SU(3) String-Flip Potential Models and Nuclear Matter

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December 22, 1993

Abstract

A Monte Carlo model for nuclear matter using a many body SUc(3) string flip potential, with fixed colour, is investigated. The potential is approximated by considering colour singlet flux tube formations that connect only three quarks at a time. The model is compared with a similar string flip model, proposed by Horowitz and Piekarewicz [1], that approximates higher order flux tube formations by connecting quarks in colour singlet chains. The former model gives an EMC nucleon “swelling” effect, whereas the latter gives an opposite effect. Possible discrepancies between the two models are discussed.

1. Introduction

There is, even after some 50 years of work on the problem, no satisfactory explanation of the properties of nuclear matter in terms of constituent nucleons. In principle, if QCD is the correct theory, the various properties of nuclear matter should be calculable in a precise fashion from a knowledge of the interactions between quarks. However, the only rigorous method is lattice QCD, which is computationally so complex that it cannot handle more than a single hadron. Hence we are forced to construct phenomenological models which satisfy most of our beliefs about the interactions between quarks but are sufficiently simple that they can be solved in a finite time.

An ideal model would have the following properties: at low densities the quarks would condense out to form isolated baryons. At a higher density, the interaction between quarks would lead to positive binding energy between nucleons, and a swelling of nucleons. At still higher densities, it is expected that the hadrons will dissolve into a quark-gluon plasma. This last is in contrast to the traditional nucleon models, which require the forces to be carefully adjusted so that they saturate at infinite density, effectively implying a hard core. Some simple models which appear to be likely candidates are string-flip potential models [1, 2], and to some extent linked cluster expansion models [3].

The cluster models are based on one-gluon exchange potentials, and use an N-body harmonic oscillator potential, i.e.

\[ V_{\text{conf}} = \frac{1}{2} k \sum_{i<j} (r_i - r_j)^2 , \quad (1) \]

to mimic quark confinement. These models are mainly used for describing short range nuclear effects, as they suffer from van der Waals forces due to the nature of the confining potential. Despite this shortcoming, they do seem quite useful in explaining local effects such as nucleon
swelling (fat nucleons) [4], quark clustering preferences, and relative strengths of the various one-gluon exchange potentials [3].

The string-flip potential models are, on the other hand, motivated by lattice QCD. They attempt to guess at how flux tubes should form amongst the quarks at zero temperature. An adiabatic assumption is made, in which the quarks move slowly enough for their fields to reconfigure themselves, such that the overall potential energy is minimized: i.e.

\[ V = \min \{ \sum_{\{q_m \ldots q_n\}} v(r_m \ldots r_n) \mid \bigcup_{\{m\ldots n\}} \{q_m \ldots q_n\} = \{q_1 \ldots q_{N_q}\} \}, \quad (2) \]

where the \( N_q \) quarks are placed in a cube of side \( L \) and subjected to periodic boundary conditions, to simulate continuous quark matter. The sum is over all gauge invariant sets \( \{q_m \ldots q_n\} \) of quarks, such that at least one element from each set lies inside a common box, whose disjoint union, \( \bigcup \), makes up the complete colour singlet set \( \{q_1 \ldots q_{N_q}\} \) of \( N_q \) quarks. It is easy to see that this potential allows for complete minimal quark clustering separability at low densities without suffering from van der Waals forces. At present these models [1, 2] are quite crude in that they do not include short range one gluon exchange phenomena and spin effects, and are flavour degenerate. Despite this they do, in general, seem quite capable of getting most of the bulk nuclear properties correct, with the exception of nuclear binding.

It is known that the \( SU(2) \) string-flip potential models do show these properties, except for the positive binding energy which probably arises from short range forces. However, the only extension to an \( SU(3) \) model [1] leads to the rather surprising result that the nucleon appears to shrink in nuclear matter. It is therefore of some interest to repeat the calculation of [1], in an attempt to see whether the approximations made there alter the solution qualitatively.

In this paper we construct a string-flip potential model for 3-quark systems. Here some simplifying assumptions about flux tube minimization are made, in order to reduce the Monte Carlo computation time. We present results for a linear potential model, \( SU_l(3) \), and a harmonic oscillator potential model, \( SU_h(3) \), in which the colour has been fixed to a given quark. Our results are compared with an \( SU_h(3) \) model proposed by Horowitz and Piekarewicz [1] in which different simplifying assumptions, about the minimal flux tube topology, were made. We also compare our earlier \( SU(2) \) results [2] with theirs [1]. The paper finishes with a discussion on future directions to pursue in attempting to get bound state nuclear matter.

### 2. \( SU(3) \) String-Flip Model

The string-flip model involves solving a Hamiltonian system of fermions governed by the potential given in equation (2). To solve this system requires the use of variational Monte Carlo techniques [5]. In order to compute any observable in a finite amount of time further assumptions about the form of the potential must be made.

