-t') G(t'-t). \quad (5.12)$$

The s' and t' integrals over the second term in the square brackets are given by

$$I_{xx} = \int G_{xy}(s-s') \delta H_0(s'-t') G_{yx}(t'-t) \frac{ds'}{2\pi} \frac{dt'}{2\pi} = - \int \frac{ds'}{2\pi} \frac{dt'}{2\pi} \left(\frac{d}{ds'} G_{xy}(s-s') \right) f_n(s') [H_0(s'-t') G_{yx}(t'-t)] - \int \frac{ds'}{2\pi} \frac{dt'}{2\pi} [G_{xy}(s-s') H_0(s'-t')] f_n(t') \frac{d}{dt'} G_{yx}(t'-t). \quad (5.13)$$

We can use the identities given in equation (5.4) and (5.5) to write I_{xx} as

$$I_{xx} = - \frac{\beta^2 \alpha^2}{(\alpha^2 + \beta^2)^2} \int \frac{ds'}{2\pi} \frac{dt'}{2\pi} G(s-s') H_0(t'-s') f_n(s') \frac{d}{ds'} G(s'-t) - \frac{\beta^2 \alpha^2}{(\alpha^2 + \beta^2)^2} \int \frac{ds'}{2\pi} \frac{dt'}{2\pi} \left[\frac{d}{dt'} G(s-s'-t') \right] f_n(t') H_0(t'-s') G(s'-t). \quad (5.14)$$

On comparing this with equation (4.26), we see that I_{xx} is proportional to the integral I that we evaluated when $\beta = 0$. Then I_{xx} is given by

$$I_{xx} = \frac{\beta^2 \alpha^2}{(\alpha^2 + \beta^2)^2} I(s, t) = \frac{\beta^2 \alpha^2}{(\alpha^2 + \beta^2)^2} \int \frac{ds' dt'}{2\pi} G(s-s') \delta_f H_0(s'-t') G(t'-t). \quad (5.15)$$

Therefore, we can write D_1 as

$$D_1(l_{ij}, P_k) = - \left[\frac{\alpha^4}{(\alpha^2 + \beta^2)^2} + \frac{\beta^2 \alpha^2}{(\alpha^2 + \beta^2)^2} \right] \times \int \frac{ds}{2\pi} \frac{ds'}{2\pi} \frac{dt}{2\pi} \frac{dt'}{2\pi} \frac{\partial}{\partial \phi(s)} \frac{\partial}{\partial \phi(t)} h_{l_{ij}, P_k}(s, t) \times [G(s-s') \delta H_0(s'-t') G(t'-t)] \quad (5.16)$$

This is just $\frac{\alpha^2}{\alpha^2 + \beta^2}$ times the expression we originally had for the third term of the Ward identity when there was no magnetic field. Now we can use our original calculations of D_P and D_L when there was no magnetic field to find the analogous terms when $\beta \neq 0$. They will just be rescaled by $\frac{\alpha^2}{\alpha^2 + \beta^2}$, so they can be written as

$$D_{P,0}(l_{ij}, P_k) = \int \frac{ds}{2\pi} \frac{ds'}{2\pi} \frac{\partial^2}{\partial \phi_\mu^2(s)} h_{l_{ij}, P_k}(s) f'_n(s') G^{\mu\mu}(s-s') \quad (5.17)$$

$$D_{P,\Delta}(l_{ij}, P_k) = - \frac{\alpha}{\alpha^2 + \beta^2} \int \frac{ds}{2\pi} \frac{dt}{2\pi} \frac{\partial^2}{\partial \phi_\mu^2(s)} h_{l_{ij}, P_k}(s) f'_n(s) \quad (5.18)$$

$$D_{L,0}(l_{ij}, P_k) = \frac{1}{2} \int \frac{ds}{2\pi} \frac{dt}{2\pi} \frac{ds'}{2\pi} \frac{dt'}{2\pi} \frac{\partial}{\partial \phi^\mu(s)} \frac{\partial}{\partial \phi^\mu(t)} h_{l_{ij}, P_k}(s, t) \times f'_n(s') [G^{\mu\mu}(s-s') + G^{\mu\mu}(t-s')] \quad (5.19)$$

and

$$D_{L,\Delta}(l_{ij}, P_k) = \frac{1}{2} \int \frac{ds}{2\pi} \frac{dt}{2\pi} \frac{ds'}{2\pi} \frac{dt'}{2\pi} \frac{\partial}{\partial \phi^\mu(s)} \frac{\partial}{\partial \phi^\mu(t)} h_{l_{ij}, P_k}(s, t) f'_n(s') \times \left[\Delta_\epsilon(t-s') \frac{d}{ds'} G^{\mu\mu}(s-s') + \Delta_\epsilon(s-s') \frac{d}{ds'} G^{\mu\mu}(t-s') \right]. \quad (5.20)$$

The diagrams for these four expressions are basically the same as the ones given in (4.37) and (4.40).

Next, we will calculate the contribution to the Ward identity when $\delta H_x(s'-t') + \delta H_y(s'-t')$ is inserted into a $G_{xy}(s-t)$ propagator. In that case, a particular graph has the value

$$D_2(l_{ij}, P_k) = - \int \frac{ds}{2\pi} \frac{dt}{2\pi} \frac{\partial}{\partial x(s)} \frac{\partial}{\partial y(t)} h_{l_{ij}, P_k}(s, t) I_{xy} = - \frac{\partial}{\partial x(s)} \frac{\partial}{\partial y(t)} \left[\text{Diagram 1} + \text{Diagram 2} \right], \quad (5.21)$$

with I_{xy} given by

$$I_{xy} = \int \frac{ds'}{2\pi} \frac{dt'}{2\pi} \left[G_{xx}(s-s') \delta H_x(s'-t') G_{xy}(t'-t) + G_{xy}(s-s') \delta H_y(s'-t') G_{yy}(t'-t) \right]. \quad (5.22)$$

We can evaluate I_{xy} in the same way we calculated I_{xx} , with the result that

$$I_{xy} = 0. \quad (5.23)$$

Therefore, inserting δH_0 into an off-diagonal propagator, $G^{\mu\nu}$ with $\mu \neq \nu$, contributes nothing new to the Ward identity.

5.3. The Zero-Mode Term

Finally, we turn our attention to the last term in the Ward identity. This term comes from connected diagrams with a truncated α_n^μ and an extra $\frac{\partial}{\partial \phi^\mu}$ derivative acting on a vertex. Because we are working in two dimensions, there are now three possibilities for these diagrams. For the connected graph given by

$$\alpha_n^\mu(s) \alpha_n^\nu(s') \left[\frac{\partial}{\partial s'} \right]^s v_{(0)} \left[\frac{\partial}{\partial s} \right]^s v_{(0)} \left[\frac{\partial}{\partial s'} \right]^s v_{(0)} \left[\frac{\partial}{\partial s} \right]^s v_{(0)} \right], \quad (5.24)$$

they are represented by

$$\begin{aligned}
 C = & \begin{array}{c} f_n(s') \quad \alpha^x(s') \\ \text{Diagram 1} \end{array} + \begin{array}{c} f_n(s') \quad \alpha^x(s') \\ \text{Diagram 2} \end{array} \\
 & + \begin{array}{c} \alpha^y(s') \quad f_n(s') \\ \text{Diagram 3} \end{array} .
 \end{aligned} \tag{5.25}$$

The first two diagrams correspond to the case when the truncated α_{-n}^μ was connected to the graph with a diagonal propagator, $G^{\mu\nu}$. Then the evaluation of the graph proceeds in exactly the same way it did when there was no magnetic field, except that G is replaced with $G^{\mu\nu}$. Once again, the contribution from these graphs cancels the “zero mode” contribution, $D_{P0} + D_{L0}$, from the third term in the Ward identity.

