1. Introduction

Several interesting phenomena are related to the discovery of Hawking radiation [1]. It is intriguing that black holes seem to obey laws of thermodynamics [2]. The information contained in the matter which made up the black hole is lost into the singularity. Hawking radiation appears in the evaporation of the hole, but the outgoing modes are not in a pure state; instead they are mixed with modes of the field that fall into the singularity. The precise significance of black hole thermodynamics, and its relation to the ordinary ideas of thermodynamics and information theory, are matters of debate.

Recently the discovery of 1+1 dimensional models for black holes [3, 4] has led to a more accurate understanding of the semiclassical features of black hole geometry and Hawking radiation. In particular the model of [4] (the RST model) can be exactly solved to yield the semiclassical geometry of a black hole formed by a shock wave of infalling matter, and evaporating by massless scalars to an endpoint (the 'thunderpop'). It may even be possible to obtain a complete quantum gravity plus matter description of the black hole evaporation process in 1+1 dimensions [5].

In this paper we study some features of the semiclassical geometry and Hawking radiation in semiclassical models. For the RST model of the evaporating hole we compute the Bogoliubov coefficients. We perform a point splitting calculation to compute the stress tensor at $T^+$. We also compute the stress tensor in the evaporating region using the anomalous trace of the matter stress tensor. The RST model is solved also for the case where the hole is formed by one shock wave, evaporation occurs for a time, and then a second shock wave increases the mass of the hole again. (This geometry is used for clarifying some aspects of the entropy produced by the hole, as discussed below.)

We then study the 'entropy of entanglement' of the Hawking radiation, by two methods. We can compute the density matrix obtained by tracing the field modes inside the horizon. This was the approach taken by Hawking and also carried out in [6] for the 1+1 dimensional case, and it yields a density matrix that is close to thermal after the initial stage of the collapse and formation of the hole. We consider the case where the hole has a finite lifetime (due to the evaporation) and thereby estimate the entropy of the entire radiation produced. We then compare this result to that obtained by using a calculation of Srednicki [7]. Srednicki considered a scalar field in flat Minkowski space in the vacuum state, and 'traced out' the degrees of freedom inside a ball of radius R.
The entropy of the reduced density matrix is the 'entropy of entanglement' between the region inside the ball and its complement. The entropy $S$ depends on $R$ and also on the ultraviolet cutoff, which gives the 'sharpness of separation' between the regions. In the one space-dimension case, the infrared cutoff also appears. We find that both for the 1+1 dimensional black hole and for the 3+1 dimensional hole the one space dimension result of Srednicki is the pertinent one to use, and the leading dependence of the entropy on the black hole mass is reproduced.

The plan of this paper is as follows. In section 2 the RST model is reviewed. In section 3 the Bogoliubov coefficients for a scalar field in the evaporating black hole background are computed. Section 4 contains the calculation of stress-tensor. Section 5 studies the two shock wave solution. We discuss the entropy of the Hawking radiation for 1+1 dimensional black holes in section 6, and for 3+1 dimensional black holes in section 7. Finally, a discussion is presented in section 8.

2. The RST model

The model of Russo, Susskind and Thorlacius (RST) [4] is a modified version of the model of two dimensional dilaton gravity coupled to quantum matter introduced in [3]. The key idea of RST was to introduce an additional counterterm which restores a global symmetry originally present in the classical dilaton gravity + matter action. This allowed them to solve the theory analytically in the large $N$ limit. The properties of the RST model have been extensively discussed in the literature [4, 8, 9] so we will just mention the facts we will need for later use.

The semiclassical effective action of RST can be written as follows

$$S = \frac{1}{2\pi} \int d^4x \sqrt{-g} [\left(e^{-2\phi} - \frac{N}{24}\phi R + 4e^{-2\phi}(|\nabla \phi|^2 + \lambda^2) - \frac{1}{2} \sum_{i=1}^{N} |\nabla \phi_i|^2] + R\phi] - N\int d^4x \sqrt{-g} \int d^2x' \sqrt{-g(x')} R(x,x') R(x''),$$

where $R$ is the 1+1 dimensional scalar curvature, $\phi$ is the dilaton field and $f_i, i = 1, \ldots, N$ are $N$ massless conformal scalar fields. $G(x,x')$ is the Green's function for the d'Alambertian in curved space. The constant $A$ plays the role of Planck mass.

The analysis of the semiclassical equations of motion that follow from the action (1) can be simplified by the following two steps. First, one can write the metric in the conformal gauge, given by

$$g_{\pm\mp} = -\frac{1}{2} e^{2\phi}, \quad g_{\pm\pm} = 0,$$

where $x^\pm = x^0 \pm x^1$. Secondly, one can make a field redefinition and introduce the fields

$$\Omega = \kappa^{-2} e^{-2\phi} + \frac{1}{4} \ln \frac{\kappa}{4},$$

$$\chi = \kappa e^{-2\phi} + \frac{1}{2} \ln(4\kappa),$$

where $\kappa \equiv \frac{N}{4}$. The coordinates $x^\pm$ can be fixed so that $\Omega = \chi$, so the dilaton field $\phi$ differs from the conformal factor of the metric $\rho$ by a constant. The matter fields are assumed to reflect from the strong coupling boundary $\Omega = \Omega_c = \frac{1}{4}$ [4]. The semiclassical equations can now be reduced to

$$\partial_\mu \partial_{\pm} \Omega = 0,$$

$$\partial_\mu \partial_\Omega = -\lambda^2,$$

$$\kappa \partial_\mu^2 \Omega = -\pi T_{\pm\pm} + \kappa \chi_\Omega (x^\pm).$$

Here $T_{\pm\pm}$ are the components for outgoing and ingoing matter energy of the stress-tensor, which is defined as follows. Since the classical matter action is written as

$$S_\chi = -\frac{1}{4\pi} \int d^4x \sqrt{-g} \sum_{i=1}^{N} |\nabla \chi|, \quad \text{(4)}$$

the stress tensor is

$$T_{\pm\pm} = -\frac{1}{2} \delta S_\chi \delta g_{\pm\pm}.$$

The components representing outgoing and ingoing matter are normalized as

$$T_{\pm\pm} = \frac{1}{2\pi} \sum_i \partial_\mu f_i \partial_\mu f_i,$$

in the conformal gauge.

The functions $\chi_\Omega(x^\pm)$ are fixed by boundary conditions. We assume that the incoming matter energy flux at $x^-$ vanishes sufficiently rapidly at early and late times. Then, $\chi_\Omega(x^\pm) = 1/4(x^\pm)^2$. The solution for the field $\Omega$ can now be found to be

$$\Omega = -\lambda^2 x^-(x^- + \frac{\pi}{\kappa\lambda^2} P_1(x^-)) + \frac{\pi}{\kappa\lambda} M(x^-) - \frac{1}{4} \ln(-4\lambda^2 x^+ x^-), \quad \text{(5)}$$

where

$$M(x^-) = \lambda \int_0^{x^-} du \ T_{++}(u),$$

$$P_1(x^+) = \int_0^{x^+} du \ T_{++}(u).$$

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Consider now an incoming matter shock wave that carries energy $M$. The stress tensor is then given by
\[ T_{\alpha\beta}(x^+) = \frac{M}{\lambda x^+} \delta(x^+ - x^+_0), \] (7)
which is the only non-vanishing component. We substitute this in the equations (6) above. Following [8], we set $\lambda x^+_0 = 1$.

After solving the RST equations one finds the following results. The Penrose diagram for the evaporating black hole spacetime is given in Fig. 1.

The spacetime is seen to be divided into three regions. The 'lowest' region I, before the incoming shockwave at $x^+_0$, is the linear dilaton vacuum, bounded from the left by the timelike strong coupling boundary $\Omega = \Omega_r$. The incoming shockwave forms the black hole by causing the boundary $\Omega = \Omega_r$ to become spacelike. It can be shown that the scalar curvature $R$ diverges at the spacelike portion of the boundary. The apparent horizon also forms after the incoming shockwave. Following [4], the apparent horizon is defined by the condition $\partial_\alpha \phi = 0$. After the black hole forms, it starts to evaporate, and the apparent horizon shrinks until it meets the singularity at the endpoint of evaporation. At the endpoint of evaporation a short (delta function) burst of negative energy is seen to emerge from the black hole. This is called the 'thunderpop' [4], and it travels along the null line $x^- = x^-_0$ to future null infinity. ($x^+_0$ is a light-cone coordinate of the endpoint to be specified later.) Thus the region II is bounded by the thunderpop and the incoming shockwave is the curved region of the evaporating black hole. After the thunderpop, the spacetime becomes again flat and the boundary $\Omega = \Omega_r$ becomes timelike. The corresponding region III is a linear dilaton vacuum.

