Free Fields and Quasi-Finite Representation of $\mathcal{W}_{1+\infty}$ Algebra

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We study quasi-finite representation of the $\mathcal{W}_{1+\infty}$ algebra recently proposed by Kac and Radul. When the central charge is integer, we show that they are represented by free fermions and bosonic ghosts. There are some nontrivial representations with vanishing central charge. We discuss that they may be described by large $N$ limit of topological models. We calculate their operator algebras explicitly.

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1. Introduction

The quantum extension of $W_{1+\infty}$ algebra appears in many places in the two-dimensional physics. In quantum gravity, it comes from Schwinger-Dyson equation of the matrix model[1]. Similar structure appears in the discrete states of quantum gravity coupled with $c = 1$ matter[2], although the relation between these two is not yet quite clear. In quantum hole effect, it appears as the realization of magnetic translation group [3]. [4] Also in turbulence in two dimensions, it might play important rôle since the fluid motion in two dimension defines the area-preserving diffeomorphism.

So far, however, the representation theory of $W_{1+\infty}$ algebra [5] [6] was not understood well compared with other loop algebras such as Virasoro algebra or Kac-Moody algebra. Only free fermion representation [7] [8] [9] is studied to a certain extent. Although the appearance of $W_{1+\infty}$ algebra itself is interesting, we do not know how to extract information from the symmetry.

$W_{1+\infty}$ algebra may be characterized by its infinite number of generator fields, $W_1$, $W_2$, $W_3$, $W_4$ . . . . Consequently, for each energy level, we may have infinite number of independent states. In such situation, it is a non-trivial question when we have finite number of non-vanishing states at each energy level (quasi-finite representation). Recently, Kac and Radul [10] gave a convincing answer to this problem. They have derived the possible form of the highest weights of $W_{1+\infty}$ module. They have also clarified the condition when the representation becomes unitary.

In this letter, we give an explicit realization of their work in terms of free fields. Roughly speaking, the representations of $W_{1+\infty}$ algebra can be classified into two groups by the value of the central charge, (1) $C \neq 0$ and (2) $C = 0$. In the first case with further condition that $C = \text{Integer}$, we show that they are essentially given by combining fermionic and bosonic ghosts. We give an explicit proof of the dimension formula and the null field conditions. In the second case, we propose that it may be described by the large $N$ limit of topological combination of free fields. Finally, we derive the operator algebra predicted by our construction.

2. Brief Sketch of Kac-Radul Formula

$W_{1+\infty}$ algebra is the quantum version of the $w_{1+\infty}$ algebra generated by polynomials of $z$ and $D = z \frac{\partial}{\partial z}$. For any function $f(w)$ which is regular at $w = 0$, we may write its
typical generator as \( z^r f(D) \). The two–cocycle which defines the quantization is explicitly given by,

\[
\Psi(z^r f(D), z^s g(D)) = -\Psi(z^s g(D), z^r f(D)) = \begin{cases} 
0 & \text{if } r = -s > 0 \\
\sum_{i \leq j \leq r} f(-j)g(r-j) & \text{if } r + s \neq 0 \text{ or } r = s = 0
\end{cases}
\]  

(2.1)

Denote the quantum generator associated with classical operator \( z^r f(D) \) as \( W(z^r f(D)) \). \( \mathcal{W}_{1+\infty} \) algebra may be written as,

\[
[W(z^r f(D)), W(z^s g(D))] = W(z^{r+s} f(D + s)g(D)) - W(z^{r+s} f(D)g(D + r)) + C \Psi(z^r f(D), z^s g(D)).
\]  

(2.2)

It is easily derived by using a useful relation \( f(D)z^r = z^r f(D + r) \).

We use \( V_k^n = W(z^k D^n) \), for generators in terms of mode. \( \mathcal{W}_{1+\infty} \) algebra in terms of lower dimensional operators are given by

\[
[V_r^0, V_s^0] = C \delta_{r,-s,} \quad [V_r^1, V_s^1] = (s-r)V_{r+s}^1 - C \frac{r^3}{6} \delta_{r,-s}.
\]

We may understand that \( V^0 \) and \( V^1 \) stand for \( U(1) \) current and Virasoro generator respectively. The central charge of Virasoro algebra is given by \( -2C \). We may freely change it by adding total derivative of \( U(1) \) currents in the definition of the Virasoro generator.

