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ON SPACE-TIME METRICS INDUCING THE RADIATION CONNECTION

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Abstract

It is shown that Lorentzian manifolds giving rise to the radiation connection, i.e. such that the canonical connection induces a unique connection on an isotropic hypersurface, are of type D or conformally flat in Petrov classification, generalizing the Robinson-Bertotti electromagnetic universe with cosmological constant. Under an holonomy hypothesis it is deduced that the underlying space-time manifold is a direct product of two two-dimensional spaces, one of them being space-like, and the other one time-like.

June 1993

CPT-93/P.2926

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I. Introduction

In previous publications [1,2] the conditions which permit to construct the bundle of adapted frames over an isotropic hypersurface (usually known as a null hypersurface but hereafter referred under the acronymous "ishyps") have been studied, which has led to introduce by this way the so-called radiative structures. But the reduction to the bundle of adapted frames of the Levi-Civita connection of the ambient space-time does not induce a connection over the ishyps. A supplementary reduction to a subbundle of the bundle of adapted frames is necessary to get a unique induced connection on the ishyps which has been called the radiation connection (also denoted by G_0 -connection for the sake of brevity in the following).

In [2] the radiation connection and the consequences on the Einstein's field equations have been studied, giving raise to a particular energy-momentum tensor.

The purpose of this paper is an extensive study of the Einstein's field equations, under the assumption of the existence of the radiation connection, using the complex vectorial version of the Newman-Penrose (N-P) formalism. This work permits to find an exact solution (already known) of the Einstein-Maxwell electrovacuum equations, which are presented in different coordinate systems. The use of these different coordinate systems, for this particular solution and a detailed study, from the holonomy group point of view, make possible the characterization of the underlying manifold. The paper is organized as follows:

In sec. 2, the radiation connection and the complex vectorial version of the N-P formalism are introduced and the constraints on the spin coefficients implied by the existence of the G_0 -connection are given. In sec. 3, the family of metrics which admit the G_0 -connection is exhibited and by solving the Einstein-Maxwell equations in vacuum an example of G_0 -space is found, namely the Robinson-Bertotti solution which is expressed in different coordinate systems. In sec. 4, a characterization of the underlying space-time manifold is given.

2. The G_0 -connection and the complex vectorial formalism

Let $(V_{3,1}, g)$ be a Lorentz space-time, the metric g being of signature -2. Let β denote the degenerate metric induced by g , on the isotropic hypersurface Σ and $[\xi]$ the line field, generating the kernel of β i.e. $[\xi] = \{\lambda\xi, \lambda \in \mathbb{R} | \beta(\xi) = 0\}$. One can suppose that ξ is the restriction to Σ of an isotropic vector field in $V_{3,1}$, also denoted by ξ , so $g(\xi, \xi) = 0$.

At each point x of Σ , the cone of the isotropic directions of x , has a first order contact with Σ , but the tangent space $T_x \Sigma$, does not contain any time-like vector, the future light cone of x being entirely on one side of Σ and the past light cone of x entirely on the other side. We shall work in the moving frames of the complex vectorial formalism [3] [4].

A covariant null tetrad of one-forms θ^a ($a = 1, 2, 3, 4$) is defined, in which the metric can be written under the form:

$$ds^2 = g = \theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1 - \theta^3 \otimes \theta^4 - \theta^4 \otimes \theta^3 \quad (1)$$

In local coordinates system (x^i) , we have ($i = 1, 2, 3, 4$)

$$\theta^1 = n_i dx^i, \quad \theta^2 = \ell_i dx^i, \quad \theta^3 = -\bar{m}_i dx^i, \quad \theta^4 = -m_i dx^i \quad (2)$$

where the covariant vectors ℓ_i and n_i are real and null and the complex null vectors m_i and $\bar{m}_i$ are complex conjugate. The dual basis to $\{\theta^a\}$ is denoted by $\{X_a\}$ and the correspondence with the "N-P operators" is given by :

$$\begin{aligned} X_1 &= D = \ell^i \partial_i & X_2 &= \Delta = n^i \partial_i \\ X_3 &= \delta = m^i \partial_i & X_4 &= \bar{\delta} = \bar{m}^i \partial_i \end{aligned} \quad (3)$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

As it has been shown in [2], the radiation connection is the reduction of the canonical Levi-Civita connection to $G_0(V_{3,1})$, the so called adapted frame bundle to the radiation connection, which is the subbundle of pseudo-orthonormal frames of $V_{3,1}$, with structure group $G_0 \approx SO(1,1) \otimes SO(2)$. Then the radiation connection one-form, which is $\mathcal{L}(G_0)$ -valued, can be written:

$$\omega = \begin{bmatrix} \omega_1^1 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 \\ 0 & 0 & \omega_3^3 & 0 \\ 0 & 0 & 0 & \omega_4^4 \end{bmatrix} \quad (4)$$

with $\omega_1^1 = \omega_2^2$ and $\omega_4^4 = -\omega_3^3$.