In the linear dilaton vacuum region I the metric is
\[ ds^2 = \left(\lambda^2 x^+ x^-\right)^{-1} dx^+ dx^- .\]
We can write it as $ds^2 = -dy^+ dy^-$ using coordinates
\[ y^- = -\frac{1}{\lambda} \ln(-\lambda x^-) \]
\[ y^+ = \frac{1}{\lambda} \ln(\lambda x^+) - y_0^- . \]
(8)
The shift $y_0^-$ is introduced to set the origin of the coordinate $y^+$ to the point $A$ where the reflected ray of the event horizon (see Fig. 1) meets $I^-$. The event horizon, the singularity and the apparent horizon meet at point $E$, the end point of evaporation. In our conventions, its coordinates are
\[ (x^+_r, x^-_r) = \left(\frac{-\lambda M}{\kappa \lambda^2}, 1 - e^{-\frac{2\lambda M}{\kappa\lambda}}\right), \frac{\kappa}{4\pi M} \left(\lambda^2 x^+_r - 1\right) . \]
(9)
We can now specify what the shift $y_0^+$ is. In region I the reflecting boundary $\Omega = \Omega_r = \frac{1}{4}$ is the line
\[ y^+ = y^- + \frac{1}{\lambda} \ln \frac{1}{4} - y_0^+ . \]
Reflecting the line $x^- = x^-_0$ (off the boundary $\Omega = \Omega_r$) back to $I^+$, we find that the point $A$ has $x^+_A = -\frac{1}{\lambda} \ln(-\lambda x^-_0) + \frac{1}{\lambda} \ln \frac{1}{4} - y_0^+$. Setting $x^+_0 = 0$ yields $y_0^+ = -\frac{1}{\lambda} \ln(-4\lambda x^-_0)$.

The region II is the curved region of the evaporating black hole. It can be joined continuously but not smoothly with region III along the line $x^- = x^-_0$, with region III being again a linear dilaton vacuum, but with the coordinate $x^-$ shifted. From the joining conditions the metric in III can be found to be
\[ ds^2 = \left(\lambda^2 x^+ (x^- + \frac{\lambda M}{\lambda^2 x^+})\right)^{-1} dx^+ dx^- . \]
If one makes the coordinate transformation
\[ \sigma^+ = \frac{1}{\lambda} \ln(\lambda x^+) \]
\[ \sigma^- = -\frac{1}{\lambda} \ln\left(\frac{\lambda x^- + \frac{\lambda M}{\lambda x^+}}{\frac{\lambda M}{\lambda x^+}}\right) , \]
this metric becomes $ds^2 = -d\sigma^+ d\sigma^-$. The coordinate $\sigma^-$ has been normalized so that the thunderpop is at $\sigma^- = 0$. The reflecting boundary $\Omega = \Omega_r$ in region III is at $\sigma^+ = \sigma^- + \frac{1}{\lambda} \ln(\lambda x^+)$.

The metric in II becomes asymptotically flat near $I^+$. We can extend the coordinates $\sigma^\pm$ from region III into region II in the neighbourhood of $I^+$. Then the metric in region II also has the asymptotic form $ds^2 \to -d\sigma^+ d\sigma^-$ near $I^+$. Thus, $\sigma^+$ are the physical coordinates near $I^+$.

Finally, we identify some points of interest in the Penrose diagram. The point $A$ where the reflection of the null line $x^- = x^-_0$ meets $I^-$ we already set to be at $y^+_A = 0$. The point $B$ is the projection of the end point $E$ along a null ray to past null infinity. We find it to be located at $y^+_B = 4\lambda M$. The point $C$ is defined by projecting the point where the apparent horizon meets the incoming shockwave along a null ray to future null infinity. It is at $\sigma^- = -\frac{1}{\lambda} \ln (\frac{\lambda M}{4\pi M} (\exp \frac{4\lambda M}{\lambda^2 M} - 1))$. Since $M \gg \kappa \lambda$, to a good accuracy $\sigma^- = -\frac{4\lambda M}{\lambda^2 M}$. The absolute value of $\sigma^-$ is thus the total (retarded) time of evaporation of the black hole.
3. Bogoliubov transformations

In this section we calculate the Bogoliubov coefficients for the relation between the natural mode functions in the 'in' region close to \( \mathcal{I}^- \) and the 'out' region close to \( \mathcal{I}^+ \). In [1] the Bogoliubov coefficients were estimated for modes travelling close to the horizon that forms in a spherical collapse of a star. In [6] the Bogoliubov coefficients were computed for the 1+1 dimensional eternal black hole geometry, i.e. without taking into account the backreaction on the metric due to the Hawking evaporation. Our notations follow those in [6] where we refer to for all introductory steps.

We choose the following form for modes at \( \mathcal{I}^- \), \( \mathcal{I}^+ \) respectively

\[
\begin{align*}
  u_\omega &= \frac{1}{\sqrt{2\omega}} e^{-i\omega \sigma^+} \quad \text{(in)} \\
  v_\omega &= \frac{1}{\sqrt{2\omega}} e^{i\omega \sigma^-} \quad \text{(out)},
\end{align*}
\]  

where the normalization factor is in agreement with the form (4) of the matter action.

We define the Bogoliubov coefficients \( \alpha_{\omega} \) and \( \beta_{\omega} \) for the relation between the 'in' and 'out' modes as follows

\[
v_\omega = \int_0^\infty d\omega' \left| \alpha_{\omega'} u_{\omega'} + \beta_{\omega'} u_{\omega'}^* \right|.
\]  

To compute the Bogoliubov coefficients, we pull the 'out' mode back to \( \mathcal{I}^- \). First we divide the mode into two parts at the point \( \sigma^- = 0 \). The 'upper' piece \( u_{\omega} = \frac{1}{\sqrt{2\omega}} e^{-i\omega \sigma^-} \delta(\sigma^-) \) reflects from the timelike boundary in region III without experiencing any blueshift. At \( \mathcal{I}^- \) it becomes

\[
u_\omega = \frac{1}{\sqrt{2\omega}} e^{-i\omega (\sigma^+ - y^+_0)} \delta(y^+ - y^+_0),
\]

where \( y^+_0 \) was defined in the previous section.

However, the 'lower' piece \( u_{\omega} = \frac{1}{\sqrt{2\omega}} e^{i\omega \sigma^-} \delta(-\sigma^-) \) gets distorted. It first experiences a blueshift when pulled back to region I. This is done by replacing the coordinate \( \sigma^- \) with the coordinate \( y^- \) using the relation

\[
\sigma^- = \frac{1}{\lambda} \ln \left( -e^{-\lambda y^-} + \pi M / \lambda \kappa \right).
\]

Then we reflect the mode from the timelike boundary in region I back to \( \mathcal{I}^- \), by replacing \( y^- \) with

\[
y^- = y^+ + y_0^+ - \frac{1}{\lambda} \ln \frac{1}{4} = y^+ - \frac{1}{\lambda} \ln(-\lambda z^+) .
\]

The final relation between the coordinates \( \sigma^- \) and \( y^+ \) can then be written as

\[
\begin{align*}
  \sigma^- &= \frac{1}{\lambda} \ln \frac{\lambda z^+ e^{-\lambda y^+} + \pi M / \lambda \kappa}{\lambda z^+ + \pi M / \lambda \kappa} \\
  &= \frac{1}{\lambda} \ln(\lambda \Delta \left( e^{-\lambda y^+} - C \right)) ,
\end{align*}
\]  

where

\[
\begin{align*}
  \lambda \Delta &= \frac{\lambda z^+}{\lambda z^+ + \pi M / \lambda \kappa} = e^{\pi M / \lambda \kappa} ,
  \\
  C &= \frac{-\pi M / \lambda \kappa}{\lambda z^+} .
\end{align*}
\]

As an important aside, notice that (13) implies a relation between a small distance \( ds^- \) at \( \mathcal{I}^+ \) centered at \( \sigma^- \) and the corresponding small distance \( ds^+ \) at \( \mathcal{I}^- \) which results from mapping the former distance along null rays which reflect from the boundary back to the past null infinity. This relation can be found to be

\[
ds^+ = \{1 + (e^{\pi M / \lambda \kappa} - 1)e^{\pi M / \lambda \kappa} \}^{-1} ds^- .
\]

The result (15) tells us that a distance \( ds^- \) at \( \mathcal{I}^+ \) (before the endpoint of evaporation) maps to a much smaller distance \( ds^+ \) at \( \mathcal{I}^- \), with the 'squeeze factor' becoming exponentially larger as the black hole evaporates. This observation will turn out to be crucial in applying the Srednicki calculation for black holes, as will be discussed in the section 6.