In this model the potential is restricted to summing over sets of colour singlet clusters of three quarks,

\[ V = \min \{ \sum_{\{q_{r,q_2,q_3}\}} v(r_r,r_{q_2},r_{q_3}) \mid \bigcup_{\{r,q_2,q_3\}} \{q_r,q_{q_2},q_{q_3}\} = \{q_1 \ldots q_{N_q}\} \}, \quad (3) \]

such that the colour of a given quark is fixed. The first assumption does have some validity as it has been shown, via a linked quark cluster model, that it is energetically more favourable for 6q systems to dissociate into two nucleons as a result of hyperfine interactions [3, 6, 7]. However, this is not necessarily the case at lower densities, as the linked cluster models are unreliable here. The second assumption, that of fixed colour, greatly reduces the number of degrees of
freedom, and therefore reduces the chance of finding an absolute minimum. At low densities this should not have any effect on the potential, as the system consists of isolated nucleons. Similarly at high densities no effect is expected, as the system consists of uncorrelated quarks. At intermediate densities some effects might be expected, particularly around any regions in which a phase transition might occur.

![Diagram](image)

**Figure 1:** Flux tube arrangements for the 3q cluster potentials
a) $v_f$ and b) $v_h$.

For the $SU_t(3)$ model, the potential, $V$, has the components [8]

$$v_f(r_r, r_g, r_b) = \sigma \begin{cases} 
 r_{br} + r_{rg} & \text{if } \angle brg \geq 120^\circ \\
 r_{rg} + r_{gb} & \text{if } \angle rgb \geq 120^\circ \\
 r_{gb} + r_{br} & \text{if } \angle gbr \geq 120^\circ 
\end{cases},$$

where $r_{ij} = r_i - r_j$, $\xi$ or

$$\xi_{r,gb} = \frac{1}{\sqrt{3}} \sqrt{r_{rg}^2 + r_{gb}^2 + r_{br}^2},$$

and

$$A = \frac{1}{4} \sqrt{(r_{rg} + r_{gb} + r_{br})(-r_{rg} + r_{gb} + r_{br})(r_{rg} - r_{gb} + r_{br})(r_{rg} + r_{gb} - r_{br})}$$

is the area inclosed by the triangle $\triangle r_{gb}$ (see figure (1.a)). For $SU_h(3)$ the components are

$$v_h(r_r, r_g, r_b) = \frac{1}{2} k \xi_{r,gb}^2$$

(see figure (1.b)). This potential was obtained by replacing the linear segments of $v_f$ by springs when the quarks, which were assumed to be of equal mass, formed a triangle with interior angles less than $120^\circ$ (see figure (5.a)). This analogue model is expected to have similar features to $SU_t(3)$ for s-wave $(qqq)$ states.

The sum in equation (3) can be reordered by restricting it to run over all sets, $\{r_{gb}\}_L$, of quark triplets contained in a central box, such that the potential $v(r_r, r_g, r_b)$ is minimized with
respect to all possible periodic permutations of the vectors \( \{ r_r, r_g, r_b \} \), with the constraint that at least one of the vectors lies inside the central box: i.e.

\[
V = \min \left\{ \sum_{\{r\}} v(r_r, r_g, r_b) \right\},
\]

where

\[
v(r_r, r_g, r_b) = \min \{ v(\mathbf{r} + \mathbf{k}, L, \mathbf{r} + k_y L, \mathbf{r} + k_b L) | k_q = -1, 0, 1 & \text{at least one } k_q = 0 \}. \quad (9)
\]

This means that for quark triplets a search of one box deep from the central box is required, giving a total of \( 27^2 \) possible permutations, in order to minimize a given \( v(r_r, r_g, r_b) \) (see figure (2)). These permutations can be reduced to 3 by requiring that at least two sides of the triangle \( \triangle rgb \), formed by a given permutation of quarks, be a minimum: i.e.

\[
v(r_r, r_g, r_b) = \min \{ v(\delta r_{rg}, \delta r_{gb}, \delta r_{rb}), v(r_{rg}, \delta r_{gb}, \delta r_{rb}), v(\delta r_{rg}, \delta r_{gb}, \delta r_{rb}) \}, \quad (10)
\]

where \( \delta r_{ij} \) is the minimum distance vector between the points \( \mathbf{r}_i \) and \( \mathbf{r}_j \), in a box of side \( L \) with periodic boundary conditions, which is given by

\[
(\delta r_{ij})_a = \begin{cases} 
(\mathbf{r}_i - \mathbf{r}_j)_a + L & \text{if } (\mathbf{r}_i - \mathbf{r}_j)_a < -L/2 \\
(\mathbf{r}_i - \mathbf{r}_j)_a & \text{if } |(\mathbf{r}_i - \mathbf{r}_j)_a| < L/2 \\
(\mathbf{r}_i - \mathbf{r}_j)_a - L & \text{if } (\mathbf{r}_i - \mathbf{r}_j)_a > L/2
\end{cases}, \quad (11)
\]

where \( a = x, y, z \). This is exact for \( SU_b(3) \); for \( SU_l(3) \), classical Monte Carlo shows that about 19% of the events deviate from the actual answer by \( \sim 0.3\% \), on average.

