BARYONS IN EFFECTIVE CHIRAL QUARK MODELS
WITH POLARIZED DIRAC SEA

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ABSTRACT

In this paper concepts and numerical calculations are reviewed in which baryons are considered as many body systems composed of three quarks coupled to a polarized Dirac sea. The interaction is the one of the Nambu-Jona-Lasinio model and of local and biquadratic form in a way which allows for spontaneous chiral symmetry breaking. The models are treated explicitly in the 0-boson and 1-quark loop approximation (Hartree mean field approach), where the real part of the effective Euclidean action is gauge invariantly regularized. In the meson sector the parameters of the models, including a cut-off for the Dirac sea, are fixed to PCAC, on-shell meson masses and decay constants. Furthermore the regularization schemes are chosen to get good values for vacuum condensates and current masses leaving the constituent mass $M$ as only free parameter. The review is focussed on the evaluation of baryon properties. Assuming a spherically symmetric hedgehog ansatz for the quark fields the solitonic solutions of the system are obtained in a selfconsistent way by solving the classical equations of motion.

After a semiclassical quantization procedure the solitons can be related to the physical states of the spin $1/2$ and $3/2$ multiplets. In this way observables and form factors can be calculated. Various types of quark fields and quark-quark couplings are investigated and reviewed. These are in SU(2) given by sigma and pion fields and in SU(3) they are complemented by kaon and eta-fields. In addition SU(2) vector mesons of $\rho$, $A_1$ and $\omega$-type are considered as well. For these models relevant observables and form factors of the nucleons are calculated. Some results on the other members of the baryon octet and decuplet are reviewed as far as they are available and of theoretical and experimental interest. For scalar and pseudoscalar couplings a clear picture emerges corresponding to localized valence quarks coupled to moderately polarized sea quarks and being separated from them by a finite energy gap. Using heat kernel and gradient expansion techniques the models of the Nambu-Jona-Lasinio type can be related to fully bosonized approaches of the Skyrme type, on the one hand, and to the chiral sigma model, representing valence quarks and dynamical meson fields, on the other.
1. Introduction

The present article deals with the description of nucleons and hyperons by means of relativistic chiral effective models. The prominent model discussed in detail is the Nambu-Jona-Lasinio (NJL) approach. It incorporates certain features of the Quantum Chromodynamics (QCD) and is related to low energy strong interaction phenomena.

Quantum Chromodynamics is generally considered to be the proper theoretical framework for the description of structures and reactions being dominated by the strong interaction. QCD is a SU(3) colour gauge theory combining quarks of $N_f$ different flavours ($u,d,s,c,b,t$) as fermions and gluons ($g$) as gauge fields. Its Lagrangian reads

$$ L_{\text{QCD}} = \sum_i \bar{q}_i \left( i \gamma^\mu D_\mu - m_i \right) q_i - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} $$

with the covariant derivative

$$ D_\mu = \partial_\mu + i g A^a_\mu \frac{1}{2} \lambda^a $$

and the field strength tensor

$$ G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu $$

Here $\lambda^a$ are the generators of $SU(3)_c$ (Gell-Mann matrices) the $f^{abc}$ are the structure coefficients of $SU(3)$ and $m_j$ are the current masses of the quarks. The basic features of QCD are (Donoghue et al. 1992) the following being all related to symmetry properties:

Universality: There is only one coupling constant $g$ for all interactions among quarks and gluons. This is a direct consequence of the local gauge invariance under the $SU(N_c)$-group, with $N_c =$ number of colours.

Asymptotic freedom: At very high energies quarks behave as free particles. In this region the effective coupling constant becomes small and perturbation theory may be applied. This property has been observed experimentally.

Confinement: Up to now neither free quarks nor gluons have been observed outside the volume occupied by a hadron. This feature has not fully been explained yet by theoretical reasoning. The idea is that only colourless systems are stable.

Chiral Symmetry: In the limit of vanishing quark masses, the QCD Lagrangian is invariant under the chiral $SU(N_f)_R \otimes SU(N_f)_L$ group of global transformations

$$ q(x) \rightarrow \exp \left( ia^a \lambda^a / 2 \right) q(x) $$

$$ q(x) \rightarrow \exp \left( i \gamma_5 \beta^a \lambda^a / 2 \right) q(x) $$

where $a = 1, \ldots, N_f^2 - 1$, so that the corresponding conserved currents read

$$ V^a_\mu (x) = \bar{q} \gamma_\mu \frac{1}{2} \lambda^a q(x) $$
In the absence of strong interactions the quarks have a current mass of electroweak origin which breaks chiral symmetry explicitly. That means

$$A^a_\mu(x) = q(x) \gamma_\mu \gamma_5 \frac{1}{2} \lambda^a$$

where the current quark mass matrix \( m \) is defined as \( m = \text{diag}(m_u, m_d, m_s, m_c, m_b, m_t) \). The attributed values are (Gasser and Leutwyler 1982) given in Tab. 1.

The chiral symmetry is well realized with respect to up- and down-quarks because of their small current mass \((\lesssim 10 \text{ MeV})\). The symmetry is spontaneously broken resulting in a chiral condensate \( \bar{u}u + \bar{d}d \approx (283 \pm 31) \text{ MeV} \) (Gasser and Leutwyler 1992). This gives the up- and down-quarks a dynamically generated constituent mass of about \( 350 - 450 \text{ MeV} \). For those quarks apparently the spontaneous breaking of chiral symmetry dominates over the explicit breaking caused by finite current masses. For heavy quarks \((c,b,t)\) the situation is rather opposite. The strange quark lies in an intermediate region and its relevance for low-energy non-strange physics is at present under debate.

Finally let us mention that in the presence of external fields chiral symmetry is also broken because of quantum corrections leading to the chiral anomaly (Adler 1969, Bell and Jackiw 1969). Vector current conservation implies an anomaly in the divergence of the axial current (Bardeen 1969), which predicts successfully the decay rate for the process \( \pi^0 \rightarrow 2\gamma \) if \( N_c = 3 \).

Scale Invariance: In the chiral limit \( m_j = 0 \) for all flavours the QCD Lagrangian does not contain any dimensional parameters. Thus it is invariant under the scale transformation \( q(x) \rightarrow \lambda^{-3/2} q(\lambda x) \). The corresponding dilatational current is not conserved because of quantum corrections leading to the trace anomaly.

The QCD has been treated in a clean way only for high energy processes, because there the asymptotic freedom (Politzer 1974, Gross and Wilczek 1973a, 1973b) facilitates the solution tremendously (perturbative regime). For low energies, in particular the structure of hadrons, the gluon self interaction \( (gf^{abc} A^a_\mu A^b_\nu A^c_\rho)^2 \) creates non-linearities and causes enormous problems (non-perturbative regime). Here the Lagrangian as such has only been treated in a regularized way on a 4-dimensional lattice in the form of lattice gauge theory (Wegner 1971, Wilson 1974). Although these calculations are potentially exact they are practically hampered by technical and conceptual problems associated with the choice of large current masses, fermion doubling, small size of lattices, etc. In addition calculations of this sort require extreme amounts of computer time (Creutz 1983). Because of this dilemma in the non-perturbative regime effective theories are very much in use. They explicitly treat those degrees of freedom, which are relevant for low energy structures, and ignore the others or parametrize them in form of coupling constants. This only makes sense if those low energy degrees of freedom are clearly separated from the high energy ones. However, such a question can be decided only by an inspection of the low energy phenomenology, which to some extend has been done already long before the advent of QCD. Of course for such a limited
theory one must not expect that all properties of QCD, discussed above, are also properties of the effective models. Hence the guideline for the effective models is the necessity to reproduce low energy phenomena and to reflect some of the QCD symmetry properties.

There are some basic phenomena of low energy hadronic physics which over the many years of experience with effective models have been identified as indispensable for the construction of an effective theory for baryon and meson structure (Cheng and Li 1984). It is clear that the pion field plays a dominant role in all hadron physics. This goes back to Yukawa’s original idea that nuclear forces are mediated by pions and corresponds to the fact that basic feature of nuclear structure at low and intermediate energies can successfully be explained in turn of nucleons and mesons instead of their constituents quarks and gluons, as they are identified at high energies. The pion is closely connected to the chiral condensate. In fact it is the most prominent Goldstone-boson (Goldstone 1961, Nambu 1960) of the spontaneously broken chiral symmetry.