Returning back to the behaviour of the modes, (13) tells us that the 'lower' part of \( u_{\omega} \) becomes

\[
v_\omega = \frac{1}{\sqrt{2\omega}} e^{i\omega \sigma^-} \delta(\sigma^-) \exp \left\{ \frac{\lambda}{\omega} \ln(\lambda \Delta \left( e^{-\lambda y^+} - C \right)) \right\} \delta \left( -y^+ \right)
\]

when pulled back to \( \mathcal{I}^- \).

It is now straightforward to proceed to find the Bogoliubov coefficients. The result is

\[
\alpha_{\omega} = \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int d\omega' \exp \left\{ i\omega' \sigma^- \right\} \left\{ e^{i\omega' y^+} \int_0^\infty ds^+ e^{i\omega' s^+} \right\}
\]

\[
+ e^{-i\omega' y^+} \int_0^\infty ds^- e^{-i(\omega + \omega') s^+} \right\} \}
\]

\[
\beta_{\omega} = \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int d\omega' \exp \left\{ i\omega' \sigma^- \right\} \left\{ e^{-i\omega' y^+} \int_0^\infty ds^+ e^{-i(\omega + \omega') s^+} \right\}
\]

\[
+ e^{i\omega' y^+} \int_0^\infty ds^- e^{i(\omega + \omega') s^+} \right\} \} .
\]
These results resemble the ones of Giddings and Nelson (GN), with the following three differences: 1) there are additional terms resulting from the 'upper' part of the mode \( v_\omega \), i.e., the part after the endpoint of evaporation, 2) \( \lambda \Delta \) is different, and 3) we have \( C = 1 - e^{-\Delta s/M} \) while GN had \( C = 1 \). (Of course, \( C \approx 1 \) since \( M/\lambda x > 1 \). However, the small difference turns out to be significant if one tries to investigate the behaviour of the modes very near the endpoint.)

The integrals in the above expressions can be evaluated further. Substituting first \( x = e^{\phi t} \) and then \( t = Cx \), we get

\[
\alpha_{\omega,\nu} = \frac{1}{2\pi} \left( \frac{\omega^\prime}{\omega} \right)^{1/2} \left\{ \left( \lambda \Delta \right)^{i\omega/\lambda} C \left( \omega/\nu \right)^{i\lambda} \int_0^\infty dt \left( 1 - t \right)^{i\omega/\lambda} t^{-1-i\left( \omega - \omega' \right)/\lambda} \right\} \left( \frac{-i}{\omega - \omega' - i\epsilon} \right) \eta. \tag{18}\]

The remaining integral can be identified as an Incomplete Beta function. The coefficient \( \beta_{\omega,\nu} \) is computed similarly. The final expressions are

\[
\alpha_{\omega,\nu} = \frac{1}{2\pi} \left( \frac{\omega^\prime}{\omega} \right)^{1/2} \left\{ \left( \lambda \Delta \right)^{i\omega/\lambda} C \left( \omega/\nu \right)^{i\lambda} B \left( -\frac{i\omega}{\lambda} + \frac{i\omega^\prime}{\lambda} + \epsilon, i + \frac{i\omega^\prime}{\lambda} \right) \right\} \left( \frac{-i\lambda e^{i\omega^\prime} \left( \omega - \omega' - i\epsilon \right)^{-1}}{\omega - \omega' - i\epsilon} \right) \tag{19}\]

\[
\beta_{\omega,\nu} = \frac{1}{2\pi} \left( \frac{\omega^\prime}{\omega} \right)^{1/2} \left\{ \left( \lambda \Delta \right)^{i\omega/\lambda} C \left( \omega/\nu \right)^{i\lambda} B \left( -\frac{i\omega}{\lambda} - \frac{i\omega^\prime}{\lambda} + \epsilon, i + \frac{i\omega^\prime}{\lambda} \right) \right\} \left( \frac{-i\lambda e^{i\omega^\prime} \left( \omega + \omega' - i\epsilon \right)^{-1}}{\omega + \omega' - i\epsilon} \right). \tag{20}\]

Note that in the semiclassical approximation the thunderpop is a delta function at \( \Delta t \), thus there is a part of the field modes that is not captured by the modes \( e^{-\nu \phi}/M \) at \( \Delta t \) for any finite range of \( \omega \). Thus the Bogoliubov coefficients computed below need to be supplemented with an infinite frequency component to completely describe the field at \( \Delta t \).

Let us now turn to the issue of examining the nature of the radiation by studying the approximate behaviour of the Bogoliubov coefficients. We expect outgoing thermal radiation at constant temperature \( T \) to be seen in the region \( \sigma^- < -\frac{1}{\Delta s/M} \) of \( \Delta t \), except perhaps at the beginning and very end of the evaporation process. The coordinate \( y^+ \) corresponding to this region is very small, so we can approximate

\[
\ln \left[ \lambda \Delta \left( e^{-\lambda y^+} - C \right) \right] \approx \ln \left[ 1 - e^{M/\lambda y^+} \right]. \tag{21}\]

If we are not too close to the endpoint, the term \( 1 \) in (20) is negligible. For the essence of Hawking radiation, we have the frequency \( \omega \sim \lambda \) at \( \Delta t \) and very high frequencies \( \omega \sim \lambda \Delta s/M + \lambda y^+ \) at \( \Delta t \). For such \( \omega, \omega \) we can ignore the second integrals in the expression (17) for \( \alpha_{\omega,\nu} \). We therefore get

\[
\alpha_{\omega,\nu} \approx \frac{1}{2\pi} \left( \frac{\omega^\prime}{\omega} \right)^{1/2} \left\{ \left( \lambda \Delta \right)^{i\omega/\lambda} C \left( \omega/\nu \right)^{i\lambda} \right\} \left( \frac{-i\lambda e^{i\omega^\prime} \left( \omega - \omega' - i\epsilon \right)^{-1}}{\omega - \omega' - i\epsilon} \right) \eta. \tag{21}\]

This means that we get the thermal relation (see [6])

\[
\alpha_{\omega,\nu} \approx -e^i\omega^\prime \beta_{\omega,\nu}. \tag{22}\]

and we see that the outgoing radiation is thermal with constant temperature \( T = \frac{\lambda}{2\pi} \) in the region \( \sigma^- < -\frac{1}{\Delta s/M} \) at \( \Delta t \).

4. The stress tensor

Since we have worked out the relations between the 'in' and 'out' modes, we can easily calculate the VEV of the stress-energy at \( \Delta t \), i.e., \( \langle 0, in \mid T_{\nu,\nu}(-\sigma^-) \mid 0, in \rangle \rangle \) by using the point-splitting method. The non-vanishing component of the stress-energy is the \( \sigma \) component. Since we are computing the 'in' VEV at \( \Delta t \), we need first the form of the 'in' mode at \( \Delta t \). For \( \sigma^- \leq 0 \) it is

\[
u_{\sigma} = \frac{1}{2\pi} \exp \left[ \frac{i\omega}{\lambda} \ln \left[ \frac{1}{\Delta s/M} \right] \right] \delta \left( -\sigma^- \right). \tag{23}\]

Again, for \( \sigma^- > 0 \) (after the endpoint) there is no redshift and it is then trivial to see that the stress-tensor vanishes in this region. We concentrate only in the region before the endpoint. Using the point-splitting method, we first calculate

\[
\left( T_{\nu,\nu}(-\sigma^-) \right) = \lim_{\sigma^- \to -\sigma^-} \frac{1}{2\pi\sigma^-} \frac{\partial}{\partial \sigma^-} \int_0^\infty d\omega \nu_{\sigma} \left( \sigma^- \right) \tag{24}\]

where the LHS means the VEV with respect to the 'in' vacuum. (We use this notation in the following.) For \( \sigma^- < 0 \) the step functions are just constant and can be ignored. Taking the partial derivatives and making a series expansion in \( (\sigma^- - \sigma^\nu) \) yields then a term which diverges quadratically in the limit, and a finite term. The divergent term must be subtracted (it is just the usual vacuum divergence) and the finite term gives the renormalized value for the stress-energy:

\[
\left( T_{\nu,\nu}(-\sigma^-) \right)_{ren} = \frac{\lambda^2}{4\pi} \left[ 1 - \frac{1}{(e^M - 1)\lambda \nu - 1} \right]. \tag{25}\]
This result is for one conformal scalar field, for \( N \) fields it must be multiplied by \( N \). Note that in the region \( \sigma^- \in (-\frac{4\pi M}{\lambda}, 0) \) we can approximate

\[
(T_\sigma)(\sigma^-)^\text{ren} \approx \frac{\lambda^2}{48\pi} - \frac{\pi}{12}(T_H)^2 ,
\]

which is the correct value for outgoing thermal radiation at temperature \( T_H = \frac{\lambda}{8\pi} \).