In the remaining case, the truncated α_{-n} was connected to the graph by an off-diagonal propagator. For example, if the truncated α_{-n}^z was connected to the graph with G_{xy} , then the expression for the graph is

$$\begin{aligned}
 C_3(l_{ij}, P_k) &= \int \frac{ds}{2\pi} \frac{dt}{2\pi} \frac{ds'}{2\pi} f'(s') G_{xy}(s' - s) \bar{h}_{ij}, P_k(s, t, \alpha) \\
 &= \text{Diagram 4} ,
 \end{aligned} \tag{5.26}$$

where now $\bar{h}_{ij}, P_k(s, t, \alpha)$ is the integral over a connected graph with an extra $\frac{\partial}{\partial y_0}$ acting on the $V(y_0)$ at the s -vertex and an extra $\frac{\partial}{\partial z_0}$ at the t vertex. For the third graph in

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(5.25), $\bar{h}_{ij}, P_k(s, t, \alpha)$ is given by

$$\begin{aligned}
 \bar{h}_{ij}, P_k(s, t, \alpha) &= \text{Diagram 5} \\
 &= \begin{array}{c} \alpha^y(s') \\ \text{Diagram 6} \end{array} .
 \end{aligned} \tag{5.27}$$

We can easily perform the s' integral, using $f_n(s') = ie^{-is's'}$ and the Fourier series for G_{xy} . It is

$$\int \frac{ds'}{2\pi} f'_n(s') G_{xy}(s' - s) = i \frac{\beta}{\alpha^2 + \beta^2} e^{-\epsilon|n|} f_n(s). \tag{5.28}$$

Then the contribution to the last term in the Ward identity due to $\bar{h}_{ij}, P_k(s, t, \alpha)$ can be written as

$$\begin{aligned}
 C_3(l_{ij}, P_k) &= i \frac{\beta}{\alpha^2 + \beta^2} e^{-\epsilon|n|} \int \frac{ds}{2\pi} \frac{dt}{2\pi} [f_n(s) - f_n(t)] \bar{h}_{ij}, P_k(s, t, \alpha) \\
 &= i \frac{\beta}{\alpha^2 + \beta^2} e^{-\epsilon|n|} \left[\text{Diagram 7} - \text{Diagram 8} \right],
 \end{aligned} \tag{5.29}$$

where we have also included the graph where an α_{-n}^y was truncated from the vertex at t .

We would like C_3 to cancel the part of $\delta_n \Gamma_C$ that came from treating the zero mode separately, namely, the part that comes from the -1 in the expression $\Delta_\epsilon - 1$ for $\partial G^{\mu\nu}(s)/\partial s$. This is given by

$$\delta_n \Gamma_Z(l_{ij}, P_k) = -i \frac{\beta}{\alpha^2 + \beta^2} \int \frac{ds}{2\pi} \frac{dt}{2\pi} [f_n(s) - f_n(t)] \frac{\partial}{\partial \phi^y(s)} \frac{\partial}{\partial \phi^x(t)} h_{ij}, P_k(s, t). \tag{5.30}$$

Once again, we have the difficulty that C is a function of α and Γ is a function of ϕ . However, if we Legendre transform $\delta_n \Gamma$, then for each $\bar{h}_{ij}, P_k(s, t, \alpha)$ we obtain a corresponding graph from a $\frac{\partial}{\partial \phi^x(s)} \frac{\partial}{\partial \phi^y(t)} h_{ij}, P_k(s, t)$. Even so, the two graphs cancel only to the leading order in ϵ , because the C_3 has an extra factor of $e^{-\epsilon|n|}$. From discussions similar to those in the following sections, we can show that $\delta_n \Gamma_Z - C_3 \rightarrow 0$ as $\epsilon \rightarrow 0$, so that, effectively, the last term in the Ward identity does cancel the zero-mode part of $\delta \Gamma$.

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5.4. Final Form of the Ward Identity

As in the case when there was no magnetic field, we must now sum over all locations of the insertion of δH and over all graphs with N vertices. Again, we sum the l_{ij} 's and P_k 's from zero to infinity and then subtract off the one-particle reducible graphs. We find that the Ward identity is given by

$$W_P + W_L + W_C + W_Z = 0. \quad (5.31)$$

W_P comes from varying the petals of a graph and is given by

$$\begin{aligned} W_P &= \sum_{s_k} \sum_{l_{ij}, P_k} \delta_n \Gamma_P(l_{ij}, P_k) - D_{P,\Delta}(l_{ij}, P_k) \\ &= \sum_k \int \prod_i \frac{ds_i}{2\pi} f'_n(s_k) \left(1 + \frac{\alpha}{\alpha^2 + \beta^2} \square_k \right) h(\phi). \end{aligned} \quad (5.32)$$

W_L comes from varying the diagonal propagators and has the form

$$\begin{aligned} W_L &= \sum_{s_k, s_m} \delta_n \Gamma_L(l_{ij}, P_k) - D_{L,\Delta}(l_{ij}, P_k) \\ &= \sum_{k \neq m} \int \prod_i \frac{ds_i}{2\pi} \frac{\partial}{\partial \phi^\mu(s_k)} \frac{\partial}{\partial \phi^\mu(s_m)} h(\phi) \\ &\quad \left\{ f_n(s_k) \frac{d}{ds_k} G^{\mu\mu}(s_k - s_m) - \int \frac{ds'}{2\pi} f_n(s') \Delta_\epsilon(s_k - s') \frac{d}{ds'} G^{\mu\mu}(s_m - s') \right\}. \end{aligned} \quad (5.33)$$

W_C is due to varying the off-diagonal propagators and is given by

$$\begin{aligned} W_C &= \sum_{s_k, s_m} \delta_n \Gamma_C(l_{ij}, P_k) - \delta_n \Gamma_Z(l_{ij}, P_k) \\ &= -\epsilon^{\mu\nu} i \frac{\beta}{\alpha^2 + \beta^2} \sum_{k,m} \int \prod_i \frac{ds_i}{2\pi} f_n(s_k) \Delta_\epsilon(s_k - s_m) \frac{\partial}{\partial \phi^\mu(s_k)} \frac{\partial}{\partial \phi^\nu(s_m)} h(\phi). \end{aligned} \quad (5.34)$$

Lastly, W_Z is due to the treatment of the zero-mode in the off diagonal propagator and is given by

$$\begin{aligned} W_Z &= \epsilon^{\mu\nu} i \frac{\beta}{\alpha^2 + \beta^2} (1 - e^{-\epsilon|n|}) \\ &\quad \times \sum_{s_k, s_m} \int \prod_i \frac{ds_i}{2\pi} [f_n(s_k) - f_n(s_m)] \frac{\partial}{\partial \phi^\mu(s_k)} \frac{\partial}{\partial \phi^\nu(s_m)} h(\phi). \end{aligned} \quad (5.35)$$

In these equations, $h(\phi)$ is due to summing over all graphs with N vertices. Before we subtract off the contribution due to disconnected and one-particle reducible graphs, it has the form

$$h(\phi) = \frac{1}{N!} \prod_{i < j} e^{\frac{G^{\mu\nu}(s_i - s_j)}{2\pi\epsilon^{\mu\nu}(i,j)} \frac{\partial}{\partial \phi^\mu(i,j)} \frac{\partial}{\partial \phi^\nu(j,i)}} \prod_i e^{\frac{1}{2} G^{\mu\mu}(0) \square_i} V(\vec{\phi}(s_i)). \quad (5.36)$$

In summary, the Ward identity is given by a sum over the following graphs:

$$\begin{aligned} 0 &= \left(1 + \frac{\alpha}{\alpha^2 + \beta^2} \square_{s_n} \right) \left[\text{Diagram 1} \right] \\ &\quad + \left[\frac{\partial}{\partial x(s_k)} \frac{\partial}{\partial x(s_m)} \left(\text{Diagram 2} - \text{Diagram 3} \right) \right] \\ &\quad - i \frac{\beta}{\alpha^2 + \beta^2} \epsilon^{xy} \frac{\partial}{\partial x(s_k)} \frac{\partial}{\partial y(s_m)} \left[\text{Diagram 4} - \text{Diagram 5} \right] \\ &\quad + \{x \leftrightarrow y\}, \end{aligned} \quad (5.37)$$

where we have ignored W_Z because it is $\propto \epsilon$, and we have obtained the last graph by interchanging k and m (and x and y) in W_C .