The pion decays via the axial current $A^a_\mu(x)$ and its decay constant $f_\pi = 93\text{MeV}$ is defined as (Cheng and Li 1984)

$$<0\left|A^a_\mu(0)\right|\pi^b(p)> = i f_\pi p_\mu \epsilon^{ab}$$

For hadrons the axial current $A^a_\mu(x)$ is partially conserved due to the small but finite current masses of the up- and down-quarks. According to the hypothesis of the partial conservation of the axial current (PCAC) (Nambu 1960, Chou 1961, Gell-Mann and Levy 1960)

$$\partial^\mu A^a_\mu(x) = f_\pi m^2_{\pi} \pi^a(x)$$

with $m_{\pi} = 139\text{MeV}$ being the mass of the pion field $\pi^a$. The iso-vector current $V^a_\mu(x)$ is conserved unless one explicitly attributes different current masses to up- and down-quarks. Both currents allow the formulation of chiral charges (Cheng and Li 1984)

$$Q^a_V(x) = \int d^3 x V^a_0(x)$$

$$Q^a_A(x) = \int d^3 x A^a_0(x)$$

and a corresponding charge algebra, which exists even if the currents are not fully conserved:

$$[Q^a_V(x), Q^b_V(x)] = i f^{abc} Q^c_V(x)$$

$$[Q^a_V(x), Q^b_A(x)] = i f^{abc} Q^c_A(x)$$

$$[Q^a_A(x), Q^b_A(x)] = i f^{abc} Q^c_V(x)$$

The relevance of this commutator algebra lies in the fact that several low energy theorems and sum rules involving strong and electro-weak processes can be derived and confronted with experimental data. A typical example is the Goldberger-Treiman relation (Goldberger and Treiman 1958), which is experimentally fulfilled to a precision of 7%:

$$g_A M_N = f_\pi g_{\pi NN}$$

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Here $M_N = 938\text{MeV}$ represents the nucleon mass, $g_A = 1.25$ is the axial coupling constant of the neutron beta-decay and $g_{\pi NN} = 13.6$ is the pion-nucleon-nucleon coupling constant. In general the violation of the chiral symmetry is less than $1\%$, and it is indeed after the isospin symmetry the best known symmetry of strong interactions at low energies. It is clear that any effective theory of hadron has to incorporate the spontaneously broken chiral symmetry and the corresponding Goldstone boson, i.e., pions in $SU(2)$ and in addition the kaons and the eta in $SU(3)$.

Another fruitful idea in the study of electromagnetic properties of hadrons is given by the vector meson dominance model, which assumes that photons do not interact directly with hadrons but rather through the virtual propagation of an intermediate massive vector meson state (Sakurai 1960, 1969). In quantum field theory such an idea can be formulated through the so-called current field identities (Kroll et al. 1967)

$$J^I=0 = \frac{m_\omega^2}{g_\omega NN} \omega(p), \quad J^I=1 = \frac{m_\rho^2}{g_\rho NN} \rho^0(p)$$

for the isoscalar and isovector components of the electromagnetic currents. Here the vector meson masses $m_\omega = 783\text{MeV}$ and $m_\rho = 770\text{MeV}$ appear together with the $\rho$ and $\omega$ exchange coupling constants of NN scattering. In most cases, the predictions of the vector meson dominance model (VDM) have been very satisfactory (see e.g. Gourdin 1974). The most famous example is provided by the prediction for the mean squared electromagnetic pion radius

$$<r^2>_{\pi} = \frac{6}{m_\rho^2} = 0.39\text{fm}^2$$

which compares reasonably well with the corresponding experimental value of $(0.44 \pm 0.01)\text{fm}^2$.

Finally, let us mention that the union of current algebra with current-field identities leads to the field algebra whose most prominent result is the KSFR relation (Kawarabayashi and Suzuki 1966, Riazuddin and Fayazuddin 1967)

$$2g^2_{\rho \pi \pi} f_\pi^\pi m_\rho^2$$

with $g_{\rho \pi \pi} = 5.48$ being the decay constant of the strong process $\rho \rightarrow \pi \pi$. The accuracy of this relation is $2\%$.

As soon as strange quarks are involved one has the Gell-Mann Okubo mass formula (Gell-Mann, 1962; Okubo 1962), which deviates from the experiment by less than $3\%:$

$$M_\Sigma - M_N = \frac{\sigma}{2}(M_\Sigma - M_N) + \frac{3}{4}(M_\Sigma - M_\Lambda)$$

and the Coleman-Glashow formula (Coleman and Glashow, 1961)

$$M_\Lambda - M_\rho + M_\Xi - M_\Xi = M_\Sigma - M_{\Sigma^+}$$

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which fits perfectly with experimental data and which was originally attributed to the electromagnetic interactions between the quarks.

An effective theory of low energy QCD degrees of freedom in the baryonic sector can only be successful if it reproduces the above phenomenological features. Therefore we concentrate in this article on models showing spontaneously broken $SU(N_f)_L \otimes SU(N_f)_R$ symmetry with a possible explicit breaking due to finite current masses. In some cases we will also consider scale invariance. We consider in detail the Nambu-Jona-Lasinio model (Nambu and Jona-Lasinio 1961a, 1961b) and will relate it to all others. The theory will be explained for scalar and pseudoscalar quark-quark interactions in the light quark sector (up, down). Generalisations towards vector interactions and/or $SU(3)$ will be done in separate chapters. Thus the NJL-Lagrangian studied in detail is

$$\mathcal{L}_{NJL} = \bar{q}(i\gamma^\mu \partial_\mu - m_0)q + \frac{G}{2} \left[ (\bar{q}q)^2 + (\bar{q}\gamma_5 q)^2 \right]$$

with $q$ having an up- and down-component and $m_0 = diag(m_u, m_d)$. In most cases we will assume $m_u = m_d$. The NJL-model is distinguished because it allows to build conceptual bridges to most of the currently used effective models for baryon structure. In such a way these models and also the NJL model can be better understood.

There are by now several attempts to relate the NJL to some low energy limit of QCD. One has been given by Diakonov and Petrov (1986) assuming the QCD vacuum to be an instanton liquid. By variational field theoretic means they obtain after various plausible approximations the zero-boson and one-quark-loop approximation to the NJL-Lagrangian. This procedure is based on previous works by 't Hooft (1976a, 1976b), Callan, Dashen and Gross (1978, 1979a, 1979b), Carlitz and Cremer (1979) and Shifman, Vainstein and Zakharov (1980a, 1980b), see also Shuryak (1980). Ball (1987) on the other hand integrates out the gluon fields in some certain approximation and obtains in the end the NJL-Lagrangian in $SU(N_f)_R \otimes SU(N_f)_L$ including vector mesonic couplings. Schaden et al. (1990) formulate QCD in terms of path integrals over the field tensor rather than the vector potential. At a certain level they perform stationary phase approximations and end up, after several further plausible arguments, at the NJL-model. Cahill, Roberts and others (Cahill and Roberts, 1985; Roberts et al., 1988; Cahill, 1992) introduce bilocal meson and diquark fields into a functional form of QCD. The NJL model appears as some pointlike approximation. There are also steps in a similar direction by Dhar and Wadia (1984, 1985), McKay and Munczek (1985), Adrianov (1985) and Chanfray et al. (1991). One should note that none of these theories is really a clean cut derivation of NJL. For example most of them are not able to say where in the chain of arguments and approximations on the way from QCD to NJL the confinement has been lost, nor can they say how the NJL-model can be improved.

Actually the effective chiral models for baryon structure can roughly be classified in three groups:

i) Chiral quark models: Here the system is dominated by three valence quarks coupled to a dynamical pion. In the chiral MIT-model (Chodos and Thorn, 1975) the quarks are confined by a bag with infinite walls and chiral symmetry is preserved by a dynamical pion field and
the continuity of the axial current at the bag's surface. The linear versions of these models can be related to the cloudy bag model (Theberge et al. 1980; Thomas AW 1983).

In the chiral $\sigma$-model (Gell-Mann and Levy 1960) we do not have confining walls but a sigma field with finite vacuum value. Here the valence quarks are localized by binding forces exerted by the sigma- and pion field (Birse and Banerjee 1984, 1985, for a review see e.g. Birse 1990 and references therein). The baryon number is carried by the three valence quarks. Altogether these models can be characterized by *valence quarks and meson fields*.

ii) Skyrme models: In these models (Skyrme 1961, Adkins et al 1983, Zahed and Brown 1986, Holzwarth and Schwesinger 1987) one does not deal with explicit quarks but solely with dynamical meson fields. They are collectively quantized in order to get the proper quantum numbers of e.g. a nucleon. The formalism is comparatively simple and hence these models enjoy great popularity. Conceptually they are based on the large $N_c$-expansion of the QCD ('t Hooft 1973, Witten 1979b). This would require an infinite set of boson fields which is in practice replaced by the pion field in the simplest form, and by vector mesonic fields in more recent versions (Meissner U G 1988). The baryon number is hidden in the topology of the Goldstone fields. At the center of the baryon the Goldstone fields must show a winding number, which is to be identified with the baryon number. These models can be characterized by *meson fields and topology*.

iii) Nambu-Jona-Lasinio models: As described above one deals solely with quarks without dynamical meson fields (Nambu and Jona-Lasinio 1961a, 1961b). Valence quarks appear in a natural way as bound, discrete and localized states (Kahana and Ripka 1984). Mesons enter the theory only as non-dynamic auxiliary fields generated by the polarization of the Dirac sea (Eguchi 1975, Kleinert 1976). In fact these models can be characterized by *valence quarks and sea quarks*.

iv) With respect to the above characterization there are some hybrid models which combine certain features. This is the chiral bag of the Stony Brook group (Brown and Rho 1979). There the pion field outside the bag shows some topological properties and carries hence a fractional baryon number. The linear chiral sigma model of Kahana and Ripka (1984) involves sea quarks and dynamic pion- and sigma fields and requires renormalization: In fact these authors are the first to treat the polarization of the Dirac sea in this context (Ripka and Kahana 1987, Soni 1987, Li et al. 1988). It shows, however, an instability and hence has never been applied to nucleon properties. Recently it was shown by Kahana and Ripka (1992) that the introduction of vector mesons in this model can cure the vacuum instability, though no numerical calculations for nucleons were performed yet.