(Comment: if one would use the approximate behaviour of the mode very near the end point, one would find that \((T_\sigma)(\sigma^-)^\text{ren} \to 0\) at roughly Planck distance from the end point. One should not, however, trust such a local treatment when dealing with modes.)

The above formula gave \((T_\sigma)(\sigma^-)\) only at \( \mathcal{I}^+ \). However, in \( 1+1 \) dimensions it is easy to calculate \((T_\mu\nu)(\sigma^-)^\text{ren} \) everywhere by using the trace anomaly, integrating the covariant conservation equation and applying the boundary conditions (see [10] for details). Using this route, we end up with the following results for the stress tensor everywhere in region II:

\[
(T_\mu\nu)(\sigma^-)^\text{ren} = \frac{N}{48\pi} \left( e^{-2\rho} + \frac{1}{4} \right) \left( \frac{\rho}{\sqrt{\lambda}^2} + \frac{1}{4} \right)^2 - 1
\]

\[
(T_\mu\nu)(\sigma^+)^\text{ren} = \frac{N}{48\pi} \left( e^{-2\rho} + \frac{1}{4} \right) \left( \frac{\rho}{\sqrt{\lambda}^2} + \frac{1}{4} \right)^2 - 1
\]

\[
(T_\mu\nu)(\sigma^-)^\text{ren} = -\frac{N\lambda^2}{24\pi} \left( e^{-2\rho} + \frac{1}{e^{\rho^2} - 1} \right) \left[ \frac{\rho}{\sqrt{\lambda}^2} + \frac{1}{4} \right] \left( \frac{\rho}{\sqrt{\lambda}^2} + \frac{1}{4} \right)^2 - 1
\]

where \( \rho \) is given implicitly via the equation

\[
e^{-\rho} = \lambda^2 \left( \frac{\rho}{\sqrt{\lambda}^2} + \frac{1}{4} \right) \left( \frac{\rho}{\sqrt{\lambda}^2} + \frac{1}{4} \right)^2 - 1.
\]

In the above the stress tensor is given in the 'Kruskal' coordinates \( x^+ \) since one is generally interested in its behaviour in both sides of the apparent horizon.

5. Two incoming shock waves

In this section we present the solution of the RST equations in the case of two incoming shock waves. We use this geometry later in section 6 where we discuss the entropy of the black hole. We want to first form a black hole of mass \( M_B \), which then starts to evaporate. Then the second shock wave at a later time carries additional energy to the black hole. For instance, it could restore the black hole back to its original mass. The question then is: How much is the entanglement entropy after the evaporation process? If the second shock wave carries just enough energy to restore the black hole back to its original mass, but not more, is this entanglement entropy related to Bekenstein entropy? We discuss this question in section 6, here we will just derive the results that we will use.

Recalling how we defined the incoming shock wave, it is clear that for two incoming shock waves we should replace \( T_\mu\nu(\sigma^-) \) with

\[
T_\mu\nu(\sigma^+) = \frac{M_B}{\lambda x_0^2} \delta(x^+ - x_0^+) + \frac{M_1}{\lambda x_1^2} \delta(x^+ - x_1^+).
\]

The first term on the RHS is the first shock wave at \( x_0^+ \) (we set again \( \lambda x_0^2 = 1 \)) carrying energy \( M_B \) and the second term is a later shock wave at \( x_1^+ \) carrying energy \( M_1 \). We do not specify \( x_1^+ \), \( M_1 \); here, but \( x_0^+ \) should be chosen so that the black hole formed by the first shock wave has not yet evaporated away when the second shock wave reaches it. On could think of \( M_1 \) being the energy needed to restore the black hole back to its original size, but for now that is not essential.

The spacetime curve for the apparent horizon can be found by solving the equation \( \partial_+ \Omega = 0 \) (this follows from setting \( \partial_+ \phi = 0 \) for the dilaton field; which is the RST definition of the apparent horizon). The solution is

\[
x^+ = \frac{1}{4\lambda} \left( x - \frac{M_B}{\lambda x_0^2} + \frac{M_1}{\lambda x_1^2} \right) \delta(x^+ - x_0^+) \delta(x^+ - x_1^+).
\]

for \( x^+ > x_0^+ \). The boundary of the spacetime is the critical line \( \Omega = \Omega_* = \frac{1}{4} \), which defines another curve \( x^+ = x^+(\sigma^-) \). The final endpoint of evaporation is where the above two curves meet. We find it to be located at

\[
x^+_* = \frac{\pi}{4\lambda x_0^2} (M_B + M_1) (1 - e^{-4\lambda (x^+_* - x_0^+)})^{-1}
\]

\[
x^+_* = \frac{\pi}{4\lambda x_0^2} (M_B + M_1) (e^{4\lambda (x^+_* - x_0^+)})^{-1}.
\]

We can now join the region II with the linear dilaton vacuum in region III with an appropriate shift of the coordinate \( x^- \). For the metric in region III, we find

\[
dx^2 = \frac{dx^+ dx^-}{\lambda^2 x^+(x^+ + x_0^+ (M_B + M_1 / \lambda x_0^2))^2}.
\]

This in turn tells us how to define the coordinates \( \sigma^+ \) which are the 'physical' coordinates near \( \mathcal{I}^+ \). We define

\[
\sigma^+ = \frac{1}{\lambda} \ln(\lambda x^+).
\]
\[ \sigma^- = \frac{1}{\lambda} \ln \left( \frac{\lambda x^+ - \lambda x^-}{\lambda x^+ + \lambda x^-} \right) \]

On the other hand, we still have the 'physical' coordinates in region 1

\[ y^+ = \frac{1}{\lambda} \ln(\lambda x^+) - y_0^+ \]  
\[ y^- = -\frac{1}{\lambda} \ln(-\lambda x^-) \]

where \( y_0^+ = -\frac{1}{\lambda} \ln(-4\lambda x^+) \), with \( x^+ \) given now by (31). The reflecting boundary in region 1 is the line \( y^+ = y^- + \frac{1}{\lambda} \ln \left( \frac{1}{1} - y_0^- \right) \).

It is straightforward to see that the relation between the coordinates \( \sigma^- \), \( y^+ \) now becomes

\[ \sigma^- = -\frac{1}{\lambda} \ln[\lambda \Delta'(e^{-\lambda y^+} - C')] \]

where

\[ \lambda \Delta' = e^{4\pi(M_0 + M_1)/\lambda \hbar} \]
\[ C' = 1 - e^{-4\pi(M_0 + M_1)/\lambda \hbar} \]

As we noticed in section 3, this implies a relation between a distance \( ds^- \) at \( I^+ \), centered at \( \sigma^- \) and the corresponding small distance \( dy^+ \) at \( I^- \). In the two shock wave case, the relation is

\[ dy^+ = \left\{ 1 + \left( e^{4\pi(M_0 + M_1)/\lambda \hbar} - 1 \right)e^{\lambda x^-} \right\}^{-1} ds^- \]

Again, a distance \( ds^- \) in \( I^+ \) (before the endpoint of evaporation) maps to a much smaller distance \( dy^+ \) in \( I^- \) and this 'squeeze factor' becomes exponentially large as the black hole evaporates. In the present case, the 'squeeze factor' reaches the value \( e^{4\pi(M_0 + M_1)/\lambda \hbar} \) which exceeds the value \( e^{4\pi M_0/\lambda \hbar} \) that would be obtained in the absence of the second shock wave. The result (36) will be used in the next section.