The highest weight state of the \( \mathcal{W}_{1+\infty} \) algebra is defined by following conditions,

\[
\begin{align*}
V_r^n |\lambda> &= 0, & \text{if } r > 0, \\
V_0^n |\lambda> &= \Delta_n |\lambda>, & n = 0, 1, 2, \ldots, \infty.
\end{align*}
\]  

(2.3)

Needless to say, unlike other loop algebras, we have infinite number of the weight vector \( \Delta_n \). For later convenience, we combine them into the following form,

\[
\Delta(x) \equiv -\sum_{n=0}^{\infty} \frac{x^n}{n!} \Delta_n.
\]  

(2.4)

This is the eigenvalue of the operator, \( W(-e^{x D}) \).

The highest weight state \(|\lambda>\) may be specified by operators which annihilate it. Those operators form a subalgebra (parabolic subalgebra) of \( \mathcal{W}_{1+\infty} \) algebra. For each level \( k \), we denote the set of its elements as \( I_k \). We may easily check that if \( z^{-k} P(D) \) is an element of \( I_k \), any element of the form \( z^{-k} P(D)f(D) \) is again the element of \( I_k \). It shows that \( I_k \) is an ideal of the polynomial algebra. We use \( \mathcal{P}_k(D) \) as the generator of ideal \( I_k \). For
quasi-finite representation, \( P_k(w) \) should be a polynomial of finite order with respect to \( w \).

The main observation of [10] is that if we determine the polynomial \( P_1(D) \equiv P(D) \) at level 1, one can derive \( P_k(D) \) for \( k > 1 \) and level 0 weight function, \( \Delta(x) \).

If the differences between each root of \( P(D) \) are not integer, level \( k \) generator is given by,

\[
P_k(D) = P(D)P(D - 1) \cdots P(D - k + 1).
\]

(2.5)

On the other hand, the information at level 0 may be extracted by multiplying \( W(zg(D)) \) from left hand side to the equation, \( W(z^{-1}P(D))\lambda >= 0 \). By using commutation relation (2.2), one finds,

\[
(W(P(D)g(D)) - W(P(D + 1)g(D + 1)) + CW(P(0)g(0)))\lambda >= 0.
\]

(2.6)

This relation holds for any \( g(D) \) and gives enough information on the highest weights \( \Delta(x) \). It is in particular useful to use the combination,

\[
F(x) \equiv C + (1 - e^x)\Delta(x)
\]

(2.7)

From (2.6), one may prove that it satisfies a differential equation, \( P(\frac{d}{dx})F(x) = 0 \). If \( P(w) \) is given by, \( P(w) = (w - s_1)^{n_1} \cdots (w - s_l)^{n_l} \), the general solution is,

\[
F(x) = \sum_{i=1}^{l} p_{n_i}(x)e^{s_i x}.
\]

(2.8)

Here \( p_{n}(x) \) is \( n - 1 \) order polynomial in \( x \). Together with (2.7), one gets the explicit form of the highest weights, which is the main result of [10],

\[
\Delta(x) = \frac{\sum_{i=1}^{l} p_{n_i}(x)e^{s_i x} - C}{e^x - 1}.
\]

(2.9)

The central charge \( C \) is determined by the requirement that there is no singularity in (2.4) when \( x \to 0 \). If we decompose (2.9), we realize that there are basically two types of representations of \( W_{1+\infty} \) algebra,

\[
\Delta(x) = C \frac{e^{sx} - 1}{e^x - 1}, \quad C \neq 0.
\]

(2.10a)

\[
\Delta(x) = \frac{p_{\ell} x^{\ell} e^{sx}}{e^x - 1}, \quad C = 0, \ell = 1, 2, 3 \ldots
\]

(2.10b)

The requirement of unitarity gives severe restriction on the polynomial \( p_{n}(x) \). It was proved [10] that the weight function should take the following form, \( \Delta(x) = \sum_{i=1}^{l} n_i (e^{s_i x} - 1)/(e^x - 1) \), with positive integer \( n_i \) and real \( s_i \). The central charge should be a positive integer, \( C = \sum_{i} n_i \).
3. Free Field Realization