Figure 2: A 2-D slice showing a typical flux tube arrangement for three quarks, placed inside a central cube, subjected to periodic boundary conditions.
The number of different elements in the set, \( \{ \sum v \} \), from which the minimum must be extracted in order to get \( V \), defined in equation (8), is \((N_n!)^2\) (where \( N_n = N_q/3 \) is the number of nucleons in the central box). For \( N_n = 7 \), say, this would be 25401600 elements! These elements can be reduced by fragmenting the set \( \{ q_1, \ldots q_{N_q} \} \) into smaller pieces, or subclusters, such that each element can find \( N_{br} \) complementary coloured pairs of quarks that are “closest” to it. These subclusters can be further fragmented, by “softening” the requirement that at least \( N_{br} \) complementary pairs exist; i.e. by searching for disjoint subclusters. These subclusters are referred to as softened subclusters. The “closeness” of quark \( q_r \) to the complementary pair \( (g_b r_b) \) is defined by the function \( \delta_{r, (gb)} = v(r_r, r_g, r_b) \), given in equation (9). The fragmented sets are thus constructed by computing an \( N_{N_q} \times N_{N_q}^2 \) matrix \( (\Delta) \) with elements \( \delta_{r, (gb)} \), and then converting it into block diagonal form \((\Delta^d)\) increasing in size from top to bottom, by swapping rows and columns such that each block diagonal element contains the elements of a fragmented set and all the off block diagonal elements are set to zero. The elements of \( \{ \sum v \} \) are now constructed by extracting permutations of elements \( \delta_{r, (gb)} \), from unique columns and rows, of the block diagonal elements of \( \Delta^d \). Further computational speed is gained by throwing away sums that start to exceed the current minimum. In general the fragmented sets, constructed from \( \{ q_1, \ldots q_{N_q} \} \), are not all disjoint from one another, and therefore the block diagonal elements of \( \Delta^d \) may overlap. The degree of overlap turns out to increase with increasing density, causing the Monte Carlo to slow down.

This fragmentation procedure, or nearest neighbour search of depth \( N_{br} \) (c.f. [2]), reduces computation time quite significantly. The cost is that rare configurations with flux tubes that stretch across the box, or across unsoftened subclusters, that give a global minimum might be missed. Preliminary Monte Carlo shows that the inclusion of softened subclusters gives no noticeable change. However, for a full \((N_n!)^2\) brute force search, doing a Monte Carlo becomes virtually impossible. A few brute force computations of the potential were made, for particles randomly thrown into a box, which seem to suggest that the fragmentation procedure is good to about 1%, with \( N_{br} \approx 4 \). SU(2) models also give similar results[2].

The validity of the fragmentation procedure can also be argued on physical grounds, for it is reasonable to assume that long flux tube configurations would tend to dissociate into \( q\bar{q} \) pairs. Therefore the fragmentation procedure can be consider as a zeroth order \( q\bar{q} \) approximation.

The choice of variational wave function should attempt to reflect the overall bulk properties of the system. Here the wave function was chosen to be of the form,

\[
\Psi(\alpha, \beta, \rho) = e^{-\frac{1}{2} \sum (\beta \xi_{r,gb})^2} \prod_{c \in \{rgb\}} |\Phi_S(\rho)|, \tag{12}
\]

where \( \alpha, \beta, \) and \( \rho (= N_n/L^3 \text{ s.t. } N_n = N_q/3) \) are variational parameters, \( \sum_{\{rgb\}} \) is over the set of quarks \( \{rgb\} \) which gives the minimal potential \( V \), and \( |\Phi_S(\rho)| \) is a Slater determinant with elements \( \phi_{ij} = \phi_i(r_j) \), which is composed of the plane wave states

\[
\phi_i(r_j) = \sin(\frac{2\pi}{L} n_i \cdot r_j + \delta_i), \tag{13}
\]

where \( \delta_i = 0 \) or \( \pi/2 \), and \((n_i)_a = 0, \pm 1, \pm 2, \ldots \) are the components of the Fermi energy level packing vector \( n_i \), for particles in a cube of side \( L \), with ordonates ranging from \(-L/2 \) to \( L/2 \), subjected to periodic boundary conditions. This particular choice of wave function mimics the overall gross features of quark matter, by giving highly correlated behaviour at low densities and uncorrelated behaviour at high densities.
The total energy for this many body system is,

\[ E(\alpha,\beta,\rho) = T_s + V, \tag{14} \]

where

\[ T_s = \frac{-\hbar^2}{4m_q} (\nabla^2 \ln \Psi) \tag{15} \]

is the kinetic energy, obtained by eliminating the surface terms from the integral \( \int \Psi^* \nabla^2 \Psi \). Thus the many body Hamiltonian system can be solved by varying the parameters \( \alpha, \beta, \) and \( \rho, \) and evaluating the expectation values by Monte Carlo integration at each step, until a minimum \( E \) is found.

