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Quantum tunneling for the Sine-Gordon potential: Energy band structure and Bogomolny-Fateyev relation

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Abstract

A path-integral calculation of the energy band structure for the sine-Gordon potential with the help of the instanton method is presented. The periodic pseudoparticle configuration obtained recently is seen to be responsible for tunneling at a finite energy which leads to the level splitting of excited states. The lowest band structure due to the tunneling of vacuum instantons is recovered from our result in the low energy limit. The above result is obtained by considering a half period of the classical solution as a kink-like configuration (with nontrivial topological charge). On the other hand if a full period of the solution is treated as a nontopological (bounce-like) pseudoparticle configuration, the imaginary parts of energy eigenvalues can be calculated. We then obtain the Bogomolny-Fateyev relation which is well known in the perturbation theory at large order.

1. Introduction

The sine–Gordon (SG) Potential has attracted considerable attention both in field theory and in condensed–matter physics. The classical SG equation has been extensively studied and the soliton obtained from it satisfying the vacuum boundary condition is perhaps the bestknown in the literature. It is well known that Coleman¹⁾ established the quantum theory by relating the SG field to the massive Thirring model.

The purpose of the present paper is to present an explicit calculation of quantum tunneling phenomena dominated by so called nonvacuum instantons at finite energy. The new pseudoparticle configurations^{2,3} characterized by nonzero energy satisfy manifestly nonvacuum boundary conditions. We do not repeat the tunneling calculation with vacuum instantons since this is just a simple matter of following the same procedure as for the calculation of the level splitting for the double–well potential⁴⁾ or the imaginary part of the energy with bounces for the inverted double–well potential⁵⁾.

In a recent investigation⁶⁾ of the rate of instanton–like transitions at high energies a solution to the classical field equation in the massless (1+1) dimensional O(3) sigma model has been found. The model has been reduced to a mechanical tunneling problem for the SG potential. Therefore the study of tunneling phenomena especially with nonvacuum instantons for the SG potential is of considerable interest in the context of efforts towards understanding the subject of baryon– and lepton–number violation at high energy⁷⁾.

2. Nonvacuum instantons and their small fluctuations

Field theoretic models with solitonic (topologically nontrivial) finite Euclidean action have been discussed for many years. More recently Manton and Samols²⁾ have discovered a new type of classical configuration of (1+1) dimensional ϕ^4 field theory by compacting the space to a circle. We then extended the case of the ϕ^4 soliton on a circle to that of the ϕ^4 –bounce and the SG–soliton³⁾ and showed that in each of the three cases the fluctuation equation is a Lamé equation. In the present paper the (1+1)–dimensional ϕ^4 SG–soliton on a circle is reduced to (1+0) dimensions for the calculations of tunneling phenomena.

The Lagrangian we consider is

$$\mathcal{L} = \frac{1}{2} \left[\frac{d\phi}{dt} \right]^2 - V(\phi) \quad (2.1a)$$

with the SG potential

$$V(\phi) = \frac{1}{g^2} (1 + \cos(g\phi)) \quad (2.1b)$$

where $g > 0$ denotes the coupling constant. The classical solution which extremizes the action is seen to satisfy the equation of motion

$$\frac{1}{2} \left[\frac{d\phi_c}{d\tau} \right]^2 - V(\phi_c) = -E_{cl} \quad (2.2)$$

where we have gone to the Euclidean time $\tau = it$. We take the mass $m_0 = 1$ and natural units $c = \hbar = 1$ throughout. Equation (2.2) can be regarded as the equation of motion of a pseudoparticle with the classical energy $E_{cl} \geq 0$ which is a constant of integration. Letting E_{cl} be confined to a region $0 \leq E_{cl} \leq \frac{2}{g^2}$ the configuration $\phi_c(\tau)$ becomes periodic such that $\phi_c(\tau + \mathcal{T}) = \phi_c(\tau)$, which now corresponds to the periodic boundary condition in the space coordinate of Ref. [3]. The classical solution is³⁾

$$\phi_c = \frac{2}{g} \sin^{-1} [k \operatorname{sn}(\tau + \tau_0)] \quad (2.3)$$