In the following, we derive explicit realizations that meet Kac-Radul Formula (2.9). Since we know free fermion (or bosonic ghost) representation, we should first check to which representation they belong. We describe free fermions and bosonic ghosts in a parallel fashion. We use the notation similar to those in [11]. Let \( b(z) \) (resp. \( c(z) \)) stands for either \( b(z) \) or \( \beta(z) \) (resp. \( c(z) \) or \( \gamma(z) \)). They are expanded as, \( c(z) = \sum_{t \in \mathbb{Z}} c_t z^{-t-s-1} \), \( b(z) = \sum_{t \in \mathbb{Z}} b_t z^{-t+s} \). We use a general representation, i.e. \( s \) may not be restricted to be half integer. As a consequence, \( c \) and \( b \) may not be invariant under \( z \rightarrow z e^{2\pi i} \). The (anti-) commutation relation is given by \( [c_n, b_m]_\varepsilon \equiv c_n b_m + \varepsilon b_m c_n = \delta_{n+m,0} \), with \( \varepsilon = 1 \) for fermions and \( \varepsilon = -1 \) for bosons. The vacuum \( |s> \) is characterized by the conditions,

\[
c_t|s> = b_{t+1}|s> = 0, \quad \ell = 0, 1, 2, \ldots \tag{3.1}
\]

The generators of \( \mathcal{W}_{1+\infty} \) algebra may be described by those \( b-c \) fields by sandwiching. For each classical generator of the form, \( z^r f(D) \), the corresponding generator is defined by,

\[
W(z^r f(D)) = \varepsilon \int \frac{dz}{2\pi i} : c(z)z^r f(D)b(z) : = \varepsilon \sum_{t \in \mathbb{Z}} : c_t f(s+\ell)b_{-t} : . \tag{3.2}
\]

It is easy to see that this gives \( \mathcal{W}_{1+\infty} \) algebra (2.2) with \( C = \varepsilon \). The commutation relation with the free fields are given by,

\[
[W(z^r f(D)), c_\ell] = c_{\ell+r} f(s+\ell), \quad [W(z^r f(D)), b_\ell] = -f(s-\ell-r) b_{\ell+s}. \tag{3.3}
\]

**Construction of Null States:** As a first check that this free field construction gives quasi-finite representation, let us examine the parabolic subalgebra that annihilates the vacuum, \( W(z^{-k} \mathcal{P}_k(D))|s> = 0 \). We use the fact that a descendant state becomes null if it satisfies the highest weight condition in terms of free fields (3.1).

Let us first investigate the null-state at level 1. Since, \( [W(z^{-1} \mathcal{P}(D)), c_\ell] \sim c_{\ell-1} \), the highest weight condition (3.1) with \( \ell = 1, 2, 3 \ldots \) holds trivially for any \( \mathcal{P}(D) \). For \( \ell = 0 \), we need to impose,

\[
[W(z^{-1} \mathcal{P}(D)), c_0] = c_{-1} \mathcal{P}(s) = 0, \quad [W(z^{-1} \mathcal{P}(D)), b_1] = -b_0 \mathcal{P}(s) = 0.
\]

The minimum choice for \( \mathcal{P}(D) \) that satisfies those two conditions is obviously,

\[
\mathcal{P}(D) = D - s. \tag{3.4}
\]
The construction of null states at higher level is similar. For level \( n \), nontrivial conditions arise from \( \ell = 0, 1, \cdots, n - 1 \). Conditions, \( \mathcal{P}_n(s + \ell) = 0 \), for \( \ell = 0, 1, \cdots, n - 1 \) should be satisfied. The minimum polynomial is

\[
\mathcal{P}_n(D) = (D - s)(D - s - 1) \cdots (D - s - n + 1). \tag{3.5}
\]

This computation manifestly confirm the general formula (2.5).

**Highest weights**: Since we derive explicitly the generators of the algebra and the null fields, the differential equation shows that the highest weight should be given in the form, \( \Delta(x) = n \frac{e^x - 1}{e^x - 1} \). with a free parameter \( n \). Actually, since \( C = \varepsilon \), \( n \) should be given by \( n = \varepsilon \). Let us confirm it explicitly.

The \( \mathcal{W}_{1+\infty} \) generator that have the eigenvalue \( \Delta(x) \) is given by,

\[
W(-e^{xD}) = -\varepsilon \int \frac{dz}{2\pi i} : c(z) e^{xD} b(z) : = -\varepsilon \int \frac{dz}{2\pi i} : c(z) b(e^x z) :. \tag{3.6}
\]

For the fermionic case, one may describe the vacuum \(|s \rangle \) through vertex operator,

\[
|s \rangle = e^{-s\phi(0)}|0 \rangle, \quad \partial_z \phi(z) = : b(z)c(z) : \tag{3.7}
\]