3. Monte Carlo Calculation

The Monte Carlo procedure uses the Metropolis algorithm [9] to generate a distribution in \( |\Psi|^2 \). The procedure then is to compute the average of an observable \( O \),

\[ \bar{O} = \frac{1}{\langle \Psi | \Psi \rangle} \int O(x) |\Psi(x)|^2 dx \approx \frac{1}{N} \sum_{n=N_0}^{N+N_0} O(x_n), \tag{16} \]

where the sum is taken from \( N \) sequential samples of the distribution \( |\Psi(\alpha,\beta,\rho;x)|^2 \) (s.t. \( \alpha, \beta, \) and \( \rho \) are fixed), after \( N_0 \) iterations have been made. The distribution in \( |\Psi|^2 \) is generated as follows: from a given configuration of particles \( x \), change all their positions randomly to a new position \( x + dx \), compute the transition probability function

\[ \tau = \min\{(|\Psi(x + dx)|/|\Psi(x)|)^2, 1\} , \tag{17} \]

and compare it with a random number \( r \in [0,1] \) — if \( \tau > r \) then accept the move by replacing \( x \) with \( x + dx \), otherwise reject the move by keeping the old \( x \), repeat the procedure until a desired \( \delta\bar{O} \) level has been reached. The initial configuration of particles, \( x = x_0 \), is generated by throwing them randomly into a box of side \( L \). All subsequent moves are constrained to the box, such that if a particle randomly moves outside, its periodic image enters from the opposite side. This algorithm, which satisfies detailed balance, is called the Metropolis algorithm and converges to the distribution \( |\Psi|^2 \) after \( N_0 \) moves have been made. The value of \( N_0 \) is determined by the point at which the statistical fluctuations in \( \sum_n O_n \) have become substantially reduced. A general rule of thumb is that convergence is more rapidly achieved if the step size, \( \delta x (= |dx|/\sqrt{3N_0}) \), is chosen such that, on average, \( \tau \approx 1/2 \). A natural length scale to use, when considering an appropriate step size, is (cf. [2])

\[ \delta x \sim \frac{rf}{\beta + \rho^{1/3}}, \tag{18} \]

where the constant \( f \approx 1/4 \). This is determined by taking several small samples from the probability distribution \( |\Psi|^2 \) and by restarting the Monte Carlo for different \( f \) values, until the desired value of \( \tau \) is reached. To ensure convergence in a finite amount of cpu time, particularly at low densities, \( rgb \) clusters of quarks (of radius order \( \delta x/2 \)) are thrown into the box randomly.

The Monte Carlo evaluation of the total energy, \( E = T_s + V \) in its current form, can
produce a significant amount of error [5]. This can be reduced by introducing a mean square “pseudo-force”,
\[ F^2 = \frac{\hbar^2}{4m_\eta N_n} (\nabla \ln \Psi)^2, \]
and re-expressing the kinetic energy as
\[ T = 2T_0 - F^2. \]
In this form, the variance of the total energy,
\[ E = 2T_0 - F^2 + V, \]
goes to zero as the wave function approaches an eigenstate of the Hamiltonian.

The variational wave function \( \Psi \) is made up of a product of a correlation piece, \( \chi \), and a Slater piece, \( \Phi \). Therefore the kinetic energy expression can be split up into three separate terms involving pure and mixed, correlation and Fermi energies: i.e.
\[ T = T_C + T_F + T_{CF}. \]
The explicit forms for these terms are: the correlation energy
\[ T_C = \frac{\alpha \beta}{8m_\eta N_n} < \sum \xi_{\eta \eta'}^{\alpha \beta} [\alpha(2 - (\beta \xi_{\eta \eta'})) + 8] >, \]
the Fermi energy
\[ T_F = \frac{2\pi^2}{m_\eta N_n L^2} \sum_{q=1}^{N_v} n_q^2, \]
and the mixed correlation-Fermi energy
\[ T_{CF} = \frac{\pi \alpha \beta}{2m_\eta N_n L} \sum_{i=1}^{N_v} \sum_{\{r_{\eta \eta'}\}} \xi_{\eta \eta'}^{\alpha \beta} \left[ \tilde{\phi}_{\eta}(r_{\eta \eta'} - r_{\eta \eta}) \phi_{\eta} + \tilde{\phi}_{\eta}(r_{\eta \eta'} - r_{\eta \eta}) \phi_{\eta} + \phi_{\eta}(r_{\eta \eta'} - r_{\eta \eta}) \phi_{\eta} \right] \cdot n_i >, \]
where \( \tilde{\phi}_{\eta} = (\phi^T)^{-1}_{\eta} \), and \( \phi_{\eta} = \phi_{\eta} + \frac{1}{m_\eta} n_i \). A detailed derivation of the above expressions can be found in the appendix.