where sn , cn , dn ... are Jacobian elliptic functions with modulus $k = \sqrt{1 - \frac{g^2 E}{2}}$ and τ_0 denotes the position of the pseudoparticle. The elliptic function $\operatorname{sn}(\tau)$ has a period $4\mathcal{K}(k)$. Then there exist critical values of $\mathcal{T}$ such that

$$\mathcal{T} = 4n\mathcal{K}(k) \quad (2.4)$$

where $n = 1, 2, 3, \dots$ are positive integers and $\mathcal{K}(k)$ is the complete elliptic integral of the first kind and so a quarter of period of $\operatorname{sn}(\tau)$. We refer to Ref. [2] for a discussion of the

significance of these critical values with regard to the behaviour of the classical solution for (1+1) dimensional ϕ^4 field theory. In our case we will see later that n is the number of instanton and anti-instanton pairs or the number of bounces. The pseudoparticle oscillates between two turning points in one of the barriers.

For zero energy $E_{cl} = 0$ ($k \rightarrow 1$) the periodic solution (2.3) reduces to the vacuum instanton configuration

$$\phi_c \xrightarrow{k \rightarrow 1} \frac{2}{g} \sin^{-1} (\tanh(\tau - \tau_0)) \quad (2.5)$$

The small fluctuation equation about the classical solution is a Lamé equation³⁾

$$\hat{M} \psi_m = E_m \psi_m \quad (2.6)$$

with operator

$$\begin{aligned} \hat{M} &= -\frac{d^2}{d\tau^2} - \cos [g\phi_c(\tau)] \\ &= -\frac{d^2}{d\tau^2} - [1 - 2k^2 \text{sn}^2(\tau)] \end{aligned} \quad (2.7)$$

The discrete eigenfunctions are³⁾ $\psi_1 = \text{cn}(\tau)$, $\psi_2 = \text{dn}(\tau)$, $\psi_3 = \text{sn}(\tau)$ with eigenvalues $E_1 = 0$, $E_2 = k^2 - 1$, $E_3 = k^2$ respectively. It is interesting to see the existence of a negative eigenvalue which would result in the imaginary part of energy by the standard instanton method⁸⁾. This reflects the Bogomoly and Fateyev observation⁹⁾ that for the large order behaviour of a perturbation series, the Green's functions have the imaginary parts at physical values of the coupling constant which are connected with tunneling to the other vacuum states.

3. Energy band structure of the SG potential and the transition amplitude for quantum tunneling

The SG potential has an infinite number of degenerate vacua. Quantum tunneling between neighbouring vacua leads to the level splitting while the levels extend to bands due to the translation symmetry expressed by $V\left[\phi + \frac{n\pi}{g}\right] = V(\phi)$. In the narrow-band approximation one finds the energy

$$E_i = \mathcal{E}_i + \sum_n J(R_m - R_n) e^{i\kappa(R_n - R_m)} \quad (3.1)$$

where $\mathcal{E}_i$ denotes the i th eigenvalue of the energy in each well for the harmonic oscillator potential $U = \frac{1}{2}(\phi - R_m)^2$ and $R_m = \frac{m\pi}{g}$ is the position of the m th minimum. In eq. (3.1)

$$J(R_m - R_n) \equiv \int \psi_i^*(\phi - R_m) [V(\phi) - U(\phi)] \psi_i(\phi - R_n) d\phi \quad (3.2)$$

is the overlap integral. ψ_i is the eigenfunction corresponding to eigenvalue $\mathcal{E}_i$. If only the contribution from the nearest neighbours is taken into account, we obtain the energy band formula

$$E = \mathcal{E}_i + 2J \cos \kappa \frac{\pi}{g} \quad (3.3)$$

The parameter J is just the level splitting resulting from quantum tunneling. To see this let us suppose $|i>_+$, $|i>_-$ are degenerate eigenstates in neighbouring wells (see Fig. 1) respectively with the same energy eigenvalue $\mathcal{E}_i$ such that $H^0|i>_{\pm} = \mathcal{E}_i|i>_{\pm}$ where H^0 is the Hamiltonian of the harmonic oscillator as the zero order approximation of our system. The degeneracy will be removed by the small tunneling effect which leads to the level splitting. The eigenstates of the Hamiltonian H become

$$|i>_0 = \frac{1}{\sqrt{2}}(|i>_+ - |i>_-), \quad |i>_e = \frac{1}{\sqrt{2}}(|i>_+ + |i>_-) \quad (3.4)$$

with eigenvalues $\mathcal{E}_i + \Delta\mathcal{E}$ and $\mathcal{E}_i - \Delta\mathcal{E}$ respectively. $\Delta\mathcal{E}$ denotes the level shift. It is obvious that

$$_+ \langle i | (\hat{H} - \hat{H}^0) | i \rangle = J = \Delta\mathcal{E} \quad (3.5)$$

In the following we calculate this energy shift $\Delta\mathcal{E}$.