It is interesting to note, that the NJL-Model (valence- and sea-quarks) can be directly related to the chiral bag models (valence-quarks and mesons) and the Skyrme-models (topology and mesons). In order to understand this, the figures 1.1-1.4 may be helpful. Fig. 1.1 shows schematically the single particles energies of the chirally symmetric plane wave vacuum as it emerges from the NJL-model in the 0-boson and 1-quark loop approximation. We concentrate on up- and down-
quarks and hence a small gap of \(2m_0 = m_u + m_d\) exists between the upper and lower continuum. In a plausible Fock-state picture the lower continuum is occupied by \(N_c = 3\) quarks for each flavour in each single particle level, as indicated by crosses. The spontaneously broken chiral vacuum is shown in Fig. 1.2. There the gap is \(2M\) and the single particle states are again of plane wave nature with the constituent mass \(M\). The vacuum in both cases is characterized by the baryon number \(B = 0\). The soliton for a "small" constituent mass \(M\) is shown in Fig. 1.3. A single particle state from the positive continuum has entered the gap and has become a bound state. It is explicitly occupied by \(N_c = 3\) valence quark and hence the baryon number of the system has increased from \(B = 0\) to \(B = 1\). The interaction of the continuum state with the valence quarks causes a polarization of the single particle states of the positive and negative continuum which is indicated by wavy lines. For large constituent masses the bound state has moved further down, see Fig. 1.4, and approaches the Dirac sea. In such a case both continua are strongly polarized and the Dirac sea incorporates another level and hence acquires the baryon number \(B = 1\). The scenarios of Fig. 1.3 and Fig. 1.4 are distinguished by the values of the constituent mass \(M\). As we will see for sigma and pionic couplings in \(SU(2)\) as well as in \(SU(3)\) the scenario of Fig. 1.3 is the one which reproduces the relevant nucleon and hyperon observables. If vector mesons are included the picture is not yet settled.

The formal techniques to relate the NJL model to the bag model and to the Skyrme model are the gradient expansion (Aitchison and Frazer 1984, 1985a and 1985b) the heat kernel expansion (Gasser and Leutwyler 1984, Ebert und Reinhardt 1986), or the expansion of Chan (1985) all applied to the fermion determinant of NJL. Actually these expansions are well converging for smoothly varying and largely extended boson fields or quark wave functions which is not always fulfilled for realistic systems. Nevertheless, if one uses only a certain number of expansion coefficients in the expansion of the effective action of the scenario of Fig. 1.3, the Lagrangian of Gell-Mann and Levi (1960) is obtained. There the baryon number is carried by the valence quarks and the polarization of the Dirac sea creates dynamic meson fields whose possible topology does not carry a baryon number. If one considers the scenario of Fig. 1.4 and terminates the expansion of the action in an appropriate way and if one ignores certain destabilizing terms then a Skyrme type Lagrangian (Skyrme 1961) is obtained. The highly polarized Dirac sea including the additional valence level is described in terms of a dynamic pion field. This exhibits a winding number to be identified with the baryon number. One has to consider these relations between the models with some care: First, the expansions are often badly converging. Second, a gradient expansion of an observable of NJL does not necessarily agree with the observable evaluated from the gradient expanded NJL Lagrangian. Nevertheless the NJL model is extremely helpful in understanding the meaning of valence quarks and topology the concepts of which are the basic features of all presently used effective chiral models.

There are two other branches relevant in the context of the NJL-model, one is the meson sector with baryon number \(B = 0\), the other is the evaluation of properties of a hot and dense medium and the embedding of a baryon in it. The meson sector has been the object of intensive studies in the last years and several review articles have been written on it (Vogl and Weise
1991, Klevansky 1992). We will discuss the meson sector in this paper only as far it is needed to explain the appearance of the chiral condensate and to adjust the parameters of NJL. The medium calculations are not discussed in the present article at all. The interesting point is that with increasing density and temperature of the medium the chiral symmetry of the system is restored. The reader is referred to Hatsuda and Kunihiro (1985, 1987a, 1987b, 1988), Meissner U G (1989a, 1989b), Bernard V et al. (1986, 1987), Jaminon et al. (1989, 1992), Ruiz Arriola E et al. (1990), Christov C V et al. (1990a, 1990b), Lutz and Weise (1991) and Hatsuda and Kunihiro (1994).
2. The Nambu-Jona-Lasinio-Model With SU(2)-Flavor: Vacuum and Mesonic Sector

The purpose of this chapter is to present the Nambu-Jona-Lasinio-model (NJL) (Nambu and Jona-Lasinio 1961) in the simplest realistic case, i.e. with scalar and pseudoscalar couplings. To begin with, we study the classical Lagrangian and its symmetries. The bosonization procedure is presented by means of path integrals and the stationary phase approximation is defined. We extract the ultraviolet structure of the model and present some different regularization schemes. The gradient and heat kernel expansions are discussed as well as the formalism for extracting vacuum and mesonic properties. The parameters of the model are fixed in order to reproduce known properties of the mesonic sector. Different ways of parameter fixings are introduced and the corresponding numerical results are shown.

2.1. The Semibosonized NJL - Effective Action

Classical Lagrangian - Symmetries

The simplest version of the Nambu-Jona-Lasinio-model is represented by the following Lagrangian

\[ \mathcal{L}_{NJL} = \mathcal{L}_{kin} + \mathcal{L}_{br} + \mathcal{L}_{int} \]  

where the kinetic, interaction and chiral breaking mass terms are given by

\[ \mathcal{L}_{kin} = \bar{q}(x)i\gamma_5 q(x) \]
\[ \mathcal{L}_{br} = -\bar{q}(x)mq(x) \]
\[ \mathcal{L}_{int} = \frac{G}{2} \left[ \left( \bar{q}(x)q(x) \right)^2 + \left( \bar{q}(x)i\gamma_5 \vec{\tau} q(x) \right)^2 \right] \]  

respectively. Here \( q(x) \) denotes a quark field with u and d flavors and \( N_c \) colors. The \( G \) is the coupling constant with dimensions of length squared. The \( \bar{m} \) represents the current quark mass matrix given by

\[ \bar{m} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = m_1 1 + \tau_3 \Delta m \]  

where the average and difference masses are defined as

\[ m_1 = \frac{1}{2}(m_u + m_d), \quad \Delta m = m_u - m_d \]  

Their attributed theoretical values at a scale of 1GeV are (Gasser and Leutwyler 1982) given in Tab. 1.1. In the following we will assume the quark mass degeneracy \( m_u = m_d = m_1 \), corresponding to exact isospin symmetry.

At the classical level the total Lagrangian is invariant under the global \( U(1) \) transformation

\[ q(x) \rightarrow q'(x) = \exp(i\delta)q(x) \]  

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The kinetic and interacting terms $L_{\text{kin}}$ and $L_{\text{int}}$ are also invariant under the $SU(2)_Y \otimes SU(2)_A$ global chiral group

$$q(x) \rightarrow q'(x) = \exp(i\vec{\tau} \cdot \vec{\sigma})q(x)$$

$$q(x) \rightarrow q'(x) = \exp(i\gamma_5 \vec{\tau} \cdot \vec{\beta})q(x)$$

These transformation properties generate the following Noether currents (Cheng and Li 1984)

$$B_\mu(x) = \bar{q}(x)\gamma_\mu q(x), \quad \text{(baryon current)}$$

$$V_\mu(x) = \bar{q}(x)\gamma_\mu \frac{\vec{\tau}}{2} q(x), \quad \text{(vector current)}$$

$$A_\mu(x) = \bar{q}(x)\gamma_\mu \gamma_5 \frac{\vec{\tau}}{2} q(x), \quad \text{(axial current)}$$

whose divergences on the classical level are given by:

$$\partial^\mu B_\mu = 0, \quad \partial^\mu V_\mu = 0, \quad \partial^\mu A_\mu = \frac{i}{2}[\bar{m}, \vec{\tau}]q$$

Corresponding to the conservation of the baryon number and isospin and the partial conservation of the axial charge respectively. Actually, even for $m = 0$ the axial current is not conserved on the quantum level in the presence of external vector and axial vector fields reflecting the chiral anomaly (Wess and Zumino 1971, Wess 1972). Finally let us mention that the total Lagrangian shows a $SU(N_c)$ global invariance, leading to the conservation of the color current. Since the four fermion interaction in eq. (2.1) is of color singlet form, this conservation does not have dynamical implications. However, there are other versions of the model which involve color octet-octet interaction in the vector and axial channels (Klimt et al. 1990; Takizawa et al. 1990) which are formally equivalent to the Lagrangian (2.2) after a Fierz rearrangement. Those have been discussed at length in other reviews (Vogl and Weise 1991; Klevansky 1992) and will not be considered here specifically.