The value of \( \alpha \) is fixed for free nucleons at \( \rho = 0 \). For the \( SU_b(3) \) model \( \alpha = 2 \), as the wave function \( \Psi \) must become that of a free 3-body harmonic oscillator. Therefore the total energy for this system is simply
\[ E_{\text{tree}}^{(h)}(\beta) = \frac{3\hbar^2}{2m_\eta} \beta^2 + \frac{3k}{2\beta^2}. \]
Minimizing this gives,
\[ E_0^{(h)} = 3\hbar \sqrt{\frac{k}{m_\eta}}, \]
where \( E_0^{(h)} = E_{\text{tree}}^{(h)}(\beta_0^{(h)}) \), and
\[ \beta_0^{(h)} = \left( \frac{m_\eta k}{\hbar^2} \right)^{1/4}. \]
$E_0^{(h)}$ and $\beta_0^{(h)}$ can be used to check the Monte Carlo. However, for the $SU_l(3)$ model such a check is not possible, as it is impossible to find $V$ analytically at $\rho = 0$. We find, by fitting our results, that

$$E_{\text{free}}^{(l)}(\beta) = g_T \frac{\hbar^2}{m_n} \beta^2 + g_V \frac{\sigma}{\beta},$$

(29)

where $g_T \approx 1.07$ and $g_V \approx 3.09$. We can also verify the virial relation $< T_l > = < V_l > /2$, which should hold at all densities. A similar check can also be done for $SU_h(3)$, with the virial relation $< T_h > = < V_h >$. For $SU_l(3)$ the parameters $\alpha$ and $\beta_0^{(l)}$ can be obtained by computing $E_{\text{free}}^{(l)}$ for different values of $(\alpha, \beta)$ until a minimum is found.

![Graph of $\beta(\rho)$ for $SU_l(3)$ and $SU_h(3)$](image)

**Figure 3:** Graph of $\beta(\rho)$ for $SU_l(3)$ and $SU_h(3)$.

A further reduction of the variational parameters is obtained by introducing the scaling transformation [2],

$$(\beta, \rho^{1/3}) \rightarrow \zeta (\cos \theta, \sin \theta),$$

(30)

where $\zeta > 0$, and $\theta$ is restricted to the interval $(0, \pi/2)$. This allows the total energy to be expressed as a polynomial in $\zeta$, which can subsequently be minimized to eliminate $\zeta$: i.e.
$\tilde{E}(\rho, \beta)$ becomes

$$E(\zeta, \theta) = \tilde{T}(\theta)\zeta^2 + \tilde{V}(\theta)/\zeta^\kappa,$$

such that $T(\zeta, \theta) = \tilde{T}(\theta)\zeta^2$, $\tilde{V}(\zeta, \theta) = \tilde{V}(\theta)/\zeta^\kappa$ and

$$\kappa = \begin{cases} 1 & \text{if } SU_t(3) \\ 2 & \text{if } SU_h(3) \end{cases},$$

which can be minimized with respect to $\zeta$ to give

$$\bar{E}(\theta) = (\kappa + 2) \left( \frac{\tilde{V}^2(\theta)\tilde{T}^\kappa(\theta)}{4\kappa^\kappa} \right)^{\frac{1}{\kappa+2}},$$

with

$$\zeta(\theta) = \left( \frac{\kappa\tilde{V}(\theta)}{2\tilde{T}(\theta)} \right)^{\frac{1}{\kappa+2}}.$$

\textbf{Figure 4:} Graph of $E_B(\rho)$ for $SU_t(3)$ and $SU_h(3)$. 

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Notice that the elimination of the parameter $\zeta$ is equivalent to imposing the virial theorem, which implies

$$<T> = \frac{k}{2} <V> .$$  \hspace{1cm} (35)

Therefore the Monte Carlo only has to be run for different $\theta$ values extracted from the “open” interval $(0, \pi/2)$. The end points are obtained by taking a limit. The $\theta = 0$ limit is equivalent to taking $\rho = 0$, which has already been discussed. The $\theta = \pi/2$ limit is equivalent to taking $\beta = 0$, which corresponds to an uncorrelated Fermi gas, with energy

$$E_{Vq}(\rho) = \left(\frac{3\pi^4}{2}\right)^{1/3} \frac{3\hbar^2}{5m_q \rho^{2/3}} + V_{Vq}(\rho) ,$$  \hspace{1cm} (36)

where

$$V_{Vq}(\rho) = \begin{cases} 
\frac{c_\alpha \sigma}{\rho^{1/3}} & \text{for } SU_{\alpha}(3) \\
\frac{c_\alpha k}{2\rho^{2/3}} & \text{for } SU_{\beta}(3) 
\end{cases} ,$$  \hspace{1cm} (37)

and $c_\alpha$ is obtained by a fit to the Monte Carlo in the $\theta = \pi/2$ limit. Thus the $\beta = 0$ limit is described by the curve $E_{Vq}(\rho)$. This curve is compared with the Monte Carlo results for $\bar{E}(\rho(\theta))$ from which a minimum energy curve $\bar{E}(\rho)$ is obtained.