The transition amplitude from the left hand well to the right at the energy E due to the instanton tunneling is seen to be

$$A_{+,-} = _+ \langle E | e^{-2\hat{H}T} | E \rangle_- = e^{-2ET} \sinh(2\Delta\mathcal{E}T) \quad (3.6)$$

which can be evaluated with the help of the path–integral method. Then

$$A_{+,-} = \int \psi_{E,+}^*(\phi_f) \psi_{E,-}(\phi_i) K(\phi_f, \tau_f; \phi_i, \tau_i) d\phi_f d\phi_i \quad (3.7)$$

where the Feynman kernel is defined as usual by

$$K(\phi_f, \tau_f; \phi_i, \tau_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\{\phi\} e^{-S} \quad (3.8)$$

with $\phi_f \equiv \phi(\tau_f)$, $\phi_i \equiv \phi(\tau_i)$ and $\tau_f - \tau_i = 2T$. What we are interested in is the limit $\phi_i \rightarrow -a$, $\phi_f \rightarrow a$ ($\pm a$ are the turning points, see Fig. 1), namely the tunneling propagator through one of the barriers. In eq. (3.8)

$$S = \int_{\tau_i}^{\tau_f} \left[\frac{1}{2} \left(\frac{d\phi}{d\tau} \right)^2 + V(\phi) \right] d\tau$$

is the Euclidean action of the pseudoparticle and $\psi_{E,+}$ and $\psi_{E,-}$ are wave functions in the barrier defined from the right and left hand wells respectively.

4. Evaluation of the transition amplitude

(1) Instanton or bounce?

In the vacuum case the instanton is considered as a stable classical configuration (kink) with nontrivial topological charge while the bounce^{5,8)} is considered to be an unstable nontopological configuration. It has been shown that all the (1+1) configurations on a circle with periodic boundary condition (periodic motion with a finite period in our 1+0 dimensional case) are unstable in view of the existence of negative eigenmodes of the small fluctuation equations^{2,3)}. Instantons and bounces are now only distinguished by the topological charge. We therefore regard the motion through half a period as the trajectory of an instanton while the motion through a full period is considered as the trajectory of a bounce or that of an instanton–antiinstanton pair. Since we consider the tunneling from one well to another, only the instanton–like solution is suitable. However, the problem is the negative eigenmode which should not contribute to the tunneling calculation so that the transition amplitude is real. It will be seen, how this difficulty is resolved.

(2) Removal of the negative eigenmode and the one instanton contribution

As in the standard path–integral calculation we expand $\phi(\tau)$ about the classical trajectory $\phi_c(\tau)$ and thus set

$$\phi(\tau) = \phi_c(\tau) + \chi(\tau) \tag{4.1}$$

where $\chi(\tau)$ is the deviation of $\phi(\tau)$ from $\phi_c(\tau)$ with fixed endpoints. The boundary condition for $\chi(\tau)$ is (with $\tau_i \rightarrow -\mathcal{K}$, $\tau_f \rightarrow +\mathcal{K}$)

$$\chi(\tau_i) = \chi(\tau_f) = 0 \tag{4.2}$$

Substitution of $\phi(\tau)$ of (4.1) into eq. (3.8) and keeping only terms containing χ up to the second order for the one loop approximation yields

$$\begin{aligned}
 K(\phi_f, \tau_f; \phi_i, \tau_i) &= \text{exp} [-S_c(\phi_f, \phi_i; \tau_f, \tau_i)] \int_{\chi(\tau_i)=0}^{\chi(\tau_f)=0} \mathcal{D}\{\chi\} \text{exp} [-\delta s] \\
 &\equiv \text{exp} [-S_c] I
 \end{aligned} \tag{4.3}$$

where the classical action S_c is evaluated along the trajectory ϕ_c so that

$$\begin{aligned}
 S_c &= \int_{\tau_i}^{\tau_f} \left[\frac{1}{2} \left[\frac{d\phi_c}{d\tau} \right]^2 + V(\phi_c) \right] d\tau \\
 &= \lim_{\substack{\phi_i \rightarrow -a \\ \phi_f \rightarrow a}} \frac{8}{g^2} [E(k) - k'^2 \mathcal{K}(k)] + 2E_{cl} T
 \end{aligned} \tag{4.4}$$