**Collective Boson Fields**

Like the Fermi theory of weak interaction the Nambu-Jona-Lasinio model is non-renormalizable, because the coupling constant of the 4-fermion interaction $G$ has the dimension $[G] = \text{mass}^2$ (Itzykson and Zuber 1980, Cheng and Li 1984). This means that with each increasing order in $G$ a new graph with a higher degree of ultraviolet divergence appears. In order to get a well defined theory it is therefore necessary to specify how the infinities of the model have to be treated.

in which order the model has to be handled.

For our later purposes it will be convenient to use a *semibosonized form* of the 4-fermion interaction eq. (2.1). The idea of *bosonization* has first been formulated in solid state physics, where it is called Hubbard-Stratonovich transformation (Negele and Orland 1987) and has been applied to the present theory by Eguchi (1976), Kikkawa (1976) and Kleinert (1978). It consists in resummarizing the graphs of the 4-fermion point interaction into a *quark-meson* interaction of the Yukawa type by introducing collective scalar-isoscalar ($\sigma$) and pseudoscalar-isovector ($\vec{\pi}$)
background fields, which carry the quantum numbers of the interaction channels ($\bar{q}q$) and ($\bar{q}i\gamma_5 q$), respectively. To this end we insert the functional identity:

$$1 = \int D\sigma D\bar{\sigma} \exp \left\{-i \int d^4 x \frac{\mu^2}{2} \left[ (\sigma + \frac{g}{\mu^2} (\bar{q}q - \frac{m_0}{G})^2) + (\bar{\sigma} + \frac{g}{\mu^2} \bar{q}i\gamma_5 q)^2 \right] \right\}$$  \hspace{1cm} (2.9)$$

into the generating functional $Z_{NJL}$:

$$Z_{NJL} = \int DqD\bar{q} e^{i \int d^4 x \left\{ \bar{q} (i\gamma_5 \gamma^\mu \sigma) + \frac{\mu^2}{2} \left[ (\bar{q}q - \frac{m_0}{G})^2 + (\bar{\sigma} + \frac{g}{\mu^2} \bar{\sigma}i\gamma_5 q)^2 \right] \right\}}$$  \hspace{1cm} (2.10)$$

and obtain:

$$Z'_{NJL} = \int \bar{q} q e^{i \int d^4 x \mathcal{L}'_{NJL}(x)}$$  \hspace{1cm} (2.11)$$

with

$$\mathcal{L}'_{NJL} = \bar{q}iDq - \frac{\mu^2}{2} (\sigma^2 + \bar{\sigma}^2) + \frac{m_0 \mu^2}{g} \sigma$$

$$iD = i\partial - g (\sigma + \bar{\sigma} \gamma_5)$$  \hspace{1cm} (2.12)$$

and:

$$G = g^2 / \mu^2$$  \hspace{1cm} (2.13)$$

Here $g$ and $\mu$ are newly introduced parameters which arise due to the present form of bosonization. Their values will be fixed in sect. 2.6.

We want to stress that on this stage the fields $\sigma$ and $\bar{\sigma}$ are non-dynamical collective background fields and no kinetic term $\frac{\mu^2}{2} \left[ \partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \bar{\sigma} \partial^\mu \bar{\sigma} \right]$ appears on the classical level.

The semibosonized Lagrangian $\mathcal{L}'_{NJL}$ remains invariant under the chiral transformation of $SU(2)_R \otimes SU(2)_L \otimes U(1)_B$, if $(\sigma, \bar{\sigma})$ transforms under the $(\frac{1}{2}, \frac{1}{2})$ representation of this group (Cheng and Li 1984). Hence we demand:

$$\sigma \rightarrow \sigma + 2\beta \bar{\sigma}$$

$$\bar{\sigma} \rightarrow \bar{\sigma} - 2\beta \sigma - 2\gamma \times \bar{\sigma}$$  \hspace{1cm} (2.14)$$

eq. (2.14) is consistent with the equations of motion for $\mathcal{L}'$.

Grassmann-Integration - Effective Chiral Action

The quark contribution $\bar{q}iDq$ in eq. (2.12) is bilinear in the quark fields $q$ and $\bar{q}$ so that the functional Grassmann integration $\int DqD\bar{q}$ can be performed analytically. After transformation to Euclidean space time (cf. appendix A) we get:

$$Z_{NJL}'' = \int D\sigma D\bar{\sigma} e^{-S_{eff}(\sigma, \bar{\sigma})}$$  \hspace{1cm} (2.15)$$
The effective chiral action \( S_{\text{eff}}(\sigma, \vec{\pi}) \) contains the fermion determinant \( S_{\text{eff}}^F(\sigma, \vec{\pi}) \) as well as a mesonic mass term \( S_{\text{eff}}^{\text{mes}}(\sigma, \vec{\pi}) \) and the chiral breaking term \( S_{\text{eff}}^{\text{br}}(\sigma, \vec{\pi}) \):

\[
S_{\text{eff}}(\sigma, \vec{\pi}) = S_{\text{eff}}^F(\sigma, \vec{\pi}) + S_{\text{eff}}^{\text{mes}}(\sigma, \vec{\pi}) + S_{\text{eff}}^{\text{br}}(\sigma, \vec{\pi})
\]

\[
S_{\text{eff}}^F(\sigma, \vec{\pi}) = -\text{Indet}(-iD) = -\text{SpIn}(-iD)
\]

\[
S_{\text{eff}}^{\text{mes}}(\sigma, \vec{\pi}) = \frac{\mu^2}{2} \int d^4x E(\sigma^2 + \vec{\pi}^2)
\]

\[
S_{\text{eff}}^{\text{br}}(\sigma, \vec{\pi}) = -\frac{m_0 \hbar^2}{g} \int d^4x E\sigma
\]

\( \text{Sp} \) denotes the total trace in functional as well as matrix space:

\[
\text{Sp} A = \int d^4x \text{Tr}_{\tau\tau'} \langle \sigma | A | x \rangle = \int d^4k \text{Tr}_{\tau\tau'} \langle k | A | k \rangle
\]

Generally the fermion determinant in Euclidean space time has a real as well as an imaginary part, reading:

\[
\text{Re} S_{\text{eff}}^F(\sigma, \vec{\pi}) = (-\frac{1}{2}) \text{SpIn}(D^\dagger D)
\]

(2.17)

and:

\[
\text{Im} S_{\text{eff}}^F(\sigma, \vec{\pi}) = (-\frac{1}{2}) \text{SpIn} \left( \frac{D}{D^\dagger} \right)
\]

(2.18)

For time independent meson fields \( \sigma \) and \( \vec{\pi} \) it will be shown in chapter 3, that the imaginary part \( \text{Im} S_{\text{eff}}^F(\sigma, \vec{\pi}) \) vanishes, because the 1- particle Hamiltonian \( h \) is hermitian. This changes if time-like vector mesons (e.g. \( \omega \) mesons) are coupled to the Lagrangian destroying the hermiticity of \( h \). In some cases the imaginary part is related to the chiral anomaly. Details will be discussed in chapter 7. In addition imaginary parts of the action occur if the system is considered in the rotating frame.

2.2. Constituent Quark Mass - Stationary Phase Approximation

The main observation of Nambu and Jona-Lasinio (1961) in their original work was that the four fermion interaction generates a dynamical mass for the fermions if the coupling constant is bigger than a certain critical value. This was done in the canonical formalism using a Bogoliubov-Valentin transformation from bare massless quarks to constituent massive quarks. The idea is that if the interaction is strong enough, the vacuum lowers its energy creating a mass gap between the positive and negative energy continua of the Dirac spectrum. Such a phenomenon is called dynamical mass generation. In the following sections we will see explicitly how a dynamical mass generation takes place within the path-integral approach to the NJL model.

Spontaneously Broken Chiral Symmetry

From eq. (2.12) one recognizes at once that for a finite vacuum expectation value \( \sigma_V \) of the \( \sigma \)-field the quarks acquire a finite dynamical or constituent mass \( M \) due to:

\[
M = \left. \frac{d^3 \mathcal{L}_{\text{NJL}}}{d^3 \sigma} \right|_V = g \sigma_V
\]

(2.19)
which means that chiral symmetry is \textit{spontaneously broken} (Cheng and Li 1984). In order to have a translation and parity invariant vacuum one has to demand that $\vec{\pi} = 0$ and that $\sigma_V$ is independent of $x$. The value of $\sigma_V$ will be determined later on. Normal ordering with respect to this vacuum is performed by subtracting the vacuum value $S_{eff}(\sigma_V, \vec{\pi}_V)$ from the effective chiral action eq. (2.16)

Stationary Phase Approximation

From now on we will assume \textit{classical fields} for $\sigma$ and $\vec{\pi}$ or in other words we perform a stationary phase approximation in 0th order with respect to the stationary points of the effective action

$$\frac{\delta S_{eff}(\sigma, \vec{\pi})}{\delta \sigma} \bigg|_{vac} = 0$$

$$\frac{\delta S_{eff}(\sigma, \vec{\pi})}{\delta \vec{\pi}} \bigg|_{vac} = 0$$

which is the \textit{Schwinger-Dyson or gap equation}. This means we have now specified the approximation in which the model will be treated. It can be shown that the stationary phase approximation of the semibosonized version is fully equivalent to the Hartree approximation in the original 4-fermion version, which has been treated by a number of authors using the Bethe-Salpeter formalism (Bernard et al. 1984, Bernard 1986, Furstl et al. 1986, Bernard et al. 1987, Providencia et al. 1987, Bernard et al. 1988, Klimt et al. 1990) and has been reviewed by Vogl and Weise (1991) and Klevansky (1992). Whereas in this chapter we restrict ourselves to the vacuum solutions $\sigma_V$ and $\vec{\pi}_V$ of eq. (2.20), the main aim of this article is to study non-translational invariant solutions of this equation corresponding to a system with finite baryon number, which will be done in the next chapter.