Figures (3) and (4) show the variational Monte Carlo results for $\beta(\rho)$ and the binding energy, $E_B(\rho) \equiv \bar{E}(\rho) - E_0$, respectively. The dashed lines on these graphs show the remnants of the minimal $\rho(\theta)$ trajectories, for $\beta$ and $E_B$, after a phase transition, at $\rho = \rho_c$, from a correlated system of quarks to an uncorrelated Fermi gas was made. The slight roughness of these lines is because the data was not fitted. In plotting these graphs it was assumed that: $N_n = 7$, $m_q = 330\, MeV$, $\sigma = 910\, MeV/fm$, and $k \approx 3244\, MeV/fm^2$. The value of $k$ was determined by setting $E_0^{(1)}$ in equation (27) to equal $E_0^{(f)}$, in the limit $N_n = 1$ and $\theta = 0.0001$. The other parameters that were determined by the Monte Carlo are given in table (1).

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<th>Table 1: Parameters Determined by Monte Carlo, with $N_n = 7$.</th>
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<td>Parameters</td>
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<td>$\alpha$</td>
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<td>$c_\alpha(\rho_{Vq}, \beta_{Vq})$</td>
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<td>$E_0(\alpha, \beta_0)$</td>
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4. Discussion

The parameter $\beta$ is related to the confinement scale for triplets of quarks. Figure (3) shows that the quarks become more deconfined as $\rho$ increases, and completely deconfined beyond the phase transition point, $\rho_c$. Thus as $\rho$ increase from 0 to $\rho_c$ the nucleon swells producing an EMC-like effect. Figure (11.b), of reference [1], shows a plot of $\lambda$ ($\sim \beta^3$) vs. $\rho$, obtained by Horowitz and Piekarewicz, which in general shows that the quarks become more confined as $\rho$ approaches $\rho_c$, and completely deconfined beyond. Therefore their model produces a result
that incorrectly explains the EMC effect [10].

\[
\begin{align*}
V &= \min\{\sum_{\{uy\}} v_{rg} + \min\{\sum_{\{gb\}} v_{gb} + \min\{\sum_{\{br\}} v_{br}\}\},
\end{align*}
\]

where \(v_{ij} = \frac{1}{2} kr_{ij}^2\), with the wave function \(\Psi = e^{-\lambda V} \Phi\) (cf. equation (12) with \(\alpha = 2\)). They have shown for \(\rho < \rho_c\) that 3-quark clusters (chains) make up more than 90% of nuclear matter, with a large remainder of these being 6-quark clusters. A closer look at their \(A \text{ u.s. } \rho\) plot shows a small dip in \(A\) around \(\rho = 0.2\). This indicates a slight swelling of the nucleon for very small

**Figure 5**: Different \(SU_3(3)\) flux tube construction schemes (RHS) motivated by their corresponding linear cousins (LHS). Figures a) and c) represented the HP construction. Figures a) and b) show a construction scheme with a more consistent weighting for \(s\)-states.

The Horowitz and Piekarewicz (HP) model approximates the higher order flux tube topologies of equation (2) with long harmonic oscillator chains that close upon themselves: i.e.
\(\rho\). In fact, in this density regime 3-quark clusters completely dominate (> 99%). The evidence from these graphs seems to suggest that too much weight is being given to higher order flux tube topologies at intermediate densities.

Lattice QCD shows that quarks like to cluster together via a linear potential. However, most phenomenological models that describe isolated hadronic matter using a harmonic oscillator potential work just as well. As can been seen from figures (3) and (4) the harmonic oscillator model gives the same overall shape as the linear one. This model was motivated by replacing each linear segment of string in a 3-quark state by a spring, with spring constant \(k\). For quarks of equal mass this reduces to a triangle of springs (see figure (5.a)). Similarly a 6-quark state would give an object that simplifies to three triangles with one of the tips from each meeting at a common vertex (see figure (5.b)). The corresponding 6-quark state for the HP model forms a closed ring which, in general, requires less energy to form (see figure (5.c)). Thus QCD motivated models would also seem to support the aforementioned claim, that HP are giving too much weight to higher order flux tube topologies.

For \(SU_h(2)\) our model [2] agrees with the HP model [1]. The graphs look similar to those shown in figures (3) and (4). Of course the fact that these models agree should be of no significance, as they both have the same potential, which only looks at combinations of \(q\bar{q}\) pairs. Also the \(SU_h(2)\) models [1, 2] when compared with \(SU_l(2)\) [2] gives similar contrasting figures to the ones presented here.