\( \phi(z) \) satisfies standard OPE \( \phi(z) \phi(0) \sim \log(z) \). Using the vertex operator representation of free fermion, \( b(z) = e^{-\phi(z)} \), \( c(z) = e^{\phi(z)} \), one may prove the Kac-Radul Formula by a direct computation,

\[
- \int \frac{dz}{2\pi i} (e^{\phi(z)} e^{-\phi(z)e^x}) e^{-s\phi(0)}|0 \rangle = - \int \frac{dz}{2\pi i} \left( \frac{e^x z)^s}{z - e^x z} e^{-s\phi(0)}|0 \rangle \right) = \frac{e^{s} e^{x}}{e^x - 1} |s \rangle >. \tag{3.8}
\]

From this expression, we need to subtract \( \frac{1}{e^x - 1} \) due to the normal ordering. This proves the formula (2.10a). To derive a similar formula for the bosonic ghost, we need to make use of the “bosonization” of bosonic ghosts [11], \( \beta = e^{-\gamma} \partial \xi \), \( \gamma = e^\gamma \eta \), with \( \xi(z)\eta(0) \sim \frac{1}{z} \) and \( \sigma(z)\sigma(0) \sim - \log(z) \). The vacuum \(|s \rangle \) is in this case represented as \(|s \rangle = e^{s\sigma(0)}|0 \rangle >. \)

The proof of the Kac-Radul Formula (with negative central charge \( C = -1 \)) can be done in the similar fashion as fermionic case.

Construction of the representation for integer \( C \) may be done by tensor products. We remark that the minimal polynomial at level one keeps its form \( \mathcal{P}(w) = w - s \) as long as we take the tensor product of the same spin fields.
**Topological Representation:** It is rather subtle to understand the second type of representation, (2.10b). In this case, the central charge vanishes, suggesting that it may be related to the topological field theory [12]. Furthermore, the minimal polynomial \( P(w) \) need to have higher zeros \( P(w) = (w - s)^n \) at \( w = s \). Since a tensor product of finite free fields gives only single zeros for \( P(w) \), we are forced to realize them in limiting procedure.

We remark that (2.10b) can be obtained from (2.10a) by differentiation with respect to spin \( s \) several times. To take subtraction in the field theory, one may combine fields with different statistics [12]. In our case, let us first combine a pair of free fermion with spin \( s' \) and a pair of bosonic ghost with spin \( s \). The Kac-Radul Formula becomes \( \Delta(x) = \frac{e^{(s+s')} - e^{sx}}{e^x - 1} \).

The minimal polynomial for this system is \( P(w) = (w - s)(w - s') \). Putting \( s' = s + \frac{p_1}{N} \) with one constant \( p_1 \), and introduce \( N \) replica of this \( bc\beta\gamma \) system, one gets

\[
\Delta(x) = N \frac{e^{x(s+rac{p_1}{N})} - e^{xs}}{e^x - 1} = \frac{p_1xe^{sx}}{e^x - 1} + O(\frac{1}{N}).
\]

Since the minimal polynomial keeps its form in the tensor product,

\[
P(w) = \lim_{N \to \infty} (w - s - \frac{p_1}{N})(w - s) = (w - s)^2.
\]

These are exactly conditions for (2.10b) with \( \ell = 1 \).

Higher representations \( \ell > 1 \) may be obtained similarly. Let us consider \( M \) copies of \( bc\beta\gamma \) system with spins, \( (s_i, t_i) \) \( (i = 1 \cdots m) \). The dimension formula in this case is,

\[
\Delta(x) = \frac{M}{e^x - 1} \sum_{\ell=1}^{m} (e^{s\ell x} - e^{t\ell x}). \tag{3.9}
\]

Consider a case when all dimensions \( (s_i, t_i) \) are very close to a fixed value, \( s \). We rewrite each dimension as \( s_i = s + C_i\delta s \), \( t_i = s + D_i\delta s \), where \( C_i \) and \( D_i \) are finite constants. Define, \( Q_\ell = \sum_{j=1}^{m} \frac{C_i - D_i}{\ell} \). (3.9) may then be expanded as follows,

\[
\Delta(x) = \frac{Me^{sx}}{e^x - 1} \sum_{\ell=1}^{\infty} Q_\ell \delta s^\ell. \tag{3.10}
\]

In order to get \( \ell \)th order representation, we need to require,

\[
Q_1 = Q_2 = \cdots = Q_{\ell-1} = 0, \quad Q_\ell \neq 0. \tag{3.11}
\]

Replacing \( M = N^\ell \) and \( \delta s = (\frac{p_1}{NQ_\ell})^\ell \), one arrives at the desired answer,

\[
\lim_{N \to \infty} \Delta(x) = \frac{p_\ell e^{sx}}{e^x - 1}.
\]
**Operator Algebra:** One merit to use free fields is that we do not need to worry about consistency of the theory. In general, it is a nontrivial task to check the Jacobi identities of the primary fields. However, in our case, it is satisfied automatically.