$E(k)$ denotes the complete elliptic integral of the second kind and $k' = \sqrt{1-k^2}$. The limit of endpoints tending to the turning points will be taken only after all the integrations have been carried out so that the singularity at the turning points is avoided.

The fluctuation action is seen to be

$$\delta S_c = \int_{\tau_i}^{\tau_f} \chi \hat{M} \chi d\tau \tag{4.5}$$

Expanding $\chi(\tau)$ in terms of the eigenmodes of $\hat{M}$, the eigenfunctions of the small fluctuation equation (2.6), i.e. setting

$$\chi(\tau) = \sum_m C_m \psi_m \tag{4.6}$$

and changing the integration variables to $\{C_m\}$, the functional integral I can be formally evaluated to be

$$I = \left| \frac{\partial \chi(\tau)}{\partial C_m} \right| \prod_m \left[\frac{\pi}{E_m} \right]^{\frac{1}{2}} \quad (4.7)$$

if all of the C_m are independent variables. In our case, however, the boundary condition (4.2) leads to constraint equations on $\{C_m\}$, that is

$$C_1 \psi_1(\pm \mathcal{K}) + C_2 \psi_2(\pm \mathcal{K}) + C_3 \psi_3(\pm \mathcal{K}) = 0$$

and so to

$$C_2 k' + C_3 = 0, \quad C_2 k' - C_3 = 0 \quad (4.8)$$

since $\text{cn } \mathcal{K} = 0$, $\text{dn } \mathcal{K} = k'$, $\text{sn } \mathcal{K} = 1$.

The only solution is $C_2 = C_3 = 0$ and hence the negative eigenvalue $E_2 = k^2 - 1$ associated with ψ_2 does not contribute to the functional integral I . The transition amplitude as well as the level splitting will therefore be real as expected.

For our present purposes we employ an alternative procedure to evaluate the functional integral I . We introduce the transformation

$$\chi(\tau) = y(\tau) + N(\tau) \int_{\tau_i}^{\tau} \frac{N(\tau')}{N^2(\tau')} y(\tau') d\tau' \quad (4.9)$$

where

$$N(\tau) = \frac{d\phi_c}{d\tau} = \frac{2}{g} k \text{cn}(\tau) \quad (4.10)$$

is the zero eigenmode of the small fluctuation equation. The boundary conditions for $y(\tau)$ obtained from those of $\chi(\tau)$ are $y(\tau_i) = 0$ and

$$y(\tau_f) + N(\tau_f) \int_{\tau_i}^{\tau_f} \frac{N(\tau')}{N^2(\tau')} y(\tau') d\tau' \equiv y(\tau_f) + f(\tau_f) = 0 \quad (4.11)$$

With the transformation (4.9) the potential term in eq. (4.5) is eliminated and the functional integral I of eq. (4.3) becomes that of a free particle propagator but with the constraint (4.11). Inserting into I the identity

$$\int dy_f \delta(y(\tau_f) + f(\tau_f)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy_f e^{-i\alpha(y(\tau_f) + f(\tau_f))} d\alpha = 1 \quad (4.12)$$

We have

$$\begin{aligned} I &= \frac{1}{2\pi} \left| \frac{\partial x}{\partial y} \right| \int dy_f \int_{y_i=0}^{y_f} \mathcal{P}\{y\} \exp \left[-\frac{1}{2} \int_{\tau_i}^{\tau_f} \left[\frac{dy}{d\tau} + i\alpha \frac{N(\tau_f)}{N(\tau)} \right]^2 d\tau \right] \\ &\quad \cdot \exp \left[-\alpha^2 \frac{N^2(\tau_f)}{2} \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2(\tau)} \right] d\alpha \\ &= \frac{1}{\sqrt{2\pi}} \left| \frac{\partial x}{\partial y} \right| \left[N^2(\tau_f) \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2(\tau)} \right]^{-\frac{1}{2}} \end{aligned} \quad (4.13)$$