Vacuum Expectation Values - Quark Condensate

In the following we need the expectation value of the bilinear density $\bar{q}(x)Kq(x)$ in the stationary phase approximation, i.e. for classical $\sigma$ and $\vec{\pi}$, where $K$ is an arbitrary matrix in spin, isospin and color space. This can be expressed by:

$$\langle \bar{q}(x)Kq(x) \rangle = \frac{\int D\bar{q}Dq e^{-\int d^4 x \bar{q}(x)(-iD)q(x)[\bar{q}(x)Kq(x)]}}{\int D\bar{q}Dq e^{-\int d^4 x \bar{q}(x)(-iD)q(x)}}$$

$$= \frac{\delta}{\delta \kappa(x)} \left. \int D\bar{q}Dq e^{-\int d^4 x \bar{q}(x)(-iD-\kappa(x)K)q(x)} \right|_{\kappa(x)=0}$$

$$= \frac{\delta}{\delta \kappa(x)} \ln \left. \int D\bar{q}Dq e^{-\int d^4 x \bar{q}(x)(-iD-\kappa(x)K)q(x)} \right|_{\kappa(x)=0}$$

$$= \langle x_E | Tr \gamma_5 (iD)^{-1} K | x_E \rangle$$

One important special case is the expectation value $\langle \bar{q}q \rangle$, the \textit{quark condensate}, which is an order parameter characterizing the strength of the spontaneous breakdown of chiral symmetry, like it does $M$ or $\sigma_V$. In the present model it can be easily determined from the general expression eq. (2.21) with $K = I$ by applying the Schwinger-Dyson equation (2.20) to eq. (2.16):

$$\langle \bar{q}q \rangle_V = \langle x_E | Tr [iDV]^{-1} | x_E \rangle = \langle - \frac{\mu^2}{g} (\sigma_V - m_0) \rangle$$

(2.22)
2.3. Divergences and Regularization

UV Divergences

In order to extract the UV divergent terms in the fermion determinant $S_{\text{eff}}(\sigma, \bar{\bar{\sigma}})$ we write its real part with eq. (2.19) in the form:

$$\text{Re} S_{\text{eff}}(\sigma, \bar{\bar{\sigma}}) = (-\frac{1}{2}) \text{Sp} \ln D^1 D$$
$$= (-\frac{1}{2}) \text{Sp} \ln [-\partial E^2 + M^2 + ig \bar{\bar{\sigma}}(\sigma + i\bar{\bar{\sigma}}\gamma_5)]$$
$$+ g^2 (\sigma^2 + \bar{\bar{\sigma}}^2 - \sigma V^2)$$

(2.23)

Introducing the abbreviations:

$$G = (-\partial E^2 + M^2)^{-1}$$
$$V = ig \bar{\bar{\sigma}}(\sigma + i\bar{\bar{\sigma}}\gamma_5) + g^2 (\sigma^2 + \bar{\bar{\sigma}}^2 - \sigma V^2)$$

(2.24)

we obtain:

$$\text{Re} S_{\text{eff}}(\sigma, \bar{\bar{\sigma}}) = \frac{1}{2} \text{Sp} \ln G - \frac{1}{2} \text{Sp} \ln (1 + GV)$$

(2.25)

The first term $\frac{1}{2} \text{Sp} \ln$ is an infinite constant independent of $\sigma$ and $\bar{\bar{\sigma}}$ and vanishes after normal ordering. The second term can be expanded in powers of $(GV)$:

$$\ln(1 + GV) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (GV)^n$$

(2.26)

In momentum space the functional trace $\text{Sp}$ gives an integral $\int \frac{d^4k}{(2\pi)^4}$. Therefore we see that all terms with $n \geq 3$ are UV convergent. The contributions $n = 1$ and $n = 2$ can be easily explicitly calculated leading to (Eguchi 1976):

$$\text{Re} S_{\text{eff}}(\sigma, \bar{\bar{\sigma}}) = [(-4N_c I_1(M))]g^2 \int d^4x E (\sigma^2 + \bar{\bar{\sigma}}^2)$$
$$+ [2N_c g^4 I_2(M)] \int d^4x E (\sigma^2 + \bar{\bar{\sigma}}^2 - \sigma V)^2$$
$$+ [(-4N_c g^2 I_2(M)] \int d^4x E \frac{1}{2} [\partial^\mu \sigma(\partial_\mu \sigma) + (\partial^\mu \bar{\bar{\sigma}})(\partial_\mu \bar{\bar{\sigma}})]$$
$$+ \tilde{S}_{\text{eff}}(\sigma, \bar{\bar{\sigma}})$$

(2.27)

where the integral

$$I_k(M) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + M^2)^k}$$

(2.28)

is quadratically divergent for $k = 1$ and logarithmically divergent for $k = 2$. The $\tilde{S}_{\text{eff}}(\sigma, \bar{\bar{\sigma}})$ contains only UV convergent terms and is of higher order in the field amplitudes $V$ and their gradients $\partial V$. A systematic expansion in $(V, \partial V)$ can be performed by the gradient or heat kernel expansion and will be discussed in the next section.

Regularization Schemes

16
The imaginary part \( \text{Im} S_{\text{eff}} \) of the fermionic action in Euclidean space is finite because it is in a fundamental way connected with the anomaly structure of the theory. The real part \( \text{Re} S_{\text{eff}} \) is always divergent corresponding to \( I_1 \) and \( I_2 \) being infinite.

The divergent integrals \( I_1 \) and \( I_2 \) have to be regularized consistently by applying a regularization scheme to \( S_{\text{eff}} \) with a finite UV cutoff \( \Lambda \), which enters as an additional parameter into our model. This is not considered to be a big problem since the model is believed to be an effective low energy approximation of QCD, with the cutoff as the relevant scale for low energy hadronic phenomena. Physically, the cutoff function models in some sense the gluon cloud around a single quark and the correct cutoff function should be derived ideally from QCD itself. Because there exists no such derivation up to now one is forced to apply general schemes which are used in the literature and fixes the cutoff \( \Lambda \) in order to describe the physics of the mesonic sector as well as possible. In practice there is no reason to prefer some scheme in favor of the other provided some constraints concerning symmetries are obeyed (Ball 1989). The question of different regularization schemes has been treated in detail by Meissner Th et al. (1990b) for the vacuum and by Blotz et al. (1990) and Doering et al. (1992) in the soliton sectors. In this review we will mainly use two of them for practical calculations: Proper time (Schwinger 1951) and Pauli-Villars (1949), as it is used in this article, both preserving Lorentz-, chiral and vector-gauge invariance (Ball 1989). Actually both can be obtained from a generalized proper-time formalism (Ball 1989).

Gauge Invariant Schemes

The generalized proper-time regularization procedure (Ball 1989) is based on the following identity for the difference of two logarithms

\[
\log \omega - \log \beta = \lim_{\Lambda \to \infty} \int_0^\infty \frac{d\tau}{\tau} \phi(\tau, \Lambda)(e^{-\tau \omega} - e^{-\tau \beta})
\]

with the additional condition \( \phi(\tau, \infty) = 1 \) and a properly chosen shape for \( \phi(\tau, \Lambda) \) in the limit of small \( \tau \). Using this representation and assuming that the vacuum contribution is subtracted, the generalized proper time regularized effective action reads

\[
\log \det D_1^\dagger D_0 \log \det D_0^\dagger D_0 = \int_0^\infty \frac{d\tau}{\tau} \phi(\tau, \Lambda) \left[ \text{tr} e^{-\tau D_1^\dagger D_0} - \text{tr} e^{-\tau D_0^\dagger D_0} \right]
\]

The original proper time method of Schwinger corresponds to the choice

\[
\phi(\tau) = \Theta \left( \frac{1}{\Lambda^2} - \tau \right)
\]

with \( \Theta \) as a single step function\(^{[1]}\). The Pauli-Villars scheme is obtained as

\[
\phi(\tau) = 1 - (1 + \Lambda^2 \tau)e^{-\tau \Lambda^2}
\]

corresponding to usual mass subtractions in Feynman diagrams.