5.5. Comments

At this point, we would like to make a few comments about the Ward identity. First, at order V_0 , it takes the form

$$0 = \int \prod_i \frac{ds_i}{2\pi} f'_n(s) \left[1 + \frac{\alpha}{\alpha^2 + \beta^2} \square_s \right] e^{\frac{1}{2} G^{\mu\mu}(0) \square_s} V(\vec{\phi}(s)). \quad (5.38)$$

This gives us the $O(V_0)$ tachyon equation of motion,

$$\left(1 + \frac{\alpha}{\alpha^2 + \beta^2} \square_s \right) V(\vec{\phi}(s)) = 0. \quad (5.39)$$

This equation of motion is satisfied whenever V is periodic in x and y with period $2\pi/k$, where k satisfies

$$\frac{\alpha}{\alpha^2 + \beta^2} k^2 = 1. \quad (5.40)$$

This is exactly the condition for the phase transition predicted by the simple renormalization group arguments in reference [6].

To this order, the Ward identity is also trivially satisfied whenever

$$e^{\frac{1}{2}G^{\mu\mu}(0)\Box}V(\vec{\phi}(s)) = 0. \quad (5.41)$$

Using the definition for $G^{\mu\mu}(t)$ in equation (3.10) and writing $V(\vec{\phi}(s))$ in terms of Fourier modes, we find that this condition is

$$e^{\frac{1}{2}\frac{\alpha}{\alpha^2+\beta^2}(\ln \epsilon^2)k^2}V_{\vec{k}}e^{i\vec{k}\cdot\vec{\phi}} = 0. \quad (5.42)$$

In the limit as ϵ goes to zero, this becomes

$$e^{\frac{\alpha}{\alpha^2+\beta^2}k^2}V_{\vec{k}} = 0. \quad (5.43)$$

If the potential is normal ordered, then the left-hand side of eqn. (5.43) is non-zero, and V must have the period given by eqn (5.40). However, if instead we choose $V_{\vec{k}}\epsilon$ to be finite for any k , then when $k^2\frac{\alpha}{\alpha^2+\beta^2} > 1$ this second condition is automatically true. This second choice for regulating $V_{\vec{k}}\epsilon$ occurs when we start with a discrete lattice for time and then pass from a finite sum over time to an integral. For calculating β -functions, this is a natural choice to make for $V_{\vec{k}}$. Therefore, in the following section we will define $T_{\vec{k}} = V_{\vec{k}}\epsilon$, and, whenever $k^2\frac{\alpha}{\alpha^2+\beta^2} > 1$, we will assume that $T_{\vec{k}}$ is finite.

More generally, as long as $V(\vec{\phi}(s))$ satisfies the equation of motion given in eqn. (5.39), then W_F , the contribution to the Ward identity due to varying a petal is always zero. Furthermore, W_L and W_C would also automatically be zero if we could immediately replace Δ , with a delta function, δ . This is actually possible for any individual graph. Because all the propagators in the graph are logarithms or sign functions, one can show by power counting that the graph diverges at most logarithmically. Thus, we only need to keep the lowest term in ϵ for Δ , so we can replace it with a δ -function. Consequently, to any finite order in α' , the Ward identity is satisfied whenever $k^2\alpha/(\alpha^2 + \beta^2) = 1$. Once we perform the sum over all numbers of links and petals, which amounts to summing to all orders in α' , we obtain the graphs described by $h(\phi)$ in equation (5.36). In these graphs, the vertices are joined by the propagators $E(t; \mu, \nu) = \exp[G^{\mu\nu}(t)]$, which go as $1/t^2$ when $\mu = \nu$. Naive power counting tells us that now the graphs can diverge as $1/\epsilon$; and in reference [6], we found that the free energy does, in fact, diverge as $1/\epsilon$. Therefore, we cannot just set $\Delta, \epsilon = \delta$, and, in the following section, we will proceed more carefully.

6. The Flows Generated by the Ward Identity

In the following sections, we will show that, to leading order in ϵ , the condition for the Ward identity to be satisfied is the same as the condition for the β -function, given by $\epsilon\frac{\partial}{\partial\epsilon}\Gamma$, to equal zero. Also, we will prove that the $\cos x$ potential satisfies the Ward identity at every order in V_0 and to all orders in α' whenever $\frac{\alpha}{\alpha^2+\beta^2} = 1$ and $\frac{\beta}{\alpha^2+\beta^2} \in \mathbb{Z}$.

To show the equivalence of the Ward identity to the equation $\epsilon\frac{\partial}{\partial\epsilon}\Gamma = 0$ for any arbitrary potential, $V(\vec{\phi}(t))$, we begin by writing $V(\vec{\phi}(t))$ in terms of its Fourier modes, $V(\vec{\phi}(t)) = \sum_{\vec{k}}(T_{\vec{k}}/\epsilon)e^{i\vec{k}\cdot\vec{\phi}(t)}$. (For simplicity, we use a sum over k , here, but the results should not change if we take an integral instead.) At order V_0 , the Ward identity for the specific modes of V labeled by $\vec{k}_1, \dots, \vec{k}_N$ is then given by

$$W_P + W_L + W_C + W_Z = 0, \quad (6.1)$$

where

$$W_P = \sum_i \int \prod \frac{ds_i}{2\pi} f_n^{s_i}(s_i) \left(1 - \frac{\alpha}{\alpha^2 + \beta^2} |\vec{k}_i|^2\right) H, \quad (6.2)$$

$$W_Z = \epsilon^{\mu\nu} i \frac{\beta}{\alpha^2 + \beta^2} (e^{-i|n|} - 1) \sum_{i < m} \int \prod \frac{ds_i}{2\pi} [f_n(s_i) - f_n(s_m)] k_i^\mu k_m^\nu H, \quad (6.3)$$

$$W_L = - \sum_{i < m} \int \prod \frac{ds_i}{2\pi} k_i^\mu k_m^\nu H \times \left\{ \sum_{\substack{a,b=(i,m)}} [f_n(s_a) - f_n(s_b)] \frac{d}{ds_a} G^{\mu\nu}(s_a - s_b) - \int \frac{ds'}{2\pi} f_n(s') \Delta_r(s_a - s') \frac{d}{ds'} G^{\mu\nu}(s_b - s') \right\}, \quad (6.4)$$

and

$$W_C = \epsilon^{\mu\nu} i \frac{\beta}{\alpha^2 + \beta^2} \sum_{i,m} \int \prod \frac{ds_i}{2\pi} f_n(s_i) \Delta_r(s_i - s_m) k_i^\mu k_m^\nu H. \quad (6.5)$$

Here, H is from the 1PI graphs with vertex factors $T_{\vec{k}} e^{i\vec{k}_j \cdot \vec{\phi}(t)}$, for $j = 1, \dots, N$. It is given by the expression

$$\tilde{h} = \frac{1}{N!} \prod_j T_{\vec{k}_j} e^{-\frac{1}{2}G^{\mu\mu}(0)k_j^2} e^{i\vec{k}_j \cdot \vec{\phi}(s_j)} \prod_{i < j} e^{-G^{\mu\nu}(s_i - s_j)k_i^\mu k_j^\nu}, \quad (6.6)$$

minus all the disconnected and one-particle reducible graphs. (For W_L and W_C , we must be a little more careful in specifying the graphs we subtract from $\tilde{h}$; we want to subtract all graphs the are one-particle reducible when we include the $\frac{d}{ds_a} G^{\mu\nu}(s_a - s_b)$ propagator for W_L and the $\Delta_r(s_i - s_m)$ propagator for W_C .) We will define the vector $\vec{q}$ to be $\vec{q} = \sum_{i=1}^N \vec{k}_i$. Then the total charge of the graph is given by $q = |\vec{q}|$.