6. Entropy for 1+1 dimensional black holes

As argued by Hawking, the process of pair creation by the gravitational field of the black hole creates a state which is 'mixed' between the regions exterior and interior to the horizon. If we compute the reduced density matrix that describes the field in the exterior region, then the entropy computed for this density matrix gives the 'entropy of entanglement' [11] between the interior and exterior regions of the hole. If we do not take into account the backreaction from the created radiation then an infinite number of particle pairs are produced and the entropy of entanglement will be found to be infinite also. But in the simple RST model we can estimate the produced entropy for the semiclassical geometry that includes backreaction. We shall do this in two ways:

1. We directly compute

\[ S = -Tr[p \ln \rho] \]

where \( p \) is the reduced density matrix describing the field in the exterior region.

Here we use the fact that \( p \) is to a good approximation a thermal density matrix.

2. We use the result of Srednicki described in the introduction. Thus a complete spacelike hypersurface is considered, and the part corresponding to the interior of the horizon is assumed to be, effectively, the traced over region considered in this approach.

1. The essential idea is to define reasonably localised regions on \( I^+ \) such that the density matrix can be described as approximately thermal in those regions. Hawking presented this analysis for the 3+1 dimensional black hole [1]; it was worked out explicitly for the 1+1 case in [6] (without backreaction). We follow the notations of [6] in the following. Define a complete set of orthonormal wavepackets on \( I^+ \):

\[ \psi_{jn} = e^{-j/2} \int_{\mu_{jn}}^{(j+1)\mu_{jn}} d\mu e^{\epsilon \omega_{jn} \mu} \psi_{\omega_{jn}} \]

where \( j = 0,1,2,\ldots \) and \( n \) are integers. The wavepacket \( \psi_{jn} \) is peaked about \( I^+ \) coordinate \( \sigma^- = 2\pi n/\epsilon \), has a spatial width \( \sim \epsilon^{-1} \) and a frequency \( \omega_{jn} \approx j \epsilon \). In this basis the reduced density matrix obtained by tracing out the field inside the black hole is

\[ \rho = N^{-1} \sum_{(n_m)} e^{-j/2} \sum_{n} \psi_{jn}(n_{jn})|\langle n_{jn} \rangle \rangle \langle \langle n_{jn} \rangle \|. \]

This has the form of a thermal density matrix. We can write

\[ \rho = \prod_{jn} \rho_{jn} \]

with

\[ \rho_{jn} = N_{jn}^{-1} \sum_{n_{rn}=0}^{n_{rn}} \tilde{r}(n_{rn})|n_{jn}\rangle \langle n_{jn}| \]

\[ r(n_{jn}) = e^{-\frac{j}{2} \omega_{rn} \epsilon} \]
We get
\[ S = \sum_{j} S_{jn} \] \hspace{1cm} (43)
where
\[ S_{jn} = -Tr_{\{\rho_{jn}, \ln \rho_{jn}\}} = - \sum_{n_{x}=0}^{\infty} r(n_{x}) \ln r(n_{x}). \] \hspace{1cm} (44)

A brief computation gives
\[ S_{jn} = \beta \omega_{j} (e^{\beta \omega_{j}} - 1)^{-1} - \ln(1 - e^{-\beta \omega_{j}}). \] \hspace{1cm} (45)
Thus
\[ S_{n} = \sum_{j} S_{jn} = \sum_{j=0}^{\infty} \{\beta \omega_{j} (e^{\beta \omega_{j}} - 1)^{-1} - \ln(1 - e^{-\beta \omega_{j}})\} \]
\[ \rightarrow \int_{0}^{\infty} d\eta \{\beta \jmath \epsilon (e^{\beta \epsilon} - 1)^{-1} - \ln(1 - e^{-\beta \epsilon})\} = \frac{\pi^{2}}{36 \epsilon}. \] \hspace{1cm} (46)

The entropy from \( N \) evaporating fields is
\[ S = N \sum_{n} S_{n}. \] \hspace{1cm} (47)

The separation between wavepackets (38) is \( \Delta \sigma = 2\pi / \epsilon. \) Thus in time \( T \) the number \( n \) ranges from \( n = 1 \) to \( n = c T / 2 \epsilon. \) From (46, 47) we get
\[ S = \frac{N \pi T}{6 \beta}. \] \hspace{1cm} (48)

The total evaporation time is \( T = \frac{M}{\lambda}. \) This gives the estimate of the total entropy created in the Hawking radiation:
\[ S_{\text{total}} = \frac{4 \pi M}{\lambda}. \] \hspace{1cm} (49)

Note that this is twice the Bekenstein entropy, which for the 1+1 dimensional hole is \( S_{\text{BH}} = 2 \pi M / \lambda. \) The result (49) is the entropy of a thermal distribution of bosons at temperature \( \beta^{-1} \) and with energy \( E = M, \) which in one dimension is given by \( S = 2 \beta E. \)

For a discussion of how the entropy of radiation at \( T^{+} \) relates to the Bekenstein entropy, see [12].

2. We now investigate the application of Srednicki's result to the black hole. We split the discussion into 3 parts:

- (i) We recall the result of [7], and discuss the issue of infrared divergence.
- (ii) We discuss how this result may be applied to the black hole, after making some plausible arguments for the required modifications.
- (iii) We carry out the calculations for the entropy.

(i) The computation of Srednicki for the one space dimension case may be described as follows.\(^1\) Consider a free scalar field on a 1-dimensional lattice, with lattice spacing \( a. \) Let this field be in the vacuum state. We select a region of length \( R \) of this lattice and trace over the field degrees of freedom outside this region. This gives a reduced density matrix \( \rho, \) from which we compute \( S = -Tr\{\rho \ln \rho\} \) which is the entropy of entanglement of the selected region with the remainder of the lattice. It is immaterial whether we trace over the interior or exterior of the selected region; since the field was in a pure state overall the entanglement entropy is the same in both cases. This entropy is given by [7]
\[ S = \kappa_{1} \ln(R/a) + \kappa_{2} \ln(\mu R), \] \hspace{1cm} (50)
for one scalar field. (For \( N \) species the result must be multiplied by \( N. \)) Here \( \mu \) is an infrared cutoff.

The infrared term is very sensitive to boundary conditions. As an example consider taking a periodic lattice, and let the scalar field be periodic as well. If the field is massless, then the zero mode of the field varies over an infinite range, in the vacuum state. If we trace over the degrees of freedom in a subregion of the lattice, then the mean value of the field inside is correlated to the mean value outside, but this mean value can take on an infinite range of values. The entanglement entropy will thus be infinite.

If we take antiperiodic boundary conditions for the scalar field then the zero mode does not exist. With this choice (50) becomes [14]
\[ S = \frac{1}{3} \ln(R/a) + \frac{1}{6} \ln(\mu R) \] \hspace{1cm} (51)
where \( I \) is the infrared cutoff coming from the finite size of the lattice. Another way to kill the zero mode of the scalar field is to have a vanishing boundary condition for the field at say \( x = 0. \) Let the traced over region extend from \( x = x_{1} \) to \( x = x_{2}. \) Now modes with wavelength much greater than \( x_{2} \) effectively vanish over the interval \( (x_{1}, x_{2}), \) so they do not serve to entangle this region with the remainder of the line.

\(^1\) Similar issues were studied in [13].
$z > 0$. Thus the entanglement entropy will be finite without the need for an explicit infrared regulator. A third way of dealing with the zero mode is to simply assume that the field has a small mass $\mu$. Then no other infrared regulation should be needed and the results should not be sensitive to choice of boundary conditions.