The radiation connection obeys the following structure equations:

$$\text{- for the torsion} \quad \Theta = 0 = d\theta + \omega \wedge \theta \quad (5a)$$

$$\text{- for the curvature} \quad \Phi = 0 = d\omega + \omega \wedge \omega \quad (5b)$$

In order to establish the complex vectorial formalism in the N-P notation, we have to find an analytical description of ω , for the representation of the Lie-algebra of the Lorentz

group, as a Lie algebra of the 3-dimensional complex rotation group. This can be done [3] - [4] in the space of complex self dual two forms, in which we can define as basis:

$$Z^1 = \theta^1 \wedge \theta^3, \quad Z^2 = \theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4, \quad Z^3 = \theta^4 \wedge \theta^2 \quad (6)$$

The components of the metric, in this basis are:

$$\gamma^{\alpha\beta} = 4 \left[\delta_1^\alpha \delta_3^\beta - \delta_2^\alpha \delta_4^\beta \right] \quad (7)$$

$\alpha, \beta = 1, 2, 3$.

The complex connection 1-forms σ_β^α are defined by:

$$dZ^\alpha = \sigma_\beta^\alpha \wedge Z^\beta \quad (8)$$

and the vectorial connection 1-forms σ_α are related to σ_β^α by:

$$\sigma_\alpha = \frac{1}{8} e_{\alpha\beta\gamma} \gamma^\delta \sigma_\delta^\beta \quad (9)$$

where $e_{\alpha\beta\gamma}$ is the three dimensional permutation symbol.

The tetrad components $\sigma_{\alpha a}$ defined by $\sigma_\alpha = \sigma_{\alpha a} \theta^a$ are 12 complex valued functions, which are none others than the N-P spin coefficients. The explicit correspondence is given by:

$$\sigma_{\alpha a} = \begin{pmatrix} \kappa & \tau & \sigma & \rho \\ \varepsilon & \gamma & \beta & \alpha \\ \pi & \nu & \mu & \lambda \end{pmatrix} \quad (10)$$

The complex curvature 2-forms Σ_β^α are defined by:

$$d\sigma_\beta^\alpha - \sigma_\gamma^\alpha \wedge \sigma_\beta^\gamma = \Sigma_\beta^\alpha \quad (11)$$

and the vectorial curvature 2-form as follows:

$$\Sigma_\alpha = \frac{1}{8} e_{\alpha\beta\gamma} \gamma^\delta \Sigma_\delta^\beta \quad (12)$$

By expanding Σ_α in the basis $\{Z^\alpha, \bar{Z}^\alpha\}$ one obtains:

$$\Sigma_\alpha = (C_{\alpha\beta} - \frac{1}{6} R \gamma_{\alpha\beta}) Z^\beta + E_{\alpha\beta} \bar{Z}^\beta \quad (13)$$

where the quantities $C_{\alpha\beta}$ and $E_{\alpha\beta}$ are related to the N-P curvature components ψ_A and ϕ_{AB} as follows:

$$C_{\alpha\beta} = \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 \\ \psi_1 & \psi_2 & \psi_3 \\ \psi_2 & \psi_3 & \psi_4 \end{pmatrix} \quad (14)$$

$$E_{\alpha\bar{\beta}} = \begin{pmatrix} \phi_{00} & \phi_{01} & \phi_{02} \\ \phi_{10} & \phi_{11} & \phi_{12} \\ \phi_{20} & \phi_{21} & \phi_{22} \end{pmatrix} \quad (15)$$

In the context of the complex vectorial formalism the fact that the radiation connection 1-form is given by (4) can be resumed by the following proposition.