Computation of the operator algebra of primary fields of type (2.10a) is quite simple since it is identical to that of vertex operator. Let us denote the primary field which corresponds to the dimension formula as $\Phi_{\{n_i,s_i\}}(z)$. One may explicitly represent it as the vertex operator which generalize (3.7). The OPE between the vertex operators is given by,

$$
\Phi_{\{n_i,s_i\}}(z) \cdot \Phi_{\{n_i,s_i'\}}(w) \sim (z-w) \sum_{i} n_i s_i s_i' \Phi_{\{n_i,(s+s')i\}}(w). 
$$  \hspace{1cm} (3.12)

On the other hand, we have to be much more cautious in calculating OPE between the primary fields of type (2.10b). Let us denote primary field which corresponds to (2.10b) by $\Psi^{(f)}_{\{s,p\}}(z)$. The precise form of the representation is not unique. In other word, the primary field with the same index $p\ell$ may be written with different $C\ell$ and $D\ell$. If we take OPE for vertex operators with $(C,D)$ and $(C',D')$ which satisfy (3.11), the composite vertex operator also need to satisfy condition (3.11),

$$
Q''_{\ell} \equiv \sum_{j=1}^{m} ((C_j - C'_j)^{i} - (D_j - D'_j)^{i})/i! = 0, \quad i = 0, 1, \cdots, \ell - 1. 
$$  \hspace{1cm} (3.13)

It is, however, difficult to find general solutions for these nonlinear equations.

In this letter, instead of using general solution for the coefficients, we use the simplest solution for (3.13), i.e. the two vectors $(C,D)$ and $(C',D')$ should be proportional. In this case, we may represent two primary fields $\Psi^{(f)}_{\{s,p\}}$, $\Psi^{(f)}_{\{s',p'\}}$ by using the same coefficients $(C,D)$ but different $\delta s$. Explicitly, $\delta s = \frac{p'_{\ell}}{NQ_{\ell}}$ for the first operator, and $\delta s' = \frac{p'_{\ell}}{NQ_{\ell}}$ for the latter. After some algebra which generalize (3.12), we obtain,

$$
\Psi^{(f)}_{\{s,p\}}(z) \Psi^{(f)}_{\{s',p'\}}(w) \sim (z-w)^{\beta(s,s',p',p') \Psi^{(f)}_{\{s+s',p',p'\}}}(w).
$$

where,

$$
p^{\ell}_{\ell} \frac{1}{1/\ell} = p^{1/\ell}_{\ell} + p^{1/\ell}_{\ell}, \quad \beta(s,s',p_{\ell},p'_{\ell}) =
\begin{cases} 
p s s'_{\ell} + p s'_{\ell} & (\ell = 1) \\
2\sqrt{p p_{\ell}^2} & (\ell = 2) \\
0 & (\ell > 2) \end{cases}
$$  \hspace{1cm} (3.14)

The dependence on $N$ or $Q_{\ell}$ disappears in these final relations, as it should be. Vanishing of the exponents $\beta$ for $\ell > 2$ cases is a simple consequence of the fact that $\Psi^{(f)}_{\{s,p\}}$ has dimension zero for these cases.

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4. Discussion

This letter is only a begining of our project and many things should be clarified in the future. First, one has to find the detail of the Hilbert space, in particular, the character formula\(^1\) The character formula for (2.10b) seems especially interesting.

Secondly, it would be desirable to give representation for \(C = 0\) by finite number of (possibly) interacting fields. As far as using large \(N\) limit, we have infinite number of redundant degree of freedom. It may be problematic in the future computation. Such representation will be also important in removing some arbitrariness in the OPE computation we have met.

Thirdly, we have to make connection with the physical situation, such as quantum gravity or quantum Hall effects. So far, only the unitary representation was discussed. However, as we have seen, \(W_{1+\infty}\) algebra has much rich structure for the nonunitary cases. Is there any place where such representation becomes useful? Turbulence may be one of the candidate it suffers the dispersion of enstrophy through viscosity.

\(^1\) We understand that the character formula for \(C = 1\) case was independently obtained by Awata, Fukuma, Odake and Quano [13]. We would like to thank M. Fukuma for explaining their work. It helps us to understand structure of the null states.
References

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