---

\(^{[1]}\)See sect. 6 for the double step proper time regularization, which offers some more degrees of freedom.
Some other schemes have been proposed, which however do not show the invariances mentioned above and will not be used in this article. Those are e.g. a relativistic four dimensional sharp cutoff method

$$\text{Tr} \log D \rightarrow \text{Tr} \{ \theta (-\partial^2 - \Lambda^2) \log D \}$$  \hspace{1cm} (2.33)

and also a three dimensional sharp cutoff

$$\text{Tr} \log D \rightarrow \text{Tr} \{ \theta (-\nabla^2 - \Lambda^2) \log D \}$$  \hspace{1cm} (2.34)

**Dynamical Mesonic Terms**

As we can see the logarithmically divergent terms are proportional to the kinetic mesonic part $$[(\partial^\mu \sigma)(\partial_\mu \sigma) + (\partial^\mu \overline{\pi})(\partial_\mu \overline{\pi})]$$ and the Mexican hat self interaction $$(\sigma^2 + \overline{\pi}^2 - \overline{\pi}^2)^2$$. These are exactly the dynamical mesonic terms appearing in the linear chiral sigma model (Gell-Mann and Levi 1960). We therefore have *generated* those parts in our model from the fermion determinant $$S_{eff} F(\sigma, \overline{\pi})$$. The proportionality constants in front of them are dependent on the cutoff $$\Lambda$$ and will therefore be determined by the physics of the mesonic sector.

### 2.4. Gradient and Heat Kernel Expansion

The effective action is a highly non-local functional in the $$\sigma$$ and $$\overline{\pi}$$ fields, i.e. it involves products of fields at all space-time points. Therefore it is rather interesting to investigate its behavior in some limiting cases. In this section we will study the case of slowly varying fields, since then the effective action reduces to a local Lagrangian. We will also see how kinetic and interacting contributions arise in this simple limit as a consequence of the polarization of the Dirac sea. This allows also to study some qualitative and quantitative features of the model. There are basically two schemes: gradient and heat kernel expansion.

In essence, the gradient expansion is an expansion in the number of Lorentz indices. The problem is merely technical and many methods have been suggested (Aitchison and Frazer 1984, 1985a, 1985b). A very elegant and powerful one has been proposed by Chan (1985), so that it will be exposed here. The method has been worked out assuming the cyclic property to be valid so that the results are automatically vector gauge invariant. Chan (1985) has applied it to second order elliptic operators up to the fourth order and more recently Caro and Salcedo (1993) up to sixth order. Hence it makes sense to apply it to the real part of the regularized effective action with

$$D^\dagger D = -\partial^2 + i\theta \overline{\Sigma} + \overline{\Sigma} \Sigma$$

with $$\Sigma = \phi(\sigma + i\overline{\pi} \gamma_1)$$. The main idea is to realize that the integral representing a one loop Feynman diagram is invariant if all external momenta are shifted by the same amount, so that one can as well average over all possible shifts. In the proper-time scheme this can be expressed as follows

$$\text{Re} S_{eff} = \int_0^\infty \frac{d\tau}{\tau} \phi(\tau, \Lambda) \text{Sp} \exp \left[ -\tau \left( -\partial^2 + i\theta \overline{\Sigma} + \overline{\Sigma} \Sigma \right) \right]$$
The next step is to expand the exponential in powers of the derivative operator $\partial$ up to the desired order and to make use of the cyclic property. This corresponds to a derivative or gradient expansion. If one further expands around the vacuum configuration $\Sigma = M$ one gets the heat kernel expansion. In other words, to get a given order of the derivative expansion one needs an infinite number of terms from the heat kernel expansion. The final expressions for the known orders are rather lengthy and will not be quoted here. For our purpose it is sufficient and instructive to restrict ourselves to the heat kernel expansion up to second order. The result is (Kleinert 1976, Ebert and Reinhardt 1986)

$$\mathcal{L} = N_c g^2 I_2(M) \text{tr}_\tau \left[ \partial_\mu \Sigma \partial^\mu \Sigma + (\Sigma \Sigma^\dagger - M^2)^2 \right] + \left( \frac{g^2}{3g^2} - 2N_c g^2 I_1(M) \right) \text{tr}_\tau (\Sigma \Sigma^\dagger - M^2)$$

(2.35)

The $I_n$-integrals are given by (2.28). The important point of this expression is that even for slightly space-time dependent fields the whole effect can be summarized in a kinetic and interaction terms for the fields $\sigma$ and $\vec{\pi}$. Thus although these did not appear explicitly in the original lagrangian, they are indeed present due to the polarization of the Dirac sea. In chapter 3 we will show how one can go beyond the limit of slowly varying fields by computing the effective action in an exact manner. The similarity of eq. (2.35) with the Gell-Mann-Levy (1960) sigma model is also very interesting and will be exploited later when fixing the parameters. The effective Lagrangian gets even simpler if the so called chiral circle condition

$$\sigma^2 + \vec{\pi}^2 = f_\pi^2$$

is imposed. Then one can use the parameterization

$$\sigma + i\vec{\pi} \cdot \vec{\pi} = f_\pi U; \quad U = e^{i\vec{\pi} \cdot \vec{\pi}}$$

with a $\text{SU}(2)$ unitary matrix $U$ and $\vec{\pi}$ as the non-linearly transforming pion field (Coleman et al. 1969). The Lagrangian reads then

$$\mathcal{L} = N_c M^2 I_2(M) \text{tr}_\tau \partial_\mu U \partial^\mu U^\dagger$$

(2.36)

which resembles the Weinberg (1967) non-linear Lagrangian. The former discussion carries along also for the currents. We just quote the final result for the vector and axial currents for linear chiral fields in the same approximation respectively

$$\vec{V}^\mu = \vec{\pi} \times \partial^\mu \vec{\pi}$$

$$\vec{A}^\mu = \sigma \partial^\mu \vec{\pi} - \vec{\pi} \partial^\mu \sigma$$

Gradient Expansion of the Imaginary Part - Wess-Zumino-Witten-Term

The gradient expansion of the imaginary part of the fermion determinant eq. (2.18) is more involved. The problem is, that the corresponding term can not be written as 4-dimensional space
time integral over a local Lagrangian. In order to obtain a closed analytical form for the gradient expansion of $\text{Im} S_{\text{eff}}^F$ one considers the change $\delta_U \text{Im} S_{\text{eff}}^F(U)$ under the variation $\delta U$ of the chiral field $U$ and performs a gradient expansion of this expression. The actual calculation (Dhar and Wadia 1984, Dhar et al. 1985, Diakonov et al. 1988) follows very closely the Goldstone-Wilczek expansion of the anomalous baryon current as it is described in App. B. The result is:

$$i \delta_U \text{Im} S_{\text{eff}}^F(U) \bigg|_{\text{grad}} = -\frac{i N_c}{48 \pi^2} \int d^4 x \epsilon_{ijk} \delta U \text{Tr}_{\text{flavor}} \left[ \left( U^\dagger \partial_i U \right) \left( U^\dagger \partial_j U \right) \left( U^\dagger \partial_k U \right) \right]$$

which can be shown to be the variation of

$$i \text{Im} S_{\text{eff}}^F(U) \bigg|_{\text{grad}} = -\frac{i N_c}{24 \pi^2} \int_{B_5} d^5 x \epsilon_{\mu\nu\rho\lambda} \text{Tr}_{\text{flavor}} \left[ \left( U^\dagger \partial_{\mu_1} U \right) \left( U^\dagger \partial_{\mu_2} U \right) \left( U^\dagger \partial_{\mu_3} U \right) \left( U^\dagger \partial_{\mu_5} U \right) \right]$$

(2.37)

where $B_5$ denotes a 5-dimensional sphere, whose boundary is the 4-dimensional space-time ($\partial B_5 = R_4$). The r.h.s. of eq. (2.37) is the famous Wess-Zumino-Witten term, which has been discovered in context of the integration of the chiral anomaly (Wess and Zumino 1971, Wess 1972). It turned out that it is indispensable to implement it in any effective chiral mesonic model in order to get a proper description of low energy mesonic phenomena (Witten 1983a). In case of an $SU(2)$ chiral field $U$ it can be shown to vanish identically as it does always, if $U$ is time independent. Further details will be discussed in the context of the $SU(3)$ NJL in chapter 6.

Goldstone and Wilczek (1981) have shown that in the lowest non-vanishing order of the derivative expansion the baryon current is proportional to the topological current

$$< B^\mu(x) > = -\frac{1}{24 \pi^2} \epsilon^{\mu\nu\sigma\beta} \text{Tr} \left( U^\dagger \partial_\nu U U^\dagger \partial_\sigma U U^\dagger \partial_\beta U \right) + \cdots$$

(2.38)

In the next chapter the relevance of such topological current in the NJL model will be discussed.