(ii) For the black hole, we start by considering a complete spacelike hypersurface through the evaporation geometry, described as follows. Starting at spatial infinity ($\sigma^- = -\infty$), we move near $I^+$ to a point with $\sigma^- = \sigma^+_1 < 0$. (The black hole vanishes at $\sigma^- = 0$.) Then we smoothly bend this hypersurface so that it enters the horizon and reaches the timelike segment of the line $\Omega = \Omega_\infty$. (This timelike segment occurs before this critical line becomes the singularity.) Thus the hypersurface is kept spacelike all through, and avoids the singularity by passing below it. An observer collecting radiation far from the black hole will see the part of this hypersurface that lies along $I^+$, and we would wish to trace over the part that cannot be observed from outside the black hole. This would give the reduced density matrix, and thereby the entropy of entanglement between the field inside and outside the hole. Let us be more precise about what we take as the ‘observed’ part. Suppose that the observer at $\sigma^+_1$ carries an instrument with her which she uses to collect the outgoing radiation. At first, near $\sigma^- = -\infty$ nothing comes out from the black hole. The observer has to wait for quite a while before the black hole starts to radiate. At some point the radiation starts and becomes approximately thermal. We have discussed earlier that this happens roughly at $\sigma^- = -\frac{\text{Area}}{4\pi\hbar}$. Correspondingly, around this point the observer turns on her instrument. The observer then collects radiation up to some retarded time $\sigma^+$, when she turns her instrument off again. Thus the part of the hypersurface between $\sigma^- \equiv \sigma^+_1$ and $\sigma^+_1$ corresponds to observations, and the rest of it will be traced over. Notice that the observer can neither start nor stop collecting radiation at an instant, but there will be a short time scale $d\sigma^-$ which she needs to turn on or shut off her instrument in a proper way. The time interval needed should be sufficiently short to give a good accuracy for specifying the turn-on point $\sigma^+_1$ and the shut-off point $\sigma^+_1$, on the other hand it should not be so small that it creates disturbances in the matter field that generate radiation comparable to the Hawking radiation collected. It is reasonable to assume that the time intervals $d\sigma^- (\sigma^-) \equiv d\sigma^- (\sigma^+_1)$ needed should be given by the typical wavelength in the outgoing thermal radiation. Thus we can assume that $d\sigma^- \sim \beta_H \sim \frac{1}{\xi}$ (our result for the entropy of the hole will turn out not to depend on this choice).

We split the contribution of different modes to the entropy, as follows: a) the modes of wavelength $\omega^{-1} \gg M/\lambda^2$: these modes are effectively constant over the observed region $(\sigma^- \equiv \sigma^+_1)$, so they may be taken as a contribution to the 'zero mode'; b) the leftmoving modes with $\omega^{-1} \lesssim M/\lambda^2$; c) the rightmoving modes with $\omega^{-1} \sim M/\lambda^2$.

Modes of type a) will give a divergence in the entanglement entropy even in flat Minkowski space (without black hole). This happens when the mass $\mu$ of the field is taken to zero, or (if the field is massless) as the observed part of the hypersurface is taken further and further away from the line $\Omega = \Omega_\infty$, where the field vanishes. Since we are interested in entropy of entanglement of the rightmoving Hawking radiation (which occurs over a distance $M/\lambda^2$), we subtract this (diverging) contribution arising from the large wavelength modes $\omega^{-1} \gg M/\lambda^2$. We also ignore the leftmoving modes b), as they do not contribute to the Hawking radiation. The rightmoving modes c) are of interest to us, but after particle pairs have been created, the state of the field is not the vacuum state for the geometry on the spacelike hypersurface under consideration. Srednicki’s result, on the other hand, applies to a vacuum state for the field. The essential idea is to follow the rightmoving (i.e. outgoing) modes of the field back from $I^+$, through the collapsing matter to the line $\Omega = \Omega_\infty$, where they reflect to left moving modes that can be followed back to $I^-$. Here these left movers are in the vacuum state, so that we may apply a Srednicki type approach to estimate the entanglement entropy in these modes.

As we follow the radiation modes back to $I^-$ in the manner indicated above, we observe the following. The region $\sigma^- \in (\sigma^- \equiv \sigma^+_1)$ in $I^+$ corresponds to a region $y^+ \in (y^+(\sigma^-), y^+(\sigma^+_1))$ in $I^-$, where the relation between $\sigma^+_1, i = 0, 1$ and $y^+(\sigma^+_1), i = 0, 1$ is given by the formula (13) in section 3. Thus, the latter region is the region of starting points for ingoing rays which will experience redshift and give rise to the collected radiation in the 'observed' region of $I^+$. Also, this region is separated from the rest of $I^-$ by 'cuts' of size $dy^+(\sigma^-)$ and $dy^+(\sigma^+_1)$, the size of which follows from the size of the corresponding cuts near $I^+$ by the relation (15) given in section 3. Now we can apply the result of [7]. We disentangle the finite region $y^+ \in (y^+(\sigma^-), y^+(\sigma^+_1))$ from the rest of $I^-$ by cuts of size $\gamma = dy^+(\sigma^-), \alpha_1 = dy^+(\sigma^+_1).$ Ignoring 'zero modes' and the leftmovers, we expect to create an entropy for each scalar field (see discussion in section 8 below)

$$S = \kappa_3 \ln \left( \frac{R}{\alpha_0} \right) + \kappa_3 \ln \left( \frac{R}{\alpha_1} \right) ,$$

(52)

where $R = y^+(\sigma^+_1) - y^+(\sigma^-)$ and $\kappa_3 = \frac{1}{\lambda^2}$. (A factor of $\frac{1}{3}$ because we are considering only rightmovers and another factor of $\frac{1}{3}$ because the contribution to $S$ is separated
over the two 'cuts'.)

(iii) Let us rewrite (52) as

$$2\kappa_{3} \ln(R\lambda) + \kappa_{3} \ln\left(\frac{1}{\sigma_{0}\lambda}\right) + \kappa_{3} \ln\left(\frac{1}{\lambda_{1}\lambda}\right). \quad (53)$$

We substitute $a_{i} = dy^{i}(\sigma_{i})$, $i = 0, 1$ given by the relation (15):

$$a_{1} = \left\{1 + \left(e^{4\lambda M/\kappa_{3}} - 1\right)e^{4\lambda M/\kappa_{3}}\right\}^{-1}d\sigma^{-} \approx e^{-4\lambda M/\kappa_{3}}e^{-\lambda_{1}\lambda} \frac{1}{\lambda},$$

$$a_{0} \approx \frac{1}{\lambda}.$$  

The former approximate formula is valid for $\sigma^{-} \in (-\frac{4\lambda M}{\kappa_{3}}, 0)$ and the latter just results from the fact that the redshift is negligible for the rays at earlier times.

After substitution we get (ignoring the infrared cut-off term)

$$S = \kappa_{3} \ln\left(e^{4\lambda M/\kappa_{3}} e^{4\lambda M/\kappa_{3}}\right) + \kappa_{3} \ln\left(\frac{1}{\lambda_{1}}\right) + 2\kappa_{3} \ln(R\lambda) \quad (55)$$

$$= \kappa_{3} \left(\frac{4\pi M}{\kappa_{3} \lambda} + \sigma^{-}\right) + 2\kappa_{3} \ln(R\lambda).$$

The second term is negligible with respect to the first term, since it can be calculated that $R \approx \frac{1}{\lambda}$. The first term is the significant term. As we take $\sigma^{-} \rightarrow 0$ closer and closer to the endpoint $\sigma^{-} = 0$ (the observer collects more outgoing radiation), the first term approaches

$$\kappa_{3} \frac{4\pi M}{\kappa_{3} \lambda}. \quad (56)$$

The above result for entropy must be multiplied by $N$, the number of scalar fields. Let us deduce the value of $\kappa_{3}$ by comparing (56) with the entropy of entanglement estimated directly from the density matrix of the outgoing radiation (eq. (49)). This gives

$$S \approx \frac{4\pi M}{\lambda} \quad \text{if} \quad \kappa_{3} = \frac{1}{12}. \quad (57)$$

If our assumptions regarding the separation of the 'zero mode', of right and left movers are correct (i.e., if $\kappa_{1} = 4\kappa_{3}$, $\kappa_{3}$ as given in (50)), then (57) agrees with the calculation of Srednicki which gave $\kappa_{1} = \frac{1}{2}$. We discuss this issue further in section 8.

Note that if we consider the evaporation right up to the endpoint, then the cutoff scale (over which the radiation measurement is switched off) must go to zero, and the entropy becomes infinite. But since we are using the semiclassical geometry, it is not justified to go below distance $\lambda^{-1}$ (or $(N\lambda)^{-1}$, for large $N$) in our analysis. With this restriction, the entropy from the cutoff scale of the endpoint is ignorable compared to the entropy of the hole, for holes that evaporate over classical time scales $4\pi M/\kappa_{3} \lambda^{2} \gg \lambda^{-1}$.

The significance of the Bekenstein entropy for a black hole is a matter of debate. One hypothesis is that the horizon behaves as a membrane with $e^{NS}$ states, so that there is an upper limit to the entanglement entropy of the matter outside the hole with the hole itself [15]. Thus if a sufficiently large amount of matter were thrown into the hole then a part of the information would have to leak back out through subtle correlations in the Hawking radiation [16].