Proposition 1. Any moving frame adapted to an isotropic hypersurface is compatible with the G_0 -radiation connection iff the N-P spin coefficients possess the following properties:

$$\begin{aligned} \kappa = \nu = \sigma = \lambda = \rho = \mu = \tau = \pi = 0 \\ \alpha = \beta = \text{imaginary} \end{aligned} \quad (16)$$

3. The Einstein-Maxwell field equations coupled to the radiation connection

By inserting relations (16) in the structure equations we get

$$\theta^1 \wedge d\theta^1 = \theta^2 \wedge d\theta^2 = \theta^3 \wedge d\theta^3 = \theta^4 \wedge d\theta^4 = 0 \quad (17)$$

and by using Frobenius theorem the metric tensor can be written:

$$ds^2 = 2F^2(u, v) du dv - 2H^2(z, \bar{z}) dz d\bar{z} \quad (18)$$

the only non zero spin coefficients being:

$$\begin{aligned} \beta = \alpha = -\frac{1}{2} \frac{H_{\bar{z}}}{H^2} \\ \varepsilon = \frac{1}{2} \frac{F_u}{F^2} \\ \gamma = -\frac{1}{2} \frac{F_u}{F^2} \end{aligned} \quad (19)$$

where $F_u = \frac{\partial F}{\partial u}$ and so on...

The N-P equations [5] are then reduced to:

$$\begin{aligned} \psi_1 = \psi_3 = \psi_0 = \psi_4 = 0 \\ \phi_{00} = \phi_{22} = \phi_{02} = \phi_{01} = \phi_{21} = 0 \\ \psi_2 + 2\Lambda = 0 \end{aligned} \quad (20)$$

$$3\Lambda - \phi_{11} = \Delta\varepsilon - D\gamma - 4\varepsilon\gamma \quad (21)$$

$$3\Lambda + \phi_{11} = \delta\alpha - \bar{\delta}\beta + 4\alpha\beta \quad (22)$$

Relations (20) on ψ_A give rise to the following proposition.

Proposition 2. Spaces admitting the G_0 -radiation connection are of type D or conformally flat in Petrov classification.

The two equations (21), (22) reduce to

$$3\Lambda - \phi_{11} = \frac{1}{F^2} \left(\frac{F_u}{F} \right)_v \quad (23)$$

$$3\Lambda + \phi_{11} = -\frac{1}{H^2} \left(\frac{H_z}{H} \right)_{\bar{z}} \quad (24)$$

and the energy momentum tensor is of the form:

$$T_{ij} = 2\phi_{11}(\ell_i n_j + n_i \ell_j + \bar{m}_i \bar{m}_j + m_i \bar{m}_j) + (\lambda - \frac{R}{4})g_{ij} \quad (25)$$

as it has been established in [2].

The spaces characterized by (18)-(24) are called "generalized Robinson-Bertotti spaces" because if the energy-momentum tensor is that of a non singular electromagnetic field, the obtained solution is the Robinson-Bertotti electromagnetic universe. In order to find solutions of Einstein-Maxwell field equations with cosmological constant, when the metric is of the form (18) and under conditions (23)-(24) one has to solve the following system

$$\begin{cases} R_{ij} - \frac{1}{2}Rg_{ij} + \lambda g_{ij} = F_{ik}F_j^k - \frac{1}{4}g_{ij}F_{k\ell}F^{k\ell} \\ \nabla^k F_{ik} = 0, \quad \nabla_{[k} F_{i]j} = 0 \end{cases} \quad (26)$$

where the electromagnetic field tensor F_{ij} is non singular and its principal null directions are aligned with the principal null directions of the Weyl tensor (ℓ and n in our case). The solution is the following:

$$\phi_{11} = B\bar{B} = B_0\bar{B}_0 = e^2 \quad (27)$$

where $B_0, \bar{B}_0$ are constants, e^2 is the charge and B is the electromagnetic potential,

$$\Lambda = -\frac{1}{6}\lambda = -\frac{R}{24} \quad (28)$$

where R is the scalar curvature.

Then by integration of eq.(23) and (24) one gets

$$F = \left(1 + \frac{1}{2}(\lambda + 2e^2)uv \right)^{-1} \quad (29)$$

$$H = \left(1 + \frac{1}{2}(\lambda - 2e^2)z\bar{z} \right)^{-1} \quad (30)$$

which corresponds to the Robinson-Bertotti electromagnetic universe with cosmological constant [6], [7], [8].