For the remainder of this paper, we will find it convenient to redefine $G^{\mu\mu}$ to be

$$G^{\mu\mu}(t) = -\frac{\alpha}{\alpha^2 + \beta^2} \ln(\epsilon^t + e^{-t} - 2 \cos t). \quad (6.7)$$

This is obtained from the original definition of $G^{\mu\mu}$ in equation (3.10) by subtracting $\frac{\alpha}{\alpha^2 + \beta^2} \epsilon$. When we use this new definition of $G^{\mu\mu}$, we must also include a factor of $\exp\left(-\frac{1}{2} \frac{\alpha}{\alpha^2 + \beta^2} \epsilon(q_x^2 + q_y^2)\right)$ in $\tilde{h}$. For brevity, we will define

$$T = \exp\left(-\frac{\epsilon}{2} \frac{\alpha}{\alpha^2 + \beta^2} (q_x^2 + q_y^2)\right) \prod_j \frac{T_j}{\epsilon} \exp\left(-\frac{1}{2} G^{\mu\mu}(0) k_j^2\right). \quad (6.8)$$

To evaluate the integrals in this expression for the Ward identity, we must Taylor expand the $e^{i\vec{k}_j \cdot \vec{\phi}(s_j)}$'s around a common point, t . Thus, we have

$$\prod_j e^{i\vec{k}_j \cdot \vec{\phi}(s_j)} = e^{i\vec{q} \cdot \vec{\phi}(t)} + \sum_j (s_j - t) \vec{k}_j \cdot \frac{\partial \vec{\phi}(t)}{\partial t} e^{i\vec{q} \cdot \vec{\phi}(t)} + \dots \quad (6.9)$$

From this expansion, we will obtain flows for coupling constants of the operators $e^{i\vec{k} \cdot \vec{\phi}(t)}$, $\frac{\partial \vec{\phi}(t)}{\partial t} e^{i\vec{q} \cdot \vec{\phi}(t)}$, and the higher degree operators. These flows will be the coefficients of these operators in the Ward identity after we have expanded all the vertices around a common point. In the following sections, we will show that these flows for the relevant and marginal operators are the same as the renormalization group flows generated by the β -function, given by $\epsilon \partial \Gamma / \partial \epsilon$. The lowest degree operators obtained from charge-0 graphs are constants and $\frac{\partial \vec{\phi}}{\partial t}$. All other operators from charge-0 graphs have additional derivatives, so they are of higher degree and should be irrelevant. Similarly, for any charge- $\vec{q}$ graph with $0 < |\vec{q}| \leq 1$, the lowest order operator generated from the Taylor series is relevant and is given by $e^{i\vec{q} \cdot \vec{\phi}}$. All higher order operators have at least one more derivative, so they are irrelevant. Lastly, we expect graphs with charge q greater than one to only generate flows for irrelevant operators.

We note that from the expansion in equation (6.9), there is no way to generate the non-local friction term or the constant gauge field [10]. As a result, we do not obtain any flows for α and β . In addition, we will show that the flow for $\partial \vec{\phi} / \partial t$ also vanishes due to symmetries of H .

7. Calculating the Flow for $e^{i\vec{q} \cdot \vec{\phi}}$

W_P . We will first evaluate the W_P term. If the potential satisfies the one-loop equation of motion (5.40), then W_P is automatically 0. For arbitrary potentials, we make the following change of variables:

$$\begin{aligned} s_+ &= s_l \\ r_j &= s_j - s_l \quad j = 1, \dots, N. \end{aligned} \quad (7.1)$$

Then we Taylor expand around s_+ . To lowest order in r_j and ϵ , we obtain

$$W_P = T \sum_l \left(1 - \frac{\alpha}{\alpha^2 + \beta^2} |\vec{k}_l|^2\right) \times \int \frac{ds_+}{2\pi} f'_n(s_+) e^{i \sum_j \vec{k}_j \cdot \vec{\phi}(s_+)}. \quad (7.2)$$

F is essentially the free energy and is defined by

$$F = \int \prod_{i=1}^N \frac{d\vec{r}_i}{2\pi} (I - R), \quad (7.3)$$

where I is the integrand for the partition function, given by

$$I = \prod_{i < j} e^{-G^{\mu\nu}(r_i - r_j) k_i^\mu k_j^\nu}, \quad (7.4)$$

and R is the integrand for all connected, one-particle reducible graphs and all disconnected graphs.

Similarly, we can expand the vertices in W_Z about a common point, s_+ . The leading term is

$$\begin{aligned} W_Z &= -\epsilon |n| i \frac{\beta}{\alpha^2 + \beta^2} T \int \frac{ds_+}{2\pi} f'_n(s_+) e^{i \sum_j \vec{k}_j \cdot \vec{\phi}(s_+)} \\ &\quad \times \sum_{l < m} \left[k_l^\mu k_m^\nu \epsilon^{\mu\nu} \int \prod_i \frac{ds_i}{2\pi} (s_m - s_l) F \right]. \end{aligned} \quad (7.5)$$

We will replace $|n| i f'_n(s_+)$ by $\pm f''_n(s_+)$ and integrate by parts. Then, for $n > 0$, the s_+ integral becomes

$$\int \frac{ds_+}{2\pi} f'_n(s_+) \frac{d}{ds_+} i \sum_j \vec{k}_j \cdot \vec{\phi}(s_+). \quad (7.6)$$

Thus, the first non-vanishing contribution to W_Z is for the flow of $\frac{d}{ds_+} e^{i\vec{q} \cdot \vec{\phi}(s_+)}$ instead of the flow of $e^{i\vec{q} \cdot \vec{\phi}(s_+)}$. Note that $W_Z = 0$ when $\vec{q} = 0$. For all other values of q , we obtain the flow for irrelevant operators and therefore still expect W_Z to vanish.

W_L . W_L is more complicated to evaluate. We have already noted that when $\epsilon \rightarrow 0$, $\Delta_\epsilon(s_a - s')$ becomes a delta function. As a result, W_L would be zero if the remaining part of the integrand weren't singular. Therefore, the non-zero part of W_L should really come from the regions of integration where the vertices are close together and the integrand is singular, which is a justification for Taylor expanding the vertices about a common point. In addition, because $\Delta_\epsilon(s_a - s')$ approaches a delta function for small ϵ , we can expand it around small ϵ and small $s_- = s_a - s'$, (since the region of integration only covers one period of Δ_ϵ).

First, we will make the following change of variables for the second term in the curly brackets in eqn. (6.4) for W_L .

$$\begin{aligned} s_- &= s' - s_a \\ s_+ &= s_a \\ t &= s_b - s_a \\ r_j &= -s_j + s_l \quad \text{when } (a, b) = (l, m) \\ \text{and } r_j &= s_j - s_l \quad \text{when } (a, b) = (m, l). \end{aligned} \quad (7.7)$$

From the definitions of r_j and t , note that

$$t = -r_m \quad \text{and} \quad r_l = 0. \quad (7.8)$$

The change of variables for the first term in the curly brackets is the same, except we do not need an s_- . We will also define

$$g_j(s) = e^{if_j \cdot \vec{\Delta}(s)}. \quad (7.9)$$

Then W_L is given by

$$\begin{aligned} W_L = & -T \sum_{l < m} k_l^\mu k_m^\nu \int \frac{ds_+}{2\pi} \frac{ds_-}{2\pi} \prod_{j=1}^N \frac{dr_j}{2\pi} \\ & \times \left\{ F(-r_1, \dots, -r_N) \prod_{i=1}^N g_i(s_+ - r_i) + F(r_1, \dots, r_N) \prod_{i=1}^N g_i(s_+ + r_i + t) \right\} \\ & \left\{ -f_n(s_+) \frac{d}{dt} G^{\mu\mu}(t) \delta(s_-) - f_n(s_+ + s_-) \Delta_\epsilon(s_-) \frac{d}{ds_-} G^{\mu\mu}(t - s_-) \right\}, \end{aligned} \quad (7.10)$$

where $F(r_1, \dots, r_N)$ is the IPI graphs with vertices at $r_1, \dots, r_N$ as defined in equation (7.3).