The Jacobian of transformation (4.9) is¹⁰⁾

$$\left| \frac{\partial x}{\partial y} \right| = \left[\frac{N(\tau_f)}{N(\tau_i)} \right]^{\frac{1}{2}} \quad (4.14)$$

so that the final result of the functional integration is

$$I = \frac{1}{\sqrt{2\pi}} \left[N(\tau_i) N(\tau_f) \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2(\tau)} \right]^{-\frac{1}{2}} \quad (4.15)$$

The next step in our calculation of the transition amplitude (3.7) is to find the proper wave functions ψ_{E+} and ψ_{E-} , and to carry out the endpoint integration. In the barrier the wave function is, of course, of the WKB form, that is

$$\psi_{E,-}(\phi_i) = \frac{C e^{-\int_{-a}^{\phi_i} N(\phi) d\phi}}{\sqrt{N(\phi_i)}}$$

$$\psi_{E,+}(\phi_f) = \frac{C e^{-\int_{\phi_f}^a N(\phi) d\phi}}{\sqrt{N(\phi_f)}} \equiv \frac{C e^{-\Omega(\phi_f)}}{\sqrt{N(\phi_f)}} \quad (4.16)$$

The normalization constant C is defined and evaluated as

$$C = \left[\frac{\frac{1}{2}}{\int_a^{a'} \frac{d\phi}{\sqrt{2(E-V)}}} \right]^{\frac{1}{2}} = \left[\frac{1}{4\kappa(k')} \right]^{\frac{1}{2}} \quad (4.17)$$

We expand $S_c(\phi_f, \phi_i, T)$ and $\Omega(\phi_f)$ as power series of $[\phi_f - \phi(T)]$ with $\phi(\tau) \rightarrow a$ up to the second order for the Gaussian approximation. Thus

$$S_c = S_c(\phi(T), \phi(-T), T) + \frac{1}{2} \frac{\partial^2 S}{\partial \phi_i^2} \Big|_{\phi_f = \phi(T)} (\phi_f - \phi(T))^2 + \dots \quad (4.18)$$

and for Ω of (4.16)

$$\Omega(\phi_f) = \frac{1}{2} \frac{\partial^2 \Omega}{\partial \phi_i^2} (\phi_f - \phi(T))^2 + \dots$$

Substituting eq. (4.15) – (4.18) into eq. (3.7), the endpoint–integration $d\phi_i$ along with $N(\phi_i)$ in the denominator becomes a Euclidean time integration and the integration of $d\phi_f$ is seen to become a Gaussian integration. The transition amplitude then is

$$A_{+,-} = \frac{2T}{4\kappa(k')} e^{-W} e^{-2E_{cl}T} \quad (4.19)$$

where we have used a formula (see Appendix of Ref. [11])

$$\frac{\partial^2 S_c}{\partial \phi_f^2} = \frac{1}{N(\tau_f)} \left[\dot{N}(\tau_f) + \frac{1}{N(\tau_f) \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2}} \right] \quad (4.20)$$

and

$$\frac{\partial^2 \Omega(\phi_f)}{\partial \phi_f^2} = - \frac{\dot{N}(\phi_f)}{N(\phi_f)} \quad (4.21)$$

which follows from the definition in eq. (4.16).

(3) Sum over contributions from instantons plus instanton–antiinstanton pairs

The configuration of one instanton plus one instanton–antiinstanton pair is given by our solution with $n = 3$ in eq. (2.4) and $T = 3K$. The trajectory for the one particle and one pair is shown in Fig. 1(b). The transition amplitude is seen to be

$$A_{+,-}^{(3)} = \int_{-T}^T d\tau_1 \int_{-T}^T d\tau_2 \int_{-T}^T d\tau \left[\frac{1}{4\mathcal{K}(k')} \right]^3 e^{-3W} e^{-2E_{cl}T} \quad (4.22)$$