Figure (4), along with similar figures given in references [1, 2], show a saturation of nuclear forces as \(\rho \rightarrow \rho_c\), followed by a phase transition to quark matter at \(\rho_c\). All of these models, however, fail to give any nuclear binding below \(\rho_c\), which would seem to suggest that the flux-tube models are incapable of giving nuclear binding. Even the \(SU_h(3)\) HP model with its long chains, which tends to underestimate the potential, indicates that this would appear to be the case [1]. HP have a 2q model [1] that would seem to suggest that even if colour were not fixed to a given quark, no nuclear binding would occur: albeit this model is for \(p\)-wave (qq) states. Thus it would appear that string-flip models, even those that include higher order flux tube topologies, or allow the colour to move from quark to quark, are insufficient to obtain nuclear binding. Therefore another mechanism for lowering the potential must be included in these models.

One way is to include one-gluon exchange interactions. As suggested by Nzr and Hoodbhoy [3], the most significiant of these are the hyperfine interactions. Other effects such as mass and isospin are expected to negligible. Relativistic effects are expected, in general, to only contribute to an overall shift down in energy.

**Conclusion**

Various string-flip potential models have been discussed in as general a setting as possible, and have been shown to be quite capable of describing the bulk properties of nuclear/quark matter with the exception of nuclear binding. At low densities they give free nucleon matter and at high densities a phase transition to free quark matter. They show an overall saturation of nuclear forces as nucleon densities are increased. At intermediate densities these models, with the exception of the HP \(SU_h(3)\) linked chain model [1], give an overall EMC-like swelling of the nucleon. It is our belief that these models with one-gluon exchange effects added on should be capable of predicting nuclear binding.

This work was supported by NSERC. The Monte Carlo calculations were performed on an 8 node DEC 5240 UNIX CPU farm, monitored by a shepherd VAX 3100, in the Carleton University Department of Physics.
Given the wave function
\[ \Psi(\alpha, \beta, \rho) = \chi(\alpha, \beta) \Phi(\rho), \] (39)
where
\[ \chi(\alpha, \beta) = e^{\sum_{\{rgh\}} (-\beta \xi_{rgh})} \] (40)
and
\[ \Phi(\rho) = \prod_{c \in \{rgh\}} |\Phi_{S,c}(\rho)| \] (41)
are the correlation and Fermi parts respectively, the kinetic energy can be split up thusly: into a correlation piece \( \overline{T}_C = 2T_{C-} - \overline{T}_C^2 \), a Fermi piece \( \overline{T}_F = 2T_{F-} - \overline{T}_F^2 \), and a mixed correlation-Fermi piece \( \overline{T}_{CF} = -2\overline{T}_{CF}^2 \), where
\[ \overline{T}_{C-} = -\frac{1}{4N_m m_q} < \sum_q (\nabla_q^2 \ln \chi) >, \] (42)
\[ \overline{T}_{F-} = -\frac{1}{4N_m m_q} < \sum_q (\nabla_q^2 \ln \Phi) >, \] (43)
\[ \overline{T}_C^2 = \frac{1}{2N_m m_q} < \sum_q (\nabla_q^2 \ln \chi)^2 >, \] (44)
\[ \overline{T}_F^2 = \frac{1}{2N_m m_q} < \sum_q (\nabla_q^2 \ln \Phi)^2 >, \] (45)
and
\[ \overline{T}_{CF} = \frac{1}{2N_m m_q} < \sum_q (\nabla_q \ln \chi) \cdot (\nabla_q \ln \Phi) >. \] (46)
The correlation pieces are straightforward to evaluate, and give the following
\[ \nabla_l \ln \chi = -\frac{\alpha}{4} \beta^{2\alpha} \sum_{\{rgh\}} \xi_{rgh}^{2\alpha-1} \nabla_l \xi_{rgh}^2 \]
\[ = -\frac{\alpha}{6} \beta^{2\alpha} \sum_{\{rgh\}} [\xi_{rgh}^{2\alpha-1}(r_{rg} - r_{br}) \delta_{rl} + (r_{gb} - r_{rg}) \delta_{gl} + (r_{br} - r_{gb}) \delta_{ld}] \] (47)
with \( \xi_{rgh} = \frac{1}{\sqrt{r_{rg}^2 + r_{gb}^2 + r_{br}^2}} \), which yields
\[ \sum_l (\nabla_l \ln \chi)^2 = -\frac{\alpha^2}{36} \beta^{2\alpha} \sum_{\{rgh\}} \xi_{rgh}^{2\alpha-4} [(r_{rg} - r_{br})^2 + (r_{gb} - r_{rg})^2 + (r_{br} - r_{gb})^2] \]
\[ = -\frac{\alpha^2}{4} \beta^{2\alpha} \sum_{\{rgh\}} \xi_{rgh}^{2\alpha-2}, \] (48)
and
\[ \sum_l (\nabla_l^2 \ln \chi) = -\frac{\alpha}{6} \beta^{\alpha} \sum_{\{rgh\}} \sum_l [\alpha \xi_{rgh}^{\alpha-4} (\nabla_l \xi_{rgh}^2) + \xi_{rgh}^{\alpha-2} \nabla_l] \]
\[ \cdot [(r_{rg} - r_{br}) \delta_{rl} + (r_{gb} - r_{rg}) \delta_{gl} + (r_{br} - r_{gb}) \delta_{ld}] \]
\[ = -\frac{\alpha}{6} \beta^{\alpha} \sum_{\{rgh\}} \left( \xi_{rgh}^{\alpha-2} [(r_{rg} - r_{br})^2 + (r_{gb} - r_{rg})^2 + (r_{br} - r_{gb})^2] + 18 \xi_{rgh}^{\alpha-2} \right) \]
\[ = -\frac{\alpha(\alpha + 4)}{2} \beta^{\alpha} \sum_{\{rgh\}} \xi_{rgh}^{\alpha-2}, \] (49)
which imply
\[ T_{C-} = \frac{\alpha(\alpha + 4)\beta^2}{8\eta_0} \sum_{\{rgb\}} \xi_{rgb}^{\alpha-2}, \] (50)
and
\[ F_C^2 = \frac{\alpha^2\beta^{2\alpha}}{8\eta_0} \sum_{\{rgb\}} \xi_{rgb}^{2\alpha-2}, \] (51)
thus giving the desired result, i.e.
\[ T_C = 2T_{C-} - F_C^2 = \frac{\alpha\beta^{\alpha}}{8\eta_0} \sum_{\{rgb\}} \xi_{rgb}^{\alpha-2} \left[ \alpha(2 - (\beta\xi_{rgb})^\alpha) + 8 \right]. \] (52)