First we concentrate on the s_- integral. $\Delta_\epsilon(s_-)$ is acting as an approximate delta function, so for a smooth enough function like f_n , we only need to keep the first few terms in the Taylor expansion for s_- . After expanding to first order in s_- , the integral of interest is

$$- \int \frac{ds_-}{2\pi} \left[f_n(s_+) \Delta_\epsilon(s_-) \frac{d}{ds_-} G^{\mu\mu}(t - s_-) + f'_n(s_+) s_- \Delta_\epsilon(s_-) \frac{d}{ds_-} G^{\mu\mu}(t - s_-) \right]. \quad (7.11)$$

The first term can be integrated exactly. It is

$$\begin{aligned} \int \frac{ds_-}{2\pi} \Delta_\epsilon(s_-) \frac{d}{ds_-} G^{\mu\mu}(t - s_-) &= -i \sum_{m \neq 0} \frac{\alpha}{\alpha^2 + \beta^2} \text{sign}(m) e^{im t} e^{-2|m|t} \\ &= -\frac{d}{dt} G_{2\epsilon}^{\mu\mu}(t) \end{aligned} \quad (7.12)$$

where $G_{m\epsilon}^{\mu\mu}(t)$ is defined to be equal to $G^{\mu\mu}(t)$ with ϵ replaced by $m\epsilon$.

To integrate the second term, we must evaluate the integral for small values of the argument of $\Delta_\epsilon(s_-)$. In addition, we only need to consider small values of t , so that the argument of $G^{\mu\mu}(t - s_-)$ is small. Otherwise, if t were large, $\frac{d}{ds_-} G^{\mu\mu}(t - s_-)$ would be a slowly varying function of s_- and integrating it against s_- times an approximate delta function would just give zero. For small s_- , ϵ , and t , the functions Δ_ϵ and $\frac{d}{ds_-} G^{\mu\mu}$ are given by equations (4.32) and (4.33) with $\alpha' = \frac{\alpha}{\alpha^2 + \beta^2}$ and the integral is

$$\begin{aligned} \int \frac{ds_-}{2\pi} s_- \Delta_\epsilon(s_-) \frac{d}{ds_-} G^{\mu\mu}(t - s_-) &\approx -\epsilon \frac{\alpha}{\alpha^2 + \beta^2} \frac{4\epsilon}{(2\epsilon)^2 + t^2} \\ &= -\frac{\alpha}{\alpha^2 + \beta^2} \epsilon \Delta_{2\epsilon}(t) (1 + O(\epsilon^2, t^2)). \end{aligned} \quad (7.13)$$

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Substituting these integrals back into the expression for W_L , using equation (7.8), and making the change of variables $r_i \rightarrow -r_i$, $t \rightarrow -t$ in the first term in the first curly brackets, we obtain

$$\begin{aligned} W_L = & -T \sum_{l < m} k_l^\mu k_m^\nu \int \frac{ds_+}{2\pi} \frac{dt}{2\pi} \prod_{j \neq m, l}^N \frac{dr_j}{2\pi} F(r_1, \dots, r_N) \\ & \left[\left\{ \prod_{i \neq l, m}^N g_i(s_+ + r_i + t) g_m(s_+) g_l(s_+ + t) - \prod_{i \neq l, m}^N g_i(s_+ + r_i) g_m(s_+ - t) g_l(s_+) \right\} \right. \\ & \times \left[-f_n(s_+) \frac{d}{dt} G^{\mu\mu}(t) + f_n(s_+) \frac{d}{dt} G_{2\epsilon}^{\mu\mu}(t) \right] \\ & \left. + \left[\prod_{i \neq l, m}^N g_i(s_+ + r_i) g_m(s_+ - t) g_l(s_+) + \prod_{i \neq l, m}^N g_i(s_+ + r_i + t) g_m(s_+) g_l(s_+ + t) \right] \right. \\ & \left. \times \frac{\alpha}{\alpha^2 + \beta^2} f'_n(s_+) \epsilon \Delta_{2\epsilon}(t) \right\}, \end{aligned} \quad (7.14)$$

where the second term in the curly brackets is one order higher in ϵ than the first term. Now we want to Taylor expand the g_i 's around $t = 0$ and $r_i = 0$. For the first term in the curly brackets, we keep the linear order terms and for the second term in the curly brackets we keep only the lowest order, constant terms, because it is already one order higher in ϵ . Concentrating on the first term in the curly brackets, we see that the zeroth order terms from the expansion in t and r all cancel. Similarly, the first order terms in r_i also all cancel, so we are left only with the first order terms in t , which are

$$\prod_{i \neq l, m}^N g_i(s_+) g'_m(s_+) g_l(s_+) t + \sum_{j \neq m}^N \prod_{i \neq l}^N g_i(s_+) g'_j(s_+) t = t \frac{d}{ds_+} \prod_{i=1}^N g_i(s_+). \quad (7.15)$$

Then, substituting this back into the expression for W_L and integrating by parts, we obtain

$$W_L = -T \sum_{l < m} k_l^\mu k_m^\nu \int \frac{ds_+}{2\pi} f'_n(s_+) \prod_{i=1}^N g_i(s_+) \int \frac{dt}{2\pi} \prod_{j \neq m, l}^N \frac{dr_j}{2\pi} F(r_1, \dots, r_N) \times E, \quad (7.16)$$

where E is given by the expression

$$E = t \frac{d}{dt} G^{\mu\mu}(t) - t \frac{d}{dt} G_{2\epsilon}^{\mu\mu}(t) + 2 \frac{\alpha}{\alpha^2 + \beta^2} \epsilon \Delta_{2\epsilon}(t). \quad (7.17)$$

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For large values of t , this expression is well defined and goes to 0 as $\epsilon \rightarrow 0$. Therefore, as long as the rest of the integrand is not too singular as $\epsilon \rightarrow 0$, we can evaluate E for small values of t and ϵ . Then E is given by

$$E = 2 \frac{\alpha}{\alpha^2 + \beta^2} \left[-\frac{t^2}{t^2 + \epsilon^2} + \frac{t^2}{t^2 + (2\epsilon)^2} + \frac{4\epsilon^2}{(2\epsilon)^2 + t^2} \right] [1 + O(t^2) + O(\epsilon^2)] \quad (7.18)$$

$$= 2 \frac{\alpha}{\alpha^2 + \beta^2} \frac{\epsilon^2}{t^2 + \epsilon^2} [1 + O(t^2) + O(\epsilon^2)].$$

It will be useful to express this in terms of a function defined on a circle, so, making use of the fact that it contributes to the integral only for small t , we will write it as

$$E \approx \epsilon \frac{\alpha}{\alpha^2 + \beta^2} \frac{e^\epsilon}{e^\epsilon + e^{-\epsilon}} \frac{1}{2 \cos t} \quad (7.19)$$

$$= -\epsilon \frac{d}{d\epsilon} G^{\mu\mu}(t) [1 + O(\epsilon^2) + O(t^2)].$$

Then we can write W_L in its final form as

$$W_L = T \sum_{l,m} k_l^\mu k_m^\mu \int \frac{ds_+}{2\pi} f_N'(s_+) \prod_{i=1}^N g_i(s_+) \quad (7.20)$$

$$\times \int \prod_{j=1}^N \frac{dr_j}{2\pi} F(r_1, \dots, r_N) \left(\epsilon \frac{d}{d\epsilon} G^{\mu\mu}(r_l - r_m) \right),$$

where we have used translational invariance to introduce r_l as a dummy variable and used the definition in equation (7.8) for t .