The total amplitude is given by the sum

$$A_{+,-} = \sum_{m=1}^{\infty} A_{+,-}^{(2m+1)} = e^{-2E_{cl}T} \sinh \left\{ \frac{T}{2\mathcal{K}(k')} e^{-W} \right\} \quad (4.23)$$

with

$$W = \frac{8}{g^2} [E(k) - k'^2 \mathcal{K}(k)] \quad (4.24)$$

Comparing (4.23) and (3.6) the level shift is given by

$$\Delta \mathcal{E} = \frac{1}{4\mathcal{K}(k')} e^{-W} = J \quad (4.25)$$

5. Low and high energy limits

For the energy far below the barrier height $E_{cl} \ll \frac{2}{g^2}$ i.e. $k^2 \rightarrow 1$, $k'^2 \rightarrow 0$ we expand the complete elliptic integrals $E(k)$ and $\mathcal{K}(k)$ as power series of k' . Up to the second order of k' we find

$$\begin{aligned} W &= \frac{8}{g^2} \left[1 - \frac{k'^2}{2} \left\{ \ln \left[\frac{4}{k'} \right] + \frac{1}{2} \right\} + O(k'^4) \right] \\ &= \frac{8}{g^2} - (n + \frac{1}{2}) \ln \left[\frac{2^5}{(n + \frac{1}{2})g^2} \right]^{n + \frac{1}{2}} - (n + \frac{1}{2}) \end{aligned} \quad (5.1)$$

so that (with $\mathcal{K}(k'=0) = \pi/2$)

$$J = \Delta\mathcal{E} = \frac{1}{2\pi} e^{-\frac{8}{g^2}} e^{n + \frac{1}{2}} \left[\frac{2^5}{(n + \frac{1}{2})g^2} \right]^{n + \frac{1}{2}} \quad (5.2)$$

Here we have used the quantization $E_{cl} = n + \frac{1}{2}$, which follows, for instance, from the oscillator approximation of the periodic potential around one of its minima and $k'^2 \simeq \frac{1}{2} g^2 E_{cl}$. For weak coupling ($g \ll 1$) and also $(n + \frac{1}{2})g^2 \ll 1$ the transition amplitude as well as the band width J indeed grow with energy exponentially. With the help of Stirling's formula in the form $n! \simeq \sqrt{2\pi} \left[\frac{n + \frac{1}{2}}{e} \right]^{n + \frac{1}{2}}$ one has

$$J = \frac{1}{\sqrt{2\pi}} e^{-\frac{8}{g^2}} \frac{1}{n!} \left[\frac{2^5}{g^2} \right]^{n + \frac{1}{2}} \quad (5.3)$$

which is in agreement with the result of Ref. (12).

It is more interesting to see the high energy limit which is related to the instanton transition at high energy in the investigation of baryon- and lepton-number violation. When the energy E_{cl} tends the top of barrier, i.e. $E_{cl} \rightarrow \frac{2}{g^2}$ and therefore $k^2 = \frac{2}{g^2} - E_{cl} \rightarrow 0$. The complete elliptic integrals now have to be expanded in rising powers of k . Then with

$$\begin{aligned} E(k) &= \frac{\pi}{2} \left[1 - \frac{1}{4} k^2 + O(k^4) \right] \\ \mathcal{K}(k) &= \frac{\pi}{2} \left[1 + \frac{1}{4} k^2 + O(k^4) \right] \end{aligned}$$

we obtain

$$W \xrightarrow[k \rightarrow 0]{} \frac{2\pi}{g^2} k^2 \rightarrow 0 \quad (5.4)$$

Thus in this limit the typical suppression factor of the transition amplitude (which follows already from vacuum instantons) disappears. However the prefactor $\frac{1}{4\mathcal{K}(k')}$ which is approximately a constant in the low energy case becomes energy dependent. This reflects the anharmonic nature of the system.

$$\frac{1}{4\mathcal{K}(k')} = \frac{1}{4\mathcal{K}\sqrt{1-k^2}} \xrightarrow[k \rightarrow 0]{} \frac{1}{4\ln(4/k)} \rightarrow 0 \quad (5.5)$$

The transition amplitude is again suppressed but this time by the prefactor. This is not surprising since the prefactor is proportional to the effective frequency of the real particle in one well. At high energies the number of impacts per unit time at the turning points approaches zero.