2.5. Mesonic spectra from effective actions

The calculation of on-shell mesonic two-point functions within the NJL model has been undertaken by several authors, both in a pure fermionic language (Blin et al. 1988; Bernard et al. 1988; Bernard and Meissner 1988; Klimt et al. 1990; Takizawa et al. 1990) and in a bosonized version (Alkofer and Zahed 1990; Jaminon et al. 1992). As we shall see in Sect. 2.7, this is actually a natural step because the parameters of the NJL model will be fixed by reproducing the meson masses. In most cases a diagrammatic procedure has been considered within a Bethe-Salpeter formalism in the ladder approximation. Similar results have been also obtained in a time-dependent Hartree-Fock approach (da Providencia 1987). The virtue of the on-shell defitition method is that no approximation is involved apart from working in the leading order of the large $N_c$ expansion. As it can be shown, the heat kernel and derivative expansions are low energy approximations to the complete on-shell two-point function. In this section we sketch the calculation of mesonic spectra in the bosonized version of the model as it has been done by Jaminon et al. (1992). This is based
on the observation, that the relevant quadratic part $S_{eff}^{F,2p}$ of the effective action (2.23) for SU(2) scalar fields can be always written in the form (Jaminon et al. 1992)

$$S_{eff}^{F,2p}(\sigma, \pi) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \left( \sigma(q) \sigma(-q) Z_\sigma(q) \left[ q^2 + m_\sigma^2(q) \right] + \pi(q) \pi(-q) Z_\pi(q) \left[ q^2 + m_\pi^2(q) \right] \right)$$

(2.39)

Therefore one can define mesonic masses from

$$\frac{1}{Z_\sigma(q^2)} \frac{1}{Z_\pi(q^2)} \frac{\delta^2 S_{eff}^{F,2p}(\sigma, \pi)}{\delta \sigma(p_1) \delta \sigma(p_2)} \bigg|_{p=p_1=-p_2, \sqrt{p^2}=q^2} = \left( p^2 + m_\sigma^2 \right) \bigg|_{p^2=q^2}$$
$$\frac{1}{Z_\sigma(q^2)} \frac{1}{Z_\pi(q^2)} \frac{\delta^2 S_{eff}^{F,2p}(\sigma, \pi)}{\delta \pi(p_1) \delta \pi(p_2)} \bigg|_{p=p_1=-p_2, \sqrt{p^2}=q^2} = \left( p^2 + m_\pi^2 \right) \bigg|_{p^2=q^2}$$

(2.40)

which effectively corresponds to a rescaling of the fields according to

$$\sigma' = Z_\sigma^{1/2}(q^2) \sigma, \quad \pi' = Z_\pi^{1/2}(q^2) \pi$$

(2.41)

The on-shell definitions of the mesons can be found then by evaluation of $Z_\sigma(-q^2 = m_\sigma^2)$ and $Z_\pi(-q^2 = m_\pi^2)$ to obtain $m_\sigma^2 = m_\sigma^2(-q^2 = m_\sigma^2)$ and $m_\pi^2 = m_\pi^2(-q^2 = m_\pi^2)$, where the Z-factors are given by the Feynman integrals of the form (Jaminon and Ripka 1993)

$$Z_\phi(q^2) = 4N_c g^2 \int_{reg} \frac{d^4k}{(2\pi)^4} G(k + q/2) G(k - q/2)$$

(2.42)

with the propagator $G(k) = 1/(k^2 + M^2)$. From these expressions it is clear that the corresponding off-shell definitions can be obtained by setting $q^2 = 0$ in all the equations. These expressions then coincide of course with the corresponding one, which one would obtain from a gradient expansion of $S_{eff}^{F}$ from the very beginning. Then the approximated analogue of (2.39) is already given by (2.27). The field redefinitions become trivial in this case and the mesonic spectra could be read off directly from the effective potential (Coleman 1985). For definiteness we present $Z_\phi(q^2)$ explicitly for the proper time regularization eq. (2.31)

$$Z_\phi(q^2) = \frac{4N_c g^2}{16\pi^2} \int dx \Gamma(0, [M^2 - x(1 - x)q^2]/\Lambda^2)$$

(2.43)

where we made use of the incomplete $\Gamma(n,x)$-function

$$\Gamma(n, x) := \int_x^\infty dt t^{n-1} e^{-t}, \quad x \geq 0$$

(2.44)
2.6. Fixing of the Parameters in the Mesonic Sector

We have mentioned already that if the coupling constant $G$ is higher than a certain critical value, the vacuum is unstable if the mass is increased. This leads eventually to the Goldstone phase. Assuming that this is the case we will fix the parameters of the model. Two ways for parameter fixing may be distinguished depending upon the particular approximation involved. We will refer to them for brevity as the on-shell and the off-shell condition. The off-shell fixing of the parameters is discussed in detail by Meissner Th et al. (1990b).

Mesonic Properties

Looking at eqs. (2.16, 2.19) we see that we have $m_0, \mu^2, \Lambda, \sigma_V$ and the constituent quark mass $M$ as the *free* parameters of the model. Thus one can impose e.g. the following *four* conditions

1) The stationary phase condition $\frac{\delta S_{eff}}{\delta \sigma}$ in the vacuum state

\[
\left. \frac{\delta S_{eff}}{\delta \sigma} \right|_{vac} = \left( \mu^2 - 8 N_c g^2 I_1(M) - \frac{m_0 \mu^2}{g \sigma_V} \right) \sigma_V = 0
\]

immediately determines $\mu^2$ in the broken phase with $\sigma_V \neq 0$.

2) Fixing of the pion decay constant $f_\pi$ is done by considering the expectation value of the axial current between a pion state and the vacuum, which in the semibosonized form corresponds to consider

\[
< 0 \mid A_\mu^a(x) \mid \pi^a > = \mathcal{N} \int Dq D\bar{q} \frac{\bar{q}(x)\gamma_\mu \gamma_5 q(x)}{\sqrt{\mathcal{Z}}} e^{i \int \bar{q}(x) \mathcal{L}}.
\]

where $\mathcal{N}$ is the normalization constant. This is a similar like eq. (2.21) but now it has to be evaluated for a plane-wave pion state and not the vacuum. In the one-fermion loop approximation we obtain then

\[
< 0 \mid A_\mu^a(x) \mid \pi^a > = \sigma_V \mathcal{Z}^{1/2}(q^2) p_\mu \pi^a(x).
\]

Comparison with the known matrix element for the pion-decay from current algebra (Cheng and Li 1984)

\[
< 0 \mid A_\mu^a(x) \mid \pi^a > = i p_\mu f_\pi(q^2) = -m_\pi^2 \pi^a(x)
\]

gives immediately the relation

\[
f_\pi(q^2) = \sigma_V \mathcal{Z}^{1/2}(q^2)
\]

for $\mathcal{Z}_\pi$ and $\sigma_V$.

3) From the definition of the pion mass in eq. (2.40) one obtains straightforwardly the explicit form

\[
m_\pi^2 = \frac{m_0 \mu^2}{g \sigma_V} \frac{1}{\mathcal{Z}_\pi}
\]

4) Now one can either choose the normalization of the pion wave function eq. (2.41) according to

\[
\mathcal{Z}_\pi(q^2 = -m_\pi^2) = 1
\]
or one can equivalently require the classical PCAC relation (Goldberger and Treiman, 1958)

\[ \partial \mu A_\mu^a = f_\pi m_\pi^2 \pi^a, \]

which in the present model gives in analogy to eq. (2.8) but now from eq. (2.12):

\[ \partial \mu A_\mu^a = \frac{m_q \mu^2}{g} \pi^a \]

(2.52)

so that together with eq. (2.50) and eq. (2.49) again eq. (2.51) follows.

This fixes all the parameters except one. Usually the constituent quark mass \( M \) is chosen as the free parameter. Then from eq. (2.51), which determines the cutoff \( A \) for a given constituent quark mass \( M \), and from eq. (2.49), one can deduce \( \sigma V = f_\pi \), so that eq. (2.50) inserted in eq. (2.45) gives \( \mu^2 \). Using again eq. (2.50) gives the current quark mass \( m_0 \). So \( M = g f_\pi \) is the only free parameter of the model.

The above description refers to the on-shell definition of the parameters, whereas the off-shell formulation corresponds to setting \( q^2 = 0 \) in eq. (2.51). Because the Goldstone bosons and especially the pions have a small mass compared to the constituent quark mass, the difference between both methods is rather small. However this is no longer the case, when heavier mesons like \( \rho, \omega \) and \( A_1 \) are taken into account.

Furthermore, using again eq. (2.22) and eq. (2.50), it follows the famous Gell-Mann, Oakes and Renner (1968) relation

\[ < \bar{q} q > m_0 = -m_\pi^2 f_\pi^2 + O(m_0) \]

(2.53)

Numerical Results

As we have said already the off-shell definition is only a low momentum approximation to the on-shell condition. For on-shell pions the external momenta are rather small so that the numerical deviations from the two methods turn out to be less than 7%. The only important difference is that in the chiral limit (\( m_0 = 0 \)) the scalar meson mass acquires a value \( m_\sigma = 2M \) in the gradient expansion, while in the on-shell method the \( \sigma \) meson lies in the quark-antiquark continuum with a finite decay width into this channel. This is the first manifestation of the lack of confinement within the model.

The dependence of \( \Lambda \) against \( M \) can be seen at Fig. 2.1. In the relevant region of \( M = 350\ldots 450 \) MeV, the numerical value of the cutoff \( \Lambda \) depends noticeably on the regularization scheme chosen.

