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QUANTUM CHARACTERISTIC EXPONENTS**

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On the existence of quantum characteristic exponents

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Abstract

The Lyapunov characteristic exponents provide a rigorous characterization of the notion of *chaos* in classical mechanics. Similar quantities have been proposed as a characterization of *quantum chaos*. In this note we find the conditions under which the existence of quantum characteristic exponents can be proved.

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In classical mechanics a dynamical system is said to be chaotic if at least one of its Lyapunov exponents is positive. To attempt a similar characterization of quantum chaos, the notion of *quantum sensitive dependence* and quantum characteristic exponent was introduced^{1,2}, namely

$$\lambda_{\mathcal{A},\{\phi_n\}} = \lim_{k \rightarrow \infty} \frac{1}{k} \lim_{n \rightarrow \infty} \ln \left\| \text{Re}(U^k \phi_n, \mathcal{A} U^k \psi) \right\| \quad (1)$$

where U is the time-evolution operator, $\mathcal{A}$ is a symmetric operator and ϕ_n a sequence of functions converging to δ'_x .

One notices that this expression has a double pointwise nature, depending both on the point x of δ'_x and the state ψ . In Ref. 2 the limit has been shown to exist in some simple models. In the cases that were computed the limit was found to be independent of ψ and to be positive in situations where there is also classical chaos. In this note, by defining an appropriate measure-theoretic setting for the quantum spaces, an existence proof is obtained for the quantum characteristic exponents.

A definition of quantum characteristic exponent of the type of (1.1) is suggested by the form of the classical Lyapunov exponent written in terms of densities $\rho(x)$ in the phase space M .

$$\lambda_{u,x} = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \left\| -u^i D_{\partial_i \delta_x} \left\{ \int dy y f^k(\psi^2(y)) \right\} \right\| \quad (2)$$

u is a vector in the tangent space TM , f is the time-evolution mapping and to emphasize the correspondence with quantum mechanics I have written $\rho(y) = \psi^2(y)$. $D_{\partial_i \delta_x}$ is the Gateaux derivative of the functional $I_k = \left\| \int dy y f^k(\psi^2(y)) \right\|$ in the space of square roots of the densities.

To check that Eq.(2) leads to the usual Lyapunov exponent definition, one uses the properties of the Perron-Frobenius operator^[3]

$$f\rho(y) = \sum_i \rho(f^{-1}y)_i \frac{1}{\|Df(f^{-1}y)_i\|}$$

to change variables and obtain

$$\lambda_{u,x} = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \left\| u^i \partial_i \left\{ f^k(x) \psi(x) \right\} \right\| \quad (3)$$

Then, if $\psi(x)$ is a C^1 -function

$$\lambda_{u,x} = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \left\| u^i \partial_i f^k(x) \right\| \quad (4)$$

Notice that, except for the regularity conditions, $\lambda_{u,x}$ does not depend on the density ψ^2 . Oseledec's theorem asserts that $\lambda_{u,x}$ is defined almost everywhere in the support of an f -invariant measure μ on M , being a constant in each ergodic component of μ .

Let U^k (k continuous or discrete) be the unitary operator of quantum time evolution. The quantum analog of Eq.(2) involves the following functional

$$u^i D_{\partial_i \delta_X} (U^k \psi, X U^k \psi) = 2 u^i \operatorname{Re} \langle U^k \partial_i \delta_X, X U^k \psi \rangle \quad (5)$$

where $U^k X U^k \psi$ is assumed to belong for example to the Schwartz space $\mathfrak{S}$. The quantum space is a space of functions on a manifold M , X being a coordinate operator in M . According to the choice for M , X may stand either for a space coordinate or a canonical momentum.