6. The Bogomolny–Fateyev relation

For the system under discussion the true vacuum state (or physical vacuum state) is stable. Only the tunneling of pseudoparticles leads to the level splitting. However, if one treats the one well as a perturbation theory vacuum, the tunneling away from the well would result in an imaginary part of energy. This phenomenon was observed long ago by Bogomoly and Fateyev in connection with the large order behaviour of perturbation theory⁹⁾.

To see how this can happen let us regard the full period of the above solution eq. (2.3) as the trajectory of a bounce with position $\tau_0 = \mathcal{K}(k)$ as shown in Fig. 1(b). Let us calculate the amplitude for the transition from one well and back to the same well due to the pseudo-particle propagator

$$A = \langle E | e^{-4HT} | E \rangle = e^{-4ET} \quad (6.1)$$

We choose $4T$ here in order to keep the same time scale as for the instanton case. The energy E then becomes a complex number as a result of the tunneling. The transition amplitude A can again be calculated with the help of the path–integral method. We have

$$A = \int \psi_E^*(\phi_f) \psi_E(\phi_i) K(\phi_f, \tau_f; \phi_i, \tau_i) d\phi_f d\phi_i \quad (6.2)$$

We are interested in the limit $\phi_f = \phi_i \rightarrow a$ (where ‘a’ is a turning point), and $\tau_f - \tau_i = 4T$ denotes a full period of motion of the bounce. With a procedure similar to that used before we have the propagator

$$K(\phi_f, \tau_f; \phi_i, \tau_i) = e^{-S_c(\phi_f, \phi_i, 4T)} I \quad (6.3)$$

where S_c is the classical action evaluated along the trajectory of the bounce ϕ_c , i.e.

$$S_c = \int_{\tau_i}^{\tau_f} \left[\frac{1}{2} \left[\frac{d\phi_c}{d\tau} \right]^2 + V(\phi_c) \right] d\tau$$

$$\xrightarrow{\phi_f = \phi_i \rightarrow a} 2W + 4E_{cl}T \quad (6.4)$$

I is again the functional integral obtained by taking the fluctuation trajectories $\chi(\tau)$ into account. The boundary condition for $\chi(\tau)$ is now

$$\chi(-2T) = \chi(2T) = 0 \quad (6.8)$$

which leads to a constraint equation on the expansion coefficients of $\chi(\tau)$ with respect to the eigenfunctions of the small fluctuation equation (2.6), i.e.

$$C_2 k' - C_3 = 0 \quad (6.6)$$

Thus C_2 and C_3 are no longer independent integration variables which result in two Gaussian integrations. The effective Gaussian integration for C_2 and C_3 becomes

$$\begin{aligned} & \int dC_2 \int dC_3 e^{-E_2 C_2^2} e^{-E_3 C_2^2} \delta(C_3 - C_2 k') \\ &= \int dC_2 e^{-C_2^2 (E_2 + k'^2 E_3)} = \int dC_2 e^{-C_2^2 k'^4} \end{aligned} \quad (6.7)$$

By rotating to the imaginary axis the integration gives rise to an imaginary value as expected. Using the same method as in the previous section for the instanton the functional integration is seen to give the same formula as eq. (4.15). The WKB wave functions are now defined only in the left hand well so that (with $N(\tau) = \frac{d\phi_c}{d\tau}$)

$$\psi_E(\phi_i) = \frac{C e^{-\int_{-a}^{\phi_i} \phi d\tau}}{\sqrt{N(\phi_i)}}, \quad \psi_E(\phi_f) = \frac{C e^{-\int_{-a}^{\phi_f} \phi d\tau}}{\sqrt{N(\phi_f)}} \quad (6.8)$$

Evaluating the endpoint-integration and summing over the infinite number of bounces we obtain

$$A = e^{-4E_{cl}T} \exp \left[-i \frac{4T}{4\mathcal{K}(k')} e^{-2W} \right] \quad (6.9)$$

where (cf. [11]) the factor $(-i)$ comes from the limit of the factor Δ of the Gaussian integration $d\phi_f$ i.e.