The Fermi pieces are also straightforward once the following identities are realized [2]:
\[ \nabla_\ell \ln |\Phi_{S_c}| = \sum_{ij} \tilde{\phi}_{ij} \nabla_\ell \phi_{ij}, \] (53)
\[ \nabla_\ell \tilde{\phi}_{ij} = -\sum_{mn} \tilde{\phi}_{im} \left( \nabla_\ell \phi_{ij}^T \right) \tilde{\phi}_{nj}, \] (54)
\[ \nabla_\ell^2 \ln |\Phi_{S_c}| = \sum_{ij} \tilde{\phi}_{ij} \nabla_\ell^2 \phi_{ij} - \sum_{mn} \phi_{im} \left( \nabla_\ell \phi_{mn} \right) \left( \nabla_\ell \phi_{ij} \right) \tilde{\phi}_{nj}, \] (55)
where \( \tilde{\phi}_{ij} \equiv (\phi^T)^{-1}_{ij} \), and \( \phi_{ij} = \sin(\frac{2\pi}{L} n_i \cdot r_j + \delta_i) \). These imply
\[ \nabla_\ell \phi_{ij} = \frac{2\pi}{L} n_i \delta_{ij} \phi'_{ij}, \] (56)
and
\[ \nabla_\ell^2 \phi_{ij} = -\frac{4\pi^2}{L^2} n_i^2 \delta_{ij} \phi_{ij}, \] (57)
where \( \phi'_{ij} \equiv \phi_i(r_j + \frac{L}{m_i^2} n_i) \). Therefore
\[ \sum_{\ell} \nabla_\ell^2 \ln |\Phi_{S_c}| = -\frac{4\pi^2}{L^2} \left[ \sum_{ij} \tilde{\phi}_{ij} n_i^2 \phi'^T_{ij} + \sum_{ijk} \tilde{\phi}_{ik} \phi_{kj} n_k \cdot n_j \phi'_{ji} \phi_{ji} \right] \] (58)
and
\[ \sum_{\ell} \left( \nabla_\ell \ln |\Phi_{S_c}| \right)^2 = \frac{4\pi^2}{L^2} \left[ \sum_{ijk} \tilde{\phi}_{ik} \phi'_{kj} n_k \cdot n_j \phi'_{ji} \phi_{ji} \right], \] (59)
imply
\[ T_{F}^{(c)} = 2T_{C-}^{(c)} - F_C^{2(c)} = \frac{2\pi^2}{m_q L^2} \sum_{ij} \tilde{\phi}_{ij} n_i^2 \phi'^T_{ji} = \frac{2\pi^2}{m_q L^2} \text{Tr}(\phi^T N_c^2) \phi = \frac{2\pi^2}{m_q L^2} \text{Tr}(N_c^2), \] (60)
where \( (N_c^2)_{ij} = n_i^2 \delta_{ij} \). Thus the desired result is obtained by summing over the quark colour degrees of freedom, i.e. \( \sum_c T_{F}^{(c)} \), which implies
\[ T_F = \frac{2\pi^2}{m_q L^2} \sum_q n_q^2. \] (61)

Finally, the mixed correlation-Fermi result, given by equation (25), is simply obtained, via equation (46), by taking the inner product of equations (47) and (53), summing over \( \ell \) and \( q \), and using the identity given by equation (56).
References