To make the limit of (5), when $k \rightarrow \infty$, to be independent of ψ (except for regularity conditions) we will need a norm for the functionals and a U -invariant measure on a space containing $\partial_i \delta_X$ is also needed to define the domain of existence of the limit. To achieve these goals one constructs a Gelfand triplet

$$E^* \supset \mathfrak{H} \supset E \quad (6)$$

E being a nuclear space and E^* its dual. It is convenient to obtain E as a countably normed space, projective limit of a sequence of Hilbert spaces $\mathfrak{H}_p$.

$$E^* \supset \dots \supset \mathfrak{H}_{-p} \supset \dots \supset \mathfrak{H}_{-1} \supset \mathfrak{H} \supset \mathfrak{H}_1 \supset \dots \supset \mathfrak{H}_p \supset \dots \supset E$$

with ordered norms

$$\|\cdot\| \leq \|\cdot\|_1 \leq \dots \leq \|\cdot\|_p \leq \dots$$

If the time evolution operator U is a strongly continuous unitary operator on $\mathfrak{H}$, by Stone's theorem, there is a self-adjoint operator A such that $U = e^{iA}$. Use A to define the sequence of norms by

$$\|f\|_p^2 = (A^p f, A^p f)$$

Furthermore A is assumed to be bounded from below and such that there is a $k \in \mathbb{N}$ for which A^{-k} is Hilbert-Schmidt. The projective limit space E is therefore a nuclear space.

The triplet (6) might be constructed with an operator other than the generator of time evolution. This particular choice, however, insures U -invariance of the norms and simplifies the proof below. Nuclearity of E is important to insure the existence of σ -additive probability measures on E^* [4].

We may now write the quantum characteristic exponent as follows

$$\lambda_{u,x} = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \| U^{-k} X U^k u^i \partial_i \delta_x \|_{-\ell} \quad (7)$$

$\| \cdot \|_{-\ell}$ being the norm in $\mathfrak{K}_{-\ell}$

$$\| \cdot \|_{-\ell} = \sup_{\phi \in E, \|\phi\|_{\ell} \leq 1} | \langle \cdot, \phi \rangle |$$

and ℓ must be greater or equal to the order of the functional $U^{-k} X U^k u^i \partial_i \delta_x$. Notice that U^k , being an element of the infinite dimensional unitary group U_{∞} , has a natural action in E^* [5].

Because E is a nuclear space, by Minlos theorem, for any continuous functional $C(\xi)$ $\xi \in E$ which is positive definite and normalized ($C(0) = 1$), there is a unique probability measure μ on $(E^*, \mathfrak{B})$ such that

$$C(\xi) = \int_{E^*} e^{i \langle x, \xi \rangle} d\mu(x)$$

$\mathfrak{B}$ is the Borel σ -algebra generated by the weak topology (or equivalently the cylinder sets) of E^* .

Let μ be such a probability measure on $(E^*, \mathfrak{B})$, invariant under the element U of the infinite-dimensional unitary group. Then, existence of the limit in Eq.(7) is a corollary of the following theorem

Theorem: Let $\phi \in \mathfrak{K}_{-p}$, $\mathfrak{K}_p$ invariant under $U^{-k} X U^k \quad \forall k$ and μ a U -invariant probability measure in $(E^*, \mathfrak{B})$ such that

$$\int_{E^*} \ln \| U^{-k} X U^k \phi \|_{-p} d\mu(\phi)$$

is finite. Then, the limit

$$\lambda_{\phi} = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \| U^{-k} X U^k \phi \|_{-p}$$

exists μ -almost everywhere in $\mathfrak{K}_{-p}$.

Proof:

$$\lambda_\phi = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \|X U^k \phi\|_{-p}$$

follows from the U -invariance of the norms. The functional $g_k(\phi) : \mathcal{H}_{-p} \rightarrow \mathbb{R}$ defined by

$$g_k(\phi) = \ln \frac{\|X U^k \phi\|_{-p}}{\|X \phi\|_{-p}} = \sum_{j=1}^k \ln \frac{\|X U^j \phi\|_{-p}}{\|X U^{j-1} \phi\|_{-p}}$$

is subadditive, $g_{k+n}(\phi) \leq g_k(\phi) + g_n(U^k \phi)$. Applying Kingman's theorem^[6] the result then follows from U -invariance of the measure and the integrability assumptions. $\square$

Provided the stated conditions are satisfied, the theorem asserts the existence μ -almost everywhere of the limit, but not its unicity. That will depend on the ergodic properties of the measure μ under the one-parameter subgroup of U_∞ generated by U .

The Gaussian (white noise) measure^[5] μ_0 given by

$$C_0(\xi) = e^{-\|\xi\|^2}$$

is invariant under the whole U_∞ group. Therefore quantum characteristic exponents will exist at least in the support of this measure, but the decomposition into ergodic components will depend on the detailed nature of the time-evolution operator U .

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