W_C . All that remains now is to evaluate W_C . Once again, we perform a change of variables given by

$$s_+ = \frac{1}{2}(s_l + s_m) \quad (7.21)$$

$$t = s_m - s_l$$

$$r_j = s_j - s_l.$$

Then W_C can be written as

$$W_C = i \frac{\beta}{\alpha^2 + \beta^2} T \sum_{(l,m)} k_l^\mu k_m^\mu \int \frac{ds_+}{2\pi} \prod_{i \neq l,m} \frac{dr_i}{2\pi} g_j(s_+ + r_j - \frac{1}{2}t) \quad (7.22)$$

$$\times \left[f_n(s_+ - \frac{1}{2}t) - f_n(s_+ + \frac{1}{2}t) \right] \Delta_\epsilon(t) F(r_1, \dots, r_N).$$

When this expression is expanded around $t = 0$ and $r_j = 0$ to first order, only one term remains, and W_C becomes

$$W_C = -i \frac{\beta}{\alpha^2 + \beta^2} T \sum_{(l,m)} k_l^\mu k_m^\mu \int \frac{ds_+}{2\pi} f_n'(s_+) \prod_{j=1}^N g_j(s_+) \quad (7.23)$$

$$\times \int \prod_{i=1}^N \frac{dr_i}{2\pi} F(r_1, \dots, r_N) (r_m - r_l) \Delta_\epsilon(r_m - r_l).$$

Now we would like to write the following expression in a slightly different form.

$$E = i \frac{\beta}{\alpha^2 + \beta^2} (r_m - r_l) \Delta_\epsilon(r_m - r_l) \quad (7.24)$$

$$= -i \frac{\beta}{\alpha^2 + \beta^2} \frac{(r_l - r_m)}{\cosh \epsilon - \cos(r_l - r_m)} \frac{\sinh \epsilon}{\cosh \epsilon - \cos(r_l - r_m)}.$$

For small ϵ , Δ_ϵ picks out only small values of its argument, $r_l - r_m$, in the integral. Therefore, to the order in ϵ that we have been doing our calculation, we can replace $(r_l - r_m) \sinh \epsilon$ with $\epsilon \sin(r_l - r_m)$. Then E is

$$E \approx -\epsilon i \frac{\beta}{\alpha^2 + \beta^2} \frac{\sin(r_l - r_m)}{\cosh \epsilon - \cos(r_l - r_m)} \quad (7.25)$$

$$= -\epsilon \frac{d}{d\epsilon} G_{xy}(r_l - r_m).$$

Substituting the expression for E back into the equation for W_C , we obtain the final form for W_C .

$$W_C = T \sum_{l < m} k_l^\mu k_m^\nu \int \frac{ds_+}{2\pi} f_n'(s_+) \prod_{j=1}^N g_j(s_+) \quad (7.26)$$

$$\times \int \prod_{i=1}^N \frac{dr_i}{2\pi} F(r_1, \dots, r_N) \left[\epsilon \frac{d}{d\epsilon} G^{\mu\nu}(r_l - r_m) \right].$$

8. The Ward Identity and the β -function

To leading order in ϵ , we have obtained the following form for the Ward identity.

$$0 = \int \frac{ds_+}{2\pi} f_n'(s_+) e^{i \sum_j k_j \cdot \vec{\phi}(s_+)} e^{-\frac{1}{2} \sum_j |\vec{k}_j|^2 G^{\mu\mu}(0)} \exp \left(-\frac{\epsilon}{2} \frac{\alpha}{\alpha^2 + \beta^2} (q_x^2 + q_y^2) \right) \quad (8.1)$$

$$\prod_j \frac{T_j}{\epsilon} \int \prod_i \frac{dr_i}{2\pi} \left\{ [I(r_1, \dots, r_N) - R(r_1, \dots, r_N)] \right.$$

$$\left. \times \left[\sum_l \left(1 - \frac{\alpha}{\alpha^2 + \beta^2} \vec{k}_l \cdot \vec{k}_l \right) + \epsilon \sum_{l < m} k_l^\mu k_m^\nu \frac{d}{d\epsilon} G^{\mu\nu}(r_l - r_m) \right] \right\},$$

where I is given by

$$I(r_1, \dots, r_N) = \prod_{i < j} e^{-G^{\mu\nu}(r_i - r_j)} k_i^\mu k_j^\nu. \quad (8.2)$$

R is the corresponding integrand for all the connected one-particle reducible and disconnected diagrams, with the extra condition that when it is multiplied by $\frac{d}{d\epsilon} G^{\mu\nu}(r_l - r_m)$, this additional propagator is included in determining whether or not a graph is 1PI or connected.

At this point we have calculated the “flow” for $\prod_{j=1}^N g_j(s) = e^{i \sum_j \vec{k}_j \cdot \vec{\phi}(s)}$. It is given by the coefficient of $\int (ds_+/2\pi) f'_n(s_+) \prod_{j=1}^N g_j(s_+)$ in equation (8.1). As we shall show next, to leading order in ϵ , this agrees exactly with the flows produced by the β -function when $\sum_j \vec{k}_j \neq 0$. We begin by showing that the expression for the Ward identity in equation (8.1) is a total derivative. From the definition of I in equation (8.2), we can see that

$$\begin{aligned} [I(r_1, \dots, r_N) - R(r_1, \dots, r_N)] \epsilon \sum_{l < m} k_l^i k_m^j \frac{d}{d\epsilon} G^{\mu\nu}(r_l - r_m) \\ = -\epsilon \frac{d}{d\epsilon} (I - R). \end{aligned} \quad (8.3)$$

To show that the other term in the Ward identity is also a derivative with respect to ϵ , we use the definition

$$G^{\mu\nu}(0) = -\frac{\alpha}{\alpha^2 + \beta^2} \ln(\epsilon^+ + \epsilon^{-\epsilon} - 2) \approx -\frac{\alpha}{\alpha^2 + \beta^2} \ln \epsilon^2 \quad (8.4)$$

to evaluate

$$\begin{aligned} e^{-\frac{1}{2} \sum_j \vec{k}_j \cdot \vec{k}_j G^{\mu\nu}(0)} \prod_j \frac{T_j}{\epsilon} \sum_i \left(1 - \frac{\alpha}{\alpha^2 + \beta^2} \vec{k}_i \cdot \vec{k}_j \right) \\ = -\epsilon \frac{d}{d\epsilon} \left[e^{-\frac{1}{2} \sum_j \vec{k}_j \cdot \vec{k}_j G^{\mu\nu}(0)} \prod_j \frac{T_j}{\epsilon} \right] (1 + O(\epsilon^2)). \end{aligned} \quad (8.5)$$

In this calculation, we have neglected the factor of $\exp\left(-\frac{1}{2} \frac{\alpha}{\alpha^2 + \beta^2} (q_x^2 + q_y^2)\right)$ in equation (8.1). However, it only contributes corrections that are proportional to $\vec{q} \cdot \vec{q}\epsilon$, which means that, to the order in ϵ that we are calculating, we can ignore this factor.

Thus, the Ward identity can be expressed as a total derivative with respect to ϵ . It is

$$0 = -\epsilon \frac{d}{d\epsilon} \int \frac{d}{ds_+} f'_n(s_+) e^{i \sum_j \vec{k}_j \cdot \vec{\phi}(s_+)} \prod_j \frac{T_j}{\epsilon} e^{-\frac{1}{2} \sum_j |\vec{k}_j|^2 G^{\mu\nu}(0)} \int \prod_i \frac{d\tau_i}{2\pi} F. \quad (8.6)$$

From this equation, we see that the Ward identity is satisfied whenever

$$\int \frac{ds_+}{2\pi} f'_n(s_+) e^{i(\sum_j \vec{k}_j) \cdot \vec{\phi}(s_+)} = 0 \quad (8.7)$$

or

$$-\epsilon \frac{d}{d\epsilon} \prod_j \frac{T_j}{\epsilon} e^{-\frac{1}{2} \sum_j |\vec{k}_j|^2 G^{\mu\nu}(0)} \int \prod_i \frac{d\tau_i}{2\pi} F = 0 \quad \text{as } \epsilon \rightarrow 0. \quad (8.8)$$

When the total charge is zero, $\sum_j \vec{k}_j = 0$. Therefore, the first condition becomes

$$\int_{-\pi}^{\pi} \frac{ds_+}{2\pi} f'_n(s_+) = 0. \quad (8.9)$$

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This is always satisfied because it is the integral of a derivative of a periodic function. When the total charge is not zero, the first condition is not satisfied. In that case, the Ward identity is satisfied if and only if the second condition, given by eqn. (8.8), is true.