$$\begin{aligned} \Delta &= \lim_{\pm 2T \rightarrow \pm 2\mathcal{K}} \frac{1}{\left[1 + N(2T) N(2T) \int_{-2T}^{2T} \frac{d\tau}{N^2} \right]^{\frac{1}{2}}} \\ &= \frac{1}{[1-2]^{\frac{1}{2}}} = -i \end{aligned} \quad (6.10)$$

where $N(\tau) = \frac{d\phi_c}{d\tau} = \frac{2}{g} \kappa \text{cn}(\tau + \tau_0)$ with $\tau_0 = \kappa(k)$

in the case of the bounce. Comparing eq. (6.1) and eq. (6.9) the imaginary part of the energy is seen to be

$$\text{Im } E = \frac{1}{4\kappa(k')} e^{-2W} \quad (6.11)$$

We now come to the interesting Bogomolny–Fateyev relation between the level splitting eq. (4.12) and the imaginary part of the energy given by (6.11), i.e.

$$\text{Im } E = 4\kappa(k') (\Delta\mathcal{E})^2 \quad (6.12)$$

For the low energy case ($k' \rightarrow 0$)

$$\text{Im } E = 2\pi(\Delta\mathcal{E})^2 \quad (6.13)$$

while

$$\Delta E = 4\pi i (\Delta\mathcal{E})^2 \quad (6.14)$$

where $\Delta E = 2i \text{Im } E$ denotes the discontinuity of the eigenenergy (low energy case) at the cut $g^2 \geq 0$. We would like to emphasize that the imaginary part of the energy, i.e. eq. (6.11), is obtained in a natural way by regarding the full period of motion as a bounce configuration. The imaginary part of the ground state energy comes as a limit of low energy. In the original paper by Bogomolny and Fateyev⁹⁾ only the ground state is discussed. It is observed that there are some difficulties in calculating the Taylor expansion of the dispersion integral, since there are no finite action solutions having equal asymptotics at $\kappa \rightarrow \pm \infty$ (but there are stable instanton–type solutions). In other words in our 1+0 dimensional case, when $T \rightarrow \pm \infty$ ($k \rightarrow 1$) the classical configuration (2.3) reduces to the kink solution (2.5). If one starts from the instanton (2.5) it would be impossible to obtain the imaginary part of the energy for the ground state. To overcome the difficulties a special technique was employed in Ref. [9] to obtain the imaginary part of the ground state energy.

The Bogomolny–Fateyev relation (6.13), (6.14) has been reinvestigated recently and discussed for physical systems with tunneling away from the perturbative minimum in Ref. [12].

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References

- 1) S. Coleman, Phys. Rev. D11, 2088 (1975).
- 2) N.S. Manton and T.S. Samols, Phys. Lett. B207, 179 (1988).
- 3) J.–Q. Liang, H.J.W. Müller–Kirsten, and D.–H. Tchrakian, Phys. Lett. B282, 105 (1992).
- 4) E. Gildener and A. Patrascioiu, Phys. Rev. D16, 423 (1977).
- 5) J.–Q. Liang and H.J.W. Müller–Kirsten, Phys. Rev. D48, 2963 (1992); Erratum Phys. Rev. D48 964 (1993).
- 6) V.A. Rubakov, D.T. Son and P.G. Tinyakov, Nuclear Phys. B404, 65 (1993).
- 7) A. Ringwald, Nucl. Phys. B330, 1(1989); also O. Espinosa, *ibid.* B343, 310 (1990).
- 8) S. Coleman, Phys. Rev. D15, 2929 (1977); C. Callan, Jr. and S. Coleman, *ibid.* 16, 1962 (1977).
- 9) E.B. Bogomoly and V.A. Fateyev, Phys. Lett. B71, 93 (1977).
- 10) See, for example. W. Dittrich and M.Reuter, "Classical and Quantum Dynamics from Classical Paths to Path Integrals" (Springer–Verlag 1992).
- 11) J.–Q. Liang and H.J.W. Müller–Kirsten, Nonvacuum bounces and quantum tunneling at finite energy, Kaiserslautern preprint KL–TH–93/18.
- 12) P. Achuthan, H.J.W. Müller–Kirsten, and A. Wiedemann, Fortschr. Phys. 38, 77 (1990).

Figure Caption

Fig. 1

- (a) Trajectories of the nonvacuum instanton (half of period) and a bounce (full period)
- (b) The case $n = 3$: one instanton plus a pair

Fig. 1