In a semiclassical treatment of the gravitational field, on the other hand, there seems to be no limit to the amount of information that can disappear into the black hole. Thus the entanglement entropy can also grow without bound when matter is repeatedly thrown into the hole and the black hole mass allowed to decrease back to $M$ by evaporation. It is possible to verify in the simple 1+1 dimensional evaporating RST solution that the entanglement entropy can indeed exceed the Bekenstein value by an arbitrary amount. We illustrate this by taking a simple example: the RST model with two incoming shock waves, discussed in the previous section.

We again apply the Srednicki approach to estimate entropy. Consider an observer at $I^{+}$ collecting radiation, who switches on the measuring device at $\sigma_{0}$ and switches it off at $\sigma_{1}$ with the corresponding switch-on-off intervals $d\sigma^{-}(\sigma_{1})$ as before. The only difference in the two shockwave case is that now the relation between the $d\sigma^{-}$ at $I^{+}$ and the corresponding $dy^{+}$’s at $I^{-}$ is different. The relation was given by the formula (36) in section 4. Now we need to substitute

$$a_{1} = \left\{1 + \left(e^{4\lambda M/\kappa_{3}} - 1\right)e^{4\lambda M/\kappa_{3}}\right\}^{-1}d\sigma^{-} \quad (58)$$

$$\approx e^{-4\lambda M/\kappa_{3}}e^{-\lambda_{1}\lambda} \frac{1}{\lambda^{-1}}$$

$$a_{0} \approx \frac{1}{\lambda}$$

into the equation (53) above. Also the distance $R$ will be different, but it is still of the order of $\frac{1}{\lambda}$ and the $R$-dependent term can thus be ignored. All this gives for the entropy (for $N$ fields)

$$S \approx N\kappa_{3} \ln\left(e^{4\lambda M/\kappa_{3}} e^{4\lambda M/\kappa_{3}}\right) + N\kappa_{3} \ln\left(\frac{1}{\lambda}\right) + \ldots \quad (59)$$

We are grateful to S. Trivedi for pointing this out to us.
\[ S \approx N \kappa_3 \frac{4\pi(M_0 + M_1)}{\kappa \lambda} + \ldots, \]

where \( + \ldots \) represents the ignored contributions from the subleading \( R \)-dependent term and the infrared cutoff term. As \( \kappa \to 0 \), the entropy becomes

\[ S \approx N \kappa_3 \frac{4\pi(M_0 + M_1)}{\kappa \lambda}. \]  

(60)

Substituting \( \kappa_3 = 1/12 \) gives then

\[ S \approx \frac{4\pi(M_0 + M_1)}{\lambda}. \]  

(61)

Note that the Bekenstein entropy, on the other hand, need not exceed \( S_{Bek} = \frac{2\pi M_0}{\lambda} \) at any time if the mass of the hole never exceeds \( M_0 \). The final entropy of entanglement (61) could be made arbitrarily large (as one could see by considering e.g. a n-shock wave case).

7. Entropy for 3+1 dimensional black holes

Let us now turn to the 3+1 dimensional black hole. We will consider only the Schwarzschild case. The Bekenstein entropy for this hole is

\[ S_{Bek} = \frac{A}{4} = \frac{4\pi(M_0 + M_1)^2}{4} = 4\pi M^2 \]  

(62)

The entropy collected in the form of radiation by an observer at \( \mathcal{I}^+ \) is also proportional to \( M^2 \) [12, 11] though it requires use of the 'transmission coefficients' \( T(\omega) \) for its computation.

One is tempted to compare such entropies to the result obtained by carrying out the flat space calculation of Srednicki for the case of 3 space dimensions. In the latter calculation one considers a massless scalar field, say, in the vacuum state in three space dimensions. One traces over the field modes inside an imaginary sphere of radius \( R \), and computes the entropy of the corresponding reduced density matrix. In this calculation it is convenient to decompose the scalar field into angular modes \( Y_{\ell m}(\theta, \phi) \), so that we obtain a 1-dimensional problem in the radial co-ordinate \( r \) for each such angular mode. The different angular modes decouple from each other, so we have to just add the entropies resulting from the computation for each mode. The radial co-ordinate is taken as a 1-dimensional lattice with lattice spacing \( a \). The entropy is found by numerical computation to be

\[ S \approx \frac{30(R/a)^3}{20} \]  

Thus we seem to reproduce the \( \sim R^2 \) dependence expected of the black hole entropy. But if we accept that (63) applies to the black hole then we are faced with the question: What should be the value of the cutoff \( a \)?

As we now argue, the result (63) is not the one relevant to the 3+1 dimensional black hole. In fact, the 1-dimensional result (59) is again the relevant one to use. To see this, consider decomposing the scalar field in the black hole geometry into modes with angular dependence \( Y_{\ell m}(\theta, \phi) \). Since we have assumed spherical symmetry, these modes decouple from each other. Thus we can consider studying the evolution of different fields (labelled by \( (\ell, m) \)) on the 1+1 dimensional geometry obtained by the spherically symmetric reduction of the 3+1 dimensional geometry. Proceeding in this way, one would need to find the 'entanglement entropy' of fields in one space dimension, which (for free fields) is given by the result (56).

At this point one sees an important difference between the flat space 3-dimensional problem and the 3+1 dimensional black hole. This difference occurs in the number of angular modes \( Y_{\ell m} \) that contribute significantly to the entropy. Consider first the flat space problem. If we had taken a lattice with lattice spacing \( a \) all over 3-space, then on the boundary sphere of radius \( R \) we could consider angular modes with \( \ell \sim R/a \). If we first reduce the Hamiltonian to a sum over modes and then put the \( r \) co-ordinate on a lattice, then again it is found that for \( l >> R/a \) the degrees of freedom on the two sides of the \( r = R \) boundary effectively decouple [7]. This happens because the radial wave equation is dominated at large \( l \) by a 'mass term' arising from the angular Laplacian, and such a term does not couple neighbouring sites of the \( r \) co-ordinate lattice. Since \(-1 \leq m \leq 1\), the number of angular modes contributing to the entropy is \( O((R/a)^3) \), which explains the leading power dependence of the entropy (63) on the cutoff \( a \). (Treating the 1-dimensional problem for each angular mode should give a \( \ln(R/a) \) dependence multiplying the dependence \( \sim (R/a)^3 \), but the above argument is too crude to distinguish the presence or absence of logarithmic terms.)

Thus we see that the difference between the \( a \) dependence of (56) and (63) can be understood in terms of the large number ( \( O((R/a)^3) \)) of angular modes contributing to the entropy in the 3-dimensional flat space problem. But in the 3+1 dimensional black hole, we know that most of the radiation comes out in only a few angular modes! In fact for a reasonable first estimate of the Hawking radiation one can require that only the \( s \)-wave modes \( (l = 0) \) emerge from the hole. We now compute the entropy of the 3+1 hole with such an approximation.
While we cannot solve the geometry of the evaporating 3 + 1 dimensional hole as accurately as for the RST model, for the purposes of our calculation we can consider the Schwarzschild metric with a time dependent mass $M$. The surface gravity of the black hole is $\kappa = (4M)^{-1}$. The temperature is $T = \kappa/2\pi = (8\pi M)^{-1}$. The luminosity in the s-wave mode is

$$L_{\omega} = \frac{dE}{dt} = \frac{1}{2\pi} \int_{0}^{\infty} \frac{d\omega \omega}{e^{\omega T} - 1} = \frac{\pi T^2}{12}$$  \hspace{1cm} (64)$$

From (64) we compute $M(y)$, the mass of the hole at the $I^+$ Schwarzschild coordinate $y$. (We take $y = 0$ at the endpoint of evaporation, so $y$ is negative in the part of $I^+$ where the Hawking radiation is being received.) We have

$$M(y) = \left(\frac{-y}{256\pi}\right)^{1/3}.$$  \hspace{1cm} (65)$$

As mentioned above, we approximate the evaporating geometry by just letting $M$ depend on time. Letting $v$ be the Minkowski co-ordinate at $I^-$. This approximation then gives

$$dv = -4M(y)dv = (ln(v_0 - v))$$  \hspace{1cm} (66)$$

denotes the co-ordinate $v$ is close to the value $v_0$ which reflects at $r = 0$ to move along the event horizon. Integrating (66) gives

$$\ln(v_0 - v_f) - \ln(v_0 - v_i) = 96\pi(M^3_f - M^3_i)$$  \hspace{1cm} (67)$$

We have $M_f = M$, and we let $M_i$ be of the order of Planck mass. Further, $ln(v_f - v_i)$ can be ignored compared to $ln(v_0 - v_f)$. Following the discussion of entropy in the 1-dimensional case, we write (for one evaporating field, one 'cut' and rightimovers only):

$$S \approx \frac{1}{12} \ln(u) = \frac{1}{12} \ln(\delta v)$$  \hspace{1cm} (68)$$

Here $\delta v$ is found from (66) by setting $dv = \Lambda$ for some chosen scale $\Lambda$ over which the observation of radiation is switched off:

$$\delta v = \frac{\Lambda}{4M_f(v_0 - v_f)} \approx \frac{\Lambda}{4M_f} e^{-96\pi M^3_f}$$  \hspace{1cm} (69)$$

Substituting in (68) we get

$$S \approx 8\pi M^2 = 2S_{Bek}$$  \hspace{1cm} (70)$$

We have ignored terms logarithmic in $M$; these corrections are smaller than the contribution of the $l \neq 0$ modes which we have also neglected.