The first conclusion to be drawn from this result is that this condition for the Ward identity to be satisfied is the same as the condition for the system to be at a zero of the beta function, where the beta function is given by $-\epsilon \partial \Gamma / \partial \epsilon$. From equations (6.6), (7.3) and (7.4), the 1-particle irreducible function for graphs with vertices $(T_j/\epsilon) e^{i \vec{k}_j \cdot \vec{\phi}(s_j)}$ for $1 \leq j \leq N$ is given by

$$\Gamma(\vec{\phi}) = \prod_i \frac{T_i}{\epsilon} e^{-\frac{1}{2} \sum_j |\vec{k}_j|^2 G^{\mu\nu}(0)} \int \prod_{i=1}^N \frac{ds_i}{2\pi} e^{i \sum_j \vec{k}_j \cdot \vec{\phi}(s_j)} F. \quad (8.10)$$

If we take a derivative with respect to ϵ , and then expand all the $\vec{\phi}(s_j)$'s about a common point, we obtain

$$-\epsilon \frac{\partial \Gamma}{\partial \epsilon} = 0 \quad (8.11)$$

if and only if

$$-\epsilon \frac{d}{d\epsilon} \int \frac{ds_+}{2\pi} e^{i \sum_j \vec{k}_j \cdot \vec{\phi}(s_+)} \prod_j \frac{T_j}{\epsilon} e^{-\frac{1}{2} \sum_j |\vec{k}_j|^2 G^{\mu\nu}(0)} \int \prod_i \frac{d\tau_i}{2\pi} F = 0. \quad (8.12)$$

Since $\int \frac{ds_+}{2\pi} e^{i \sum_j \vec{k}_j \cdot \vec{\phi}(s_+)} \neq 0$ for arbitrary $\vec{\phi}(s_+)$, we find that the beta function equals zero only when equation (8.8) is satisfied. Therefore, when $q \neq 0$, the conditions for the Ward identity to be satisfied and for the system to lie at a zero of the beta function are the same, at least in the limit as $\epsilon \rightarrow 0$.

Secondly, we note that the Ward identity was an infinite set of symmetries; for each $f_n(s) = i e^{i n s}$ there is a separate identity. However, we have reduced each of these equations down to one single condition, given by equation (8.8). As a result, if the system possesses just one of these symmetries (or if it is at a zero of the β -function) it will possess all of the symmetries, as in the case of scale invariance leading to conformal invariance in 2D theories.

9. Validity of Approximations

At this point, we will review the approximations that were made in obtaining equation (8.6) and show that they should be valid as ϵ goes to zero. The first comes from expanding the vertices around a common point, s_+ . This approximation involved keeping only the lowest order terms in the Taylor series for $\prod_j g_j(s_+ + r_j)$ and $f'_n(s_+ + t)$. The higher order terms include higher derivatives on g_j and f_n . We can always integrate by parts so that all these extra derivatives act on the g_j 's. That means we are calculating the flows for higher dimension operators, such as $\frac{dg}{dt} e^{i \vec{q} \cdot \vec{\phi}(t)}$, $\frac{d^2 g}{dt^2} e^{i \vec{q} \cdot \vec{\phi}(t)}$, etc. As long as $q > 0$, all of these operators are irrelevant, so we expect them to go to zero as $\epsilon \rightarrow 0$. The only remaining relevant operator is $\frac{dg}{dt}$, which we will calculate in the following section.

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The second kind of approximation comes from expanding propagators around small values of their arguments and small ϵ . Before we evaluate the integrals, all these approximations give corrections that are $O(\epsilon^2)$ or $O(\epsilon^4)$. The $O(\epsilon^2)$ corrections are higher order in ϵ , as claimed, but, when calculating W_L and W_C , we must be careful about the $O(\epsilon^2)$ corrections. (For W_P , we can directly evaluate the $O(\epsilon^2)$ corrections to the integral in equation (4.35), with the result that they go as at most as $O(\epsilon)$.) Whenever we obtained a correction that was $O(\epsilon^2)$, it came from expanding a product of an approximate delta function times a propagator. If only the short-distance behavior of the whole graph is important, then this correction goes as $O(\epsilon^2)$ after we integrate over the graph. However, if the large distance contribution is bigger than the short distance contribution, then the correction goes at most as ϵ , where the ϵ comes from the approximate δ -function and the c is a finite number because for large separations of the vertices the integrals are bounded.

The third approximation involved ignoring the factor of $\exp(-\frac{1}{2} \frac{\alpha}{\alpha^2 + \beta^2} (q_x^2 + q_y^2))$ in equation (8.1). It gives corrections that are $O(\epsilon)$.

To see whether these corrections should vanish as ϵ goes to zero, we will look at the superficial degree of divergence of the graphs. Recall that $\exp(-k_i k_j G^{\mu\nu}(t)) \sim (\epsilon^2 + t^2)^{k_i k_j}$ for small ϵ and t , and $|\exp(-k_i k_j G^{\mu\nu})| = 1$ for $\mu \neq \nu$. For convenience, we will let $\gamma = \frac{\alpha}{\alpha^2 + \beta^2}$. Then the vertex factors and self-interactions in equation (8.8) go as

$$A = \prod_{i=1}^N (T_i / \epsilon) \gamma |\vec{k}_i|^2. \quad (9.1)$$

The short-distance contributions from the integrals go as

$$S = \epsilon^{N-1+\gamma} [q^2 - \sum |\vec{k}_i|^2], \quad (9.2)$$

and the large-separation contributions goes as

$$L = \epsilon c \quad \text{and} \quad L = c_p, \quad (9.3)$$

for $W_L + W_C$ and W_P , respectively. From equation (7.2), we note that when all the $\gamma |\vec{k}_i|^2 = 1$, then $c_p = 0$. Also, as noted in the previous paragraphs, the correction to L should go as $\epsilon(c + c_p)$.

If we renormalize the potential by keeping $V_i^R = T_i$ finite, then the graphs go as

$$\prod_{i=1}^N V_i^R \left[\epsilon^{\gamma q^2 - 1} + (c\epsilon + c_p) \epsilon^{\sum_{i=1}^N (\gamma |\vec{k}_i|^2 - 1)} \right]. \quad (9.4)$$

Ideally, for calculating the renormalization group flow, we would like only the short-distance behavior to be important. In that case, the graph goes as $\epsilon^{\gamma q^2 - 1}$ and the corrections go at most as $\epsilon^{\gamma q^2}$. Therefore, for $q \neq 0$, they vanish as $\epsilon \rightarrow 0$. Unfortunately, for arbitrary potentials, the large-distance part of the graph can be non-negligible. As long as $\gamma |\vec{k}_i|^2 \geq 1$, the long distance part goes to 0 as $\epsilon \rightarrow 0$, so the graphs should be proportional to $\epsilon^{\gamma q^2 - 1}$ and our approximations should be valid. In some special cases, even when $\gamma |\vec{k}_i|^2 < 1$, the

large-distance part of the graph can be renormalized away [11] so that $c = c_p = 0$ and the graph is still proportional to $V^N \epsilon^{\gamma q^2 - 1}$. However, in the general case, we will normal order any vertex that has $\gamma |\vec{k}_i|^2 < 1$. In that case, $V_i^R = T_i \epsilon^{\gamma |\vec{k}_i|^2 - 1}$ is kept finite. This is similar to what is done in the Sine-Gordon model [12]. Then the graph goes as

$$\prod_{i=1}^N V_i^R \left[\epsilon^{\gamma q^2 - 1} \epsilon^{\sum_{i=1}^M (1 - \gamma |\vec{k}_i|^2)} + (c\epsilon + c_p) \epsilon^{\sum_{i=M+1}^N (\gamma |\vec{k}_i|^2 - 1)} \right], \quad (9.5)$$

where the vertices with $1 \leq i \leq M$ have $\gamma |\vec{k}_i|^2 < 1$ and those with $M+1 \leq i \leq N$ have $\gamma |\vec{k}_i|^2 \geq 1$. The short-distance contribution is now at most $\epsilon^{\gamma q^2 - 1}$, which means that for $q \neq 0$ the corrections to the short-distance behavior still go to zero as $\epsilon \rightarrow 0$. The long distance behavior can still be finite, but the corrections now go at most as $\epsilon(c + c_p)$, so they also vanish as $\epsilon \rightarrow 0$. These estimates ignore the possibility that we could get even more divergent contributions from the regions of integration where only some of the variables are close. We are assuming that once the disconnected diagrams are subtracted, this is not the case. Given this assumption, we conclude that when $q \neq 0$, all our approximations should be valid and the Ward identity is equivalent to the β -function as $\epsilon \rightarrow 0$.