The same metric in different coordinate systems can be written as:

$$ds^2 = \frac{2dudv}{\frac{1}{2}(\lambda + 2e^2)\sin h^2(v-u)} - \frac{2dzd\bar{z}}{\frac{1}{2}(\lambda - 2e^2)\cos h^2(z+\bar{z})} \quad (31)$$

and by a "à la Kruskal" transformation:

$$ds^2 = L(y)dt^2 - \frac{1}{L(y)}dy^2 - V(x)dw^2 - \frac{dx^2}{V(x)} \quad (32)$$

where:

$$L(y) = (\lambda + 2e^2)y^2 + f_1y + f_0 \quad (33)$$

$$V(x) = (\lambda - 2e^2)x^2 + g_1x + g_0 \quad (34)$$

The form (32) of the metric appears among the Schrödinger separable solutions of Einstein-Maxwell equations found by Carter [9] and Debever and McLennaghan [4]. It is a metric product of two 2-dimensional spaces of constant curvature and hence admits a six parameter group of motions [10].

By comparing (29), (30) and (32) we have to remark the form (32) of the solution is "stationnary", while the form defined by (29)-(30) is not "stationnary" for the null coordinate u which characterizes the isotropic hypersurface ($u = \text{constant}$).

To summarize this section, we can say that spaces admitting the G_0 -radiation connection admit metrics of the form (18) with an energy-momentum tensor defined by (25) and a member of this family is the Bertotti-Robinson electromagnetic universe with cosmological constant.

4. What about the space-time manifold ?

Let us recall that the existence of the radiation connection is linked to the property of the orthogonal frames bundle to be reducible to the G_0 -bundle of the adapted frames to the

ishyps. In terms of holonomy this corresponds to the situation for which at each point of the orthogonal frames bundle the holonomy group of the Levi-Civita connection is reducible to $G_0 \simeq SO(2) \otimes SO(1,1)$. In fact in such a case the decomposition theorem of de Rham type for simply connected pseudo-Riemannian manifolds can be applied [11] and as noted in [12], $SO(2) \otimes SO(1,1)$ as holonomy group at each point of the space-time manifold $V_{3,1}$ gives rise to a pair of invariant two-dimensional spaces: one time-like and one space-like and hence to two two-dimensional smooth integrable distributions on $V_{3,1}$. Then according to the above-mentioned reduction theorem, $V_{3,1}$ is isometric to the direct product of the maximal integral manifolds, namely $V_{2,0} \times V_{1,1}$ of the parallel distributions. So the space-time metric splits into two global, smooth, covariantly constant, second-order symmetric tensor, as follows $g = Ag_{2,0} + Bg_{1,1}$ where A and B are arbitrary constants [12], $V_{2,0}$ is a Riemannian manifold and $V_{1,1}$ a Lorentzian one in accordance to the prescribed holonomy type, which is also compatible with Proposition II. Let us note that the Ricci tensor is of algebraic type $[2S - 2T]_{(2)}$ in Plebanski notation or $\{(1,1)(11)\}$ in Segre notation, or their degeneracy.

As a final remark it is worth noticing that in the above described space-time the isotropic hypersurface Σ can be viewed as the direct product $\Sigma = V_{2,0} \times \gamma$, where γ is any isotropic curve in $V_{1,1}$.

REFERENCES

- [1] G. Burdet and M. Perrin, *Radiation Structures on Isotropic Hypersurfaces*, Preprint CPT-90/P.2479.
- [2] G. Burdet and M. Perrin, *Gravitational Waves without Gravitons*, Lett. in Math. Physics **25** (1992) 39.
- [3] M. Cahen, R. Debever, L. Defrise, *A Complex Vectorial Formalism in General Relativity*, Jour. of Math. and Mechanics **16** (1967) 7.
- [4] R. Debever, R.G. McLenaghan, *Orthogonal Transitivity and Null Geodesic Separability in Type D Electrovac Solutions of Einstein's Field Equations with Cosmological Constant*, Jour. of Math. Physics **22** (1981) 1711-1726.
- [5] E. Newman and R. Penrose, *An Approach to Gravitational Radiation by a Method of Spin Coefficients*, Jour. of Math. Physics **3** (1962) 3.
- [6] J. Plebanski, S. Hacyan, *Some Exceptional Electrovac Type D Metrics with Cosmological Constant*, J. Math. Phys. **20** (1979) 1004-1010.
- [7] I. Robinson, Bull. Acad. Pal. Sci. Math. Astr. Physics **7** (1959) 351.
- [8] B. Bertotti, Phys. Rev. **116** (1959) 1331.
- [9] B. Carter, Comm. Math. Phys. **10** (1968) 280.
- [10] D. Kramer, H. Stephani, E. Herlt, M. Maccallum, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, (1980) p.102.
- [11] H. Wu, Illinois Jour. Math. **8** (1964) 291-311.
- [12] G.S. Hall, *General Relativity and Gravitation*, **20** (1988) 399-406.