We can now compare the result (70) for the entanglement entropy with the thermodynamic entropy collected at $I^+$. Following [12], we compare the change in the thermodynamic entropy received at $I^+$ to the change in the Bekenstein entropy of the hole:

$$R = \frac{dS}{dS_{Bek}} = \frac{\int_{0}^{\infty} d\omega \frac{\sigma(\omega)}{\omega} (\frac{e^{\omega T} - 1}{\omega}) - \ln(e^{\omega T} - 1))}{\int_{0}^{\infty} d\omega \frac{\sigma(\omega)}{\omega}}$$  \hspace{1cm} (71)$$

Here

$$\sigma(\omega) = \sum_{l,m} \Gamma_{lm}(\omega) [2\pi M^3]$$  \hspace{1cm} (72)$$

is the absorptivity per unit area of the black hole. In the above used approximation to Hawking radiation we used only the $l = 0$ mode, and let the 'transmission coefficient' $\Gamma$ be unity for all $\omega$. This gives $\sigma(\omega) = 1/[2\pi M^3]$. Substituting this in (71) gives $R = 2$, in accordance with (70). For more realistic models, taking into account the transmission coefficients $\Gamma_{lm}(\omega)$, one obtains $R \approx 1.3 - 1.6$ [17]. To reproduce the effect of nontrivial $\Gamma(\omega)$ we would need to extend the Srednicki calculation to fields with position dependent 'mass term'; we do not investigate this further here.

We can also compare the above derivations to the direct computation of the entropy of the density matrix obtained in the evaporation process; i.e., carry out the calculation analogous to (39) to (40). We again have $S_n = \pi^2 T^3 3c$, but now $T = (8\pi M_f)^{-1} = T_c$. Following the evaporation process we find $M_n = (n/128\pi)^{1/3}$. This gives, as expected,

$$S = \sum_{n=1}^{\infty} \frac{\pi^2 T^3}{3c} \approx 8\pi M^2 = 2S_{Bek}.$$  \hspace{1cm} (73)$$

In our above application of Srednicki's result, we found that the one space dimensionnal formulation was more relevant, rather than the three space dimensional one. On the other hand if an observer stands near the horizon of the black hole, she sees thermal radiation with power in a large number of angular momentum modes. Then it is possible that by carrying out the above calculations with a different choice of hypersurface (e.g. with the 'outside' part of the hypersurface corresponding to a static frame near the horizon) one would find relevant the equation (63).

8. Discussion

In this paper we have carried out the computation of Bogoliubov coefficients, stress tensor and the entanglement entropy for an evaporating black hole. Before one had explicit models of evaporating geometries, such quantities had been calculated only in
the absence of backreaction. With backreaction, it is possible to obtain for example the stress tensor in the region between the event horizon and the apparent horizon.

Concerning the application of Srednicki’s result to the black hole entanglement entropy, we have made two assumptions. First, we have assumed that the infrared divergence comes from modes with wavelength very large compared to the system size; after these modes are removed, the entropy can be split into a contribution from rightmoving modes and a contribution from leftmoving modes. Second, we assumed that when we cut the region $R$ out of a line, the entropy of system, say, can be split into two parts, one coming from each of the two cuts at the two ends of $R$. What we do now is offer some heuristic arguments to justify these assumptions.

First we wish to understand the appearance of the logarithmic dependence in the entanglement entropy. Consider a segment of the real line, $0 \leq z \leq I_1 + I_2$, divided into two regions near $z = I_1$ by a ‘cut’ of length $\alpha < I_1, I_2$. Further, assume $I_1 < I_2$. The scalar field we take to vanish at $z = 0$, $x = I_1 + I_2$. What is the entanglement entropy of $I_1$ with $I_2$ (with cutoff scale $\alpha$)?

Suppose $I_1 = I_2$. The field modes have wave numbers

$$k = \frac{n\pi}{I_1 + I_2}, \quad n = 1, 2, \ldots$$

(74)

The mode $k = \pi/(I_1 + I_2)$ is ‘shared’ between the two sides $I_1, I_2$, and we assume that it gives an entropy $s$. Now consider modes with $1 \leq n \lesssim (I_1 + I_2)/\alpha$. For any scale $\omega^{-1}$ of the wavelength we can make localized wavepackets just as was done in section 6 (eq. (38)). The number of such wavepackets is $\sim n$. Only the wavepacket that overlaps both sides of the cut at $z = I_1$ contributes to the entanglement entropy; and we again take this entropy to be $s$. (This is an assumption.) Equivalently, we find that each mode (74) contributes $\frac{1}{n}$ to the entropy, which becomes

$$S \approx s \sum_{n=1}^{I_1/\alpha} \frac{1}{n} \approx s \ln \frac{I_1}{\alpha}$$

(75)

Now suppose instead that $I_1 << I_2$. For $k^{-1} >> I_1$, the mode essentially vanishes over $I_1$, and so cannot entangle this region with $I_2$. For wavelengths $k^{-1} \lesssim I_1$, we make localized wavepackets in the region $0 < x < \alpha z$, $\alpha \lesssim 1$, just as in the discussion above, and thus find again eq. (75), where we note that $I_1$ is the smaller segment.

Now we address a more complicated case. We have the segments

- $I_1: \ 0 < x < I_1$

- $R: \ I_1 < x < I_1 + R$

- $I_2: \ I_1 + R < x < I_1 + I_2 + R$

Let $R << I_1 << I_2$. $I_1$ and $R$ are separated by a cut of size $\alpha_1 << R$, and $I$ and $I_2$ are separated by $\alpha_2 <<< R$. We want the entanglement entropy of $R$ with the remainder (i.e., $I_1 \cup I_2$). Again, we do not need to consider modes with wavelength $k^{-1} >> I_1$, since these effectively vanish over $R, I_1$. The modes with $R < k^{-1} < I_1$ give a contribution $S \approx s \ln \frac{R}{\alpha_1}$. The modes with $k^{-1} \lesssim R$ can be made into wavepackets that get ‘partitioned’ at two places; $x = I_1, x = I_1 + R$. These give the contributions $S \approx s \ln \frac{R}{\alpha_2}, S \approx s \ln \frac{R}{\alpha_3}$. Overall, we then get

$$S \approx s \ln \frac{R}{\alpha_1} + s \ln \frac{R}{\alpha_2} + s \ln \frac{I_1}{R}$$

(76)

If $\alpha_1 = \alpha_2 = \alpha$, $I_1 = I_2 \equiv I$, we can write

$$S \approx 2s \ln \frac{R}{\alpha} + s \ln \frac{I}{R}$$

(77)

which resembles (50) with $\kappa = -\frac{1}{2}\kappa_1$. The value of $s$ we can fix by the direct calculation of the entropy in the 1+1 dimensional black following eq. (49). For the modes with $k^{-1} \lesssim R$, one can clearly make a breakup between right and left movers. Thus we conclude $s = \frac{1}{4}$, after doubling up the obtained answer for the right movers alone.

The above gives a heuristic understanding of the entropy of entanglement, which should be useful in applying the result with a variety of boundary conditions.

In conclusion, we note that the ‘exponential expansion’ of coordinates near the horizon gives rise to thermal radiation, as was shown by Hawking. By using the result of Srednicki, the same coordinate transformation gives the entanglement entropy produced by the thermal radiation. Thus we seem to be one step closer to understanding the nature of black hole thermodynamics.

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References


[14] M. Srednicki (private communication)


Figure caption

Figure 1. The black hole geometry formed by an incoming shock wave (thick line with arrows). Evaporation occurs in region II; regions I and III are linear dilaton vacua.