10. Charge-0 graphs and the flow for $d\tilde{\phi}/dt$

The case when $\vec{q} = 0$ is special. The leading order contribution gives the flow for $e^{\tilde{\phi}} \tilde{\phi}(t) = 1$. Because it is multiplied by equation (8.7), the Ward identity for this operator is always equal to zero, even when we include the corrections due to expanding the propagators and δ -functions. The next most relevant operator obtained from these graphs is $d\tilde{\phi}(t)/dt$. If we repeat the calculations of the Ward identity and β -functions of the previous sections, keeping the next higher order term in the Taylor expansion of $f_n(s) \prod_{i=1}^N g(s_i)$, we find that the flow for $\vec{k}_j \cdot d\tilde{\phi}(t)/dt$ is

$$\int \prod_i \frac{d\vec{r}_i}{2\pi} \left\{ r_j F(r_1, \dots, r_N) \times \left[\sum_i \left(1 - \frac{\alpha}{\alpha^2 + \beta^2} \vec{k}_i \cdot \vec{k}_i \right) + \epsilon \sum_{l < m} k_l^\mu k_m^\nu \frac{d}{d\epsilon} G^{\mu\nu}(r_l - r_m) \right] \right\}, \quad (10.1)$$

and all our approximations should be valid.

For each graph that contributes to this integral, there is another graph with the signs of all the k_i reversed. Both graphs have the same value of $F(r_1, \dots, r_N)$, and when the total charge of the graph is 0, both graphs contribute to the flow of $d\tilde{\phi}/dt$. Therefore, when we add the expressions from equation (10.1) due to these two graphs, we obtain zero. We conclude that the coupling constant for $d\tilde{\phi}/dt$ does not flow.

11. Critical cosine potential

Finally, we return to the special case of the on-shell cosine potential which satisfies $\frac{\alpha}{\alpha^2 + \beta^2} |\vec{k}_i|^2 = 1$. From the discussion in the previous sections, we expect that the charge-one graphs for Γ should go as ϵ^0 . This means they can have a logarithmic divergence in ϵ .

It follows that when we take $\epsilon \partial \Gamma / \partial \epsilon$ in order to evaluate the β -function, we can obtain a non-zero answer. Therefore, for arbitrary $\bar{k}_i$, we do not expect the theory to be at a zero of the β -function and we would expect the coupling constant of $e^{i\bar{k}\cdot\bar{\psi}(t)}$ to flow. In fact, in reference [13], the β -function for arbitrary tachyon potentials and zero gauge-field was calculated, and the potential which satisfies the β -function at one-loop does not satisfy it at higher orders [14]. This calculation, though, does not apply to the special case of the cosine potential given in equation (2.9), because the integrals with the regulator in reference [13] are undefined in the limit as $\frac{\alpha}{\alpha+g^2}k_\mu^2 \rightarrow 1$ for any individual component of each $\bar{k}$. For the special choice of potential given in equation (2.9), whenever $\frac{\alpha}{\alpha+g^2}k_z^2 = \frac{\alpha}{\alpha+g^2}k_y^2 = 1$ and $\beta/\alpha \in \mathbb{Z}$, we can say much more if we use the regulator defined in this paper instead.

According to references [5] and [6], we know that in this case the free energy has no logarithmic divergences. In reference [15], we prove that the free energy of charge zero graphs goes as $1/\epsilon$ (with no logarithmic subdivergences.) This means that all our approximations for charge-0 graphs are valid, and they satisfy the Ward identity. Similar calculations [16] prove that for all charged graphs, the free energy, F^* , is always finite, as we expected, and that all the approximations we made in calculating the flow for the coupling constant of $e^{i\bar{k}\cdot\bar{\psi}(t)}$ are valid. Thus $\epsilon \partial F^* / \partial \epsilon = 0$, as ϵ goes to zero, for all the charged graphs. The only difference between the graphs for F^* and Γ is that in the latter we must subtract the 1PR graphs. For the critical cosine potential, using the results of reference [15], it is straightforward to show that all such 1PR graphs give 0 contribution to Γ and the Ward identity as $\epsilon \rightarrow 0$. We conclude that this potential is at a zero of the β -function and satisfies the Ward identity to all orders in α' , at every order in V .

12. Conclusions

In this paper, we have shown that when the non-local, 1-D field theory has a constant gauge field and arbitrary scalar potential, the reparametrization invariance Ward identities are equivalent to the β -function as the cutoff goes to zero. One consequence of this result is that if the theory is scale invariant, it also exhibits an infinite set of symmetries which come from the reparametrization invariance of the underlying string theory. The proof presented in this paper is rather involved, so we hope in the future one might find a simpler way to demonstrate the equivalence. Also, the question still remains of whether, for any boundary state or dissipative quantum system, scale invariance always implies that the Ward identities are satisfied.

Secondly, combined with results from reference [15] and generalizations, we have proved that the cosine potential of equation (2.9) with $\frac{\alpha}{\alpha+g^2}k_z^2 = \frac{\alpha}{\alpha+g^2}k_y^2 = 1$ and $\beta/\alpha \in \mathbb{Z}$ satisfies the Ward identity. Therefore, for this particular choice of on-shell tachyon potential, the one-loop solution to the β -function remains a solution to all orders in V_0 and α' . This differs from the case for the general on-shell open string tachyon potential (as in references [13] and [14]) and for the sine-Gordon theory, where the β -function gets corrections at higher orders in V_0 , so that the value of the period flows.

These results demonstrate that at these critical values of the cosine potential and gauge field, the system is a conformal field theory, and, if we include the remaining 24 dimensions, should give solutions to open string theory with non-trivial backgrounds. These

backgrounds are of interest, firstly because, using the methods in [15], we can find exact solutions for some of the correlation functions and can easily do computations to low orders in V_0 . Secondly, there is a whole fractal network of circles in the $\alpha - \beta$ plane where one-loop renormalization group calculations and duality symmetries show that dissipative Hofstadter model is critical. We still do not know if these theories are really critical to all orders in the potential. If they are, the results in this paper would suggest that these theories also give solutions to string theory. Thus we would have a complex picture of what happens in open string theory as we vary some of the background fields.

To apply the results in this paper to dissipative quantum systems, we must integrate over the zero mode of $\bar{X}$ in equation (4.4). Then the Ward identities for $n = 0, \pm 1$ says that the correlation functions for the special critical theories are $SU(1,1)$ covariant in addition to satisfying the remaining Ward identities. This means that the critical theory is not only invariant under scaling and time translations, but also under taking the time, t , to $1/t$. From the point of view of dissipative quantum systems, this enhanced symmetry is unexpected. These results are helpful in solving for the correlation functions of the multi-critical dissipative Hofstadter model, but, by themselves, do not contain enough information for solving the theory. In a future paper, we will combine these results with other symmetries of the dissipative Hofstadter model to find exact solutions for some of the correlation functions.

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