From eq. (2.22) we can now evaluate the quark condensate and compare it with the generally accepted value (Shuryak 1986) of

\[ < \frac{1}{2}(\bar{u}u + \bar{d}d)^{1/2} > \approx -(225 \pm 24) \text{MeV} \]

(2.54)

The results for the vacuum sector, i.e. quark condensate, current quark mass and the vacuum energy density (de Grand et al. 1975) can be seen at Tab. 2.1. For these observables, i.e. for the quark condensate and the current quark mass a plateau has been observed for constituent masses above 400 MeV.
3. Systems with Finite Baryon Numbers - Solitonic Solutions

In this chapter we consider static (time independent) meson field configurations. First we show the expression for the total energy (sect. 3.1). A system with baryon number $B = 1$ is obtained by adding $N_c = 3$ valence quarks, which is formally done by introducing a thermochemical potential $\mu$. Furthermore we discuss the connection between baryon number $B$ and topological winding number $n$ of the $\mathbb{R}$ field (sect.3.2). The mean field equations of motion for the non-linear model (meson fields restricted to the chiral circle) are derived and solved for a $B = 1$ system (sect.3.3). Systems with higher baryon and winding number are briefly discussed (sect.3.4). Finally we show that for the linear Nambu–Jona-Lasinio model (off the chiral circle) no solitonic solution exists but the system collapses to a zero energy and zero size configuration (sect.3.5).

3.1. Static Meson Field Configurations - Energy and Static Expectation Values

1-Particle Hamiltonian $h$ and Dirac Spectrum

In order to construct a system with finite baryon number $B$ one has to consider meson field configurations different from the vacuum. Because we are interested in static properties of the baryon, we will restrict ourselves to time independent meson fields $\sigma$ and $\vec{\pi}$. For those the Euclidean Dirac operator $i\mathcal{D}$ can be separated into a trivial time derivative $\frac{\partial}{\partial \tau}$ as well as the time-independent 1-quark hamiltonian $h$:

$$-i\mathcal{D} = \beta(\frac{\partial}{\partial \tau} + h)$$

where

$$h = \frac{\alpha_\pi}{2} + g\beta [\sigma(\vec{r}) + i\vec{\pi}(\vec{r})\gamma_5]$$

and in the vacuum:

$$h_V = \frac{\alpha_\pi}{2} + \beta M$$

It is our aim to express the operator $\text{Sp}n(-i\mathcal{D})$ through the eigenvalues of $h$. One should note that $h$ is hermitian and traceless: $\text{Tr} h = 0$. Furthermore we make the hedgehog ansatz for $\sigma$ and $\vec{\pi}$:

$$\sigma(\vec{r}) = \sigma(r)$$
$$\vec{\pi}(\vec{r}) = i\pi(r)$$

For chiral models which use a gradient or heat kernel expansion for the highly non local fermion determinant $\text{Sp}n(-i\mathcal{D})$ like the Skyrme model (cf. sect.8.1) or the Gell-Mann–Levi diurnal sigma model (cf. sect.8.2) it has been proven (Ruiz-Arriola et al. 1989) that the hedgehog shape eq. (3.4) is a necessary condition for the meson fields to be a solution of the time independent classical mean field equations of motion for a baryonic system. One has to admit that there exists up to now no such proof for the case if $\text{Sp}n(-i\mathcal{D})$ is treated exactly as it is done in the present approach.

Due to the hedgehog ansatz (3.4) the 1-particle hamiltonian $h$ (3.2) commutes with the Grand spin $G^2$ (where $\vec{G} = \vec{J} + \vec{T}$, $\vec{J}$: total spin, $\vec{T}$: isospin), its $z$-component $G_z$ as well as the parity...
II and the 4 observables $(h, G^2, G_z, II)$ form a complete set of commuting operators. Therefore the eigenstates $| \lambda \rangle$ of $h$:

$$h | \lambda \rangle = \epsilon_\lambda, \quad \langle \hat{\tau} | \lambda \rangle = \phi_\lambda(\hat{\tau})$$

$$h_V | \lambda V \rangle = \epsilon_{\lambda V}, \quad \langle \hat{\tau} | \lambda V \rangle = \phi_{\lambda V}(\hat{\tau})$$

(3.5)

can be characterized by the 4 quantum numbers $\epsilon_\lambda, G, G_z$ and $(-)^l$, where $G_z$ is degenerated. Following Kahana and Ripka (1984) the eigenvalue problem eq. (3.5) can be solved numerically by putting the system in a large but finite sphere with radius $D$ and appropriate boundary conditions for the radial part of the free eigenfunctions at $D$. In doing so one obtains a basis consisting of discretized wave numbers $k_n$, which belong to a given grand spin $G$ and parity $(-)^l$. For the numerical diagonalization of the matrix of $h$ one takes wave numbers smaller than a given numerical cutoff frequency $K_{max}$ into account. $K_{max}$ and $D$ have to be chosen large enough so that the value of any calculated observable does not change any more by further increasing $K_{max}$ and $D$. It should be emphasized that $K_{max}$ is a purely numerical cutoff and has nothing to do with the model intrinsic physical UV cutoff $\Lambda$ introduced and fixed in Chap. 2 in order to reproduce the pion decay constant $f_\pi$.

Fig. 3.1 shows the spectrum of $h$ for a meson profile on the chiral circle (non-linear realization of chiral symmetry) as example:

$$\sigma^2 + \pi^2 = \sigma^2_V + \pi^2_V = f_\pi^2$$

$$\sigma(r) = f_\pi \cos \theta(r)$$

$$\pi(r) = f_\pi \sin \theta(r)$$

(3.6)

where the chiral angle $\theta(r)$ is parameterized by

$$\theta(r) = -n \pi e^{-\hat{\tau}}$$

(3.7)

with the topological winding number $n = 1$ (Kahana et al. 1984). The parameter $R$ characterizes the size of the meson profile and in a way describes the magnitude of the deviation between the actual and the the vacuum field configuration ($R = 0$). If $R$ is large enough ($R \cdot M \geq 0.5$) one finds bound orbitals both in the positive and the negative spectrum. The positive bound state with lowest single particle energy and quantum numbers $G^P = 0^+$, will be called the valence orbit $| \lambda \rangle = | val \rangle$. Its single particle energy decreases with increasing $R$, switches sign and gets finally part of the negative spectrum. This level originates from the positive continuum and gets bound and localized by interacting with the negative continuum (Dirac sea). This simple non-linear mechanism creates the soliton.

**Static Energy**

Because the meson fields are time independent, the trace over the Euclidean time coordinate $\tau$ in the fermion determinant can be performed trivially. Assuming anti-periodic boundary conditions in the Euclidean time interval $[-T, T]$ the Euclidean time gradient $\frac{\partial}{\partial \tau}$ has the spectrum $u_n = \frac{(2n+1)}{2} \pi$, $n = 0, \pm 1, \pm 2, \ldots$. Therefore the expression for any functional $F$ of the form $F \left[ \frac{\partial}{\partial \tau}, f(\hat{\tau}) \right]$
simplifies in the zero temperature limit \((T \rightarrow \infty, \Delta u_n \rightarrow 0)\), which leads to the the ground state of the system, to:

\[
\text{Sp} \frac{\partial}{\partial \tau} \left[ F_{\tau, \tau} \right] = \int d^4 x E \text{Tr} \left[ F_{\tau, \tau} \frac{\partial}{\partial \tau} \right] \mid x_E > = T \int d^3 x E \int \frac{du}{2\pi} \text{Tr} F \left[ -iu, f(\bar{\tau}) \right] \tag{3.8}
\]

Especially if we take for \(F\) the effective action \(S_{\text{eff}}\) we find for the imaginary part:

\[
\text{Im} S_{\text{eff}}(\sigma, \bar{\tau}) = \text{Im} S_{\text{eff}}^F(\sigma, \bar{\tau}) = -\sum_{\lambda} \int \frac{du}{2\pi} \text{Im} \text{ln}(-iu + \epsilon_\lambda) = \sum_{\lambda} \int \frac{du}{2\pi} \arctan \frac{u}{\epsilon_\lambda} = 0 \tag{3.9}
\]

which is based on the fact that \(\hbar\) is hermitian and therefore \(\epsilon_\lambda\) real. As we have already mentioned this is not true if \(\omega\)-mesons are included in the model (cf. chap. 7). We therefore have:

\[
S_{\text{eff}}(\sigma, \bar{\tau}) = \text{Re} S_{\text{eff}}(\sigma, \bar{\tau}) = T \cdot E \tag{3.10}
\]

where \(E\) is the total static energy of the system. The explicit expressions for the various parts of \(E\) in terms of the eigenvalues \(\epsilon_\lambda\) read:

\[
E = E_{\text{sea}} + E_{\text{mes}} + E_{\text{br}} \tag{3.11a}
\]

with:

\[
E_{\text{sea}} = \left( -\frac{N_c}{2} \right) \left[ \sum_{\lambda} R_1(\epsilon_\lambda, A) - \sum_{\lambda V} R_1(\epsilon_{\lambda V}, A) \right] \tag{3.11b}
\]

\[
E_{\text{mes}} = \frac{\hbar^2}{2} \int d^3 r (\sigma^2 + \bar{\sigma}^2 - f^2_z) \tag{3.11c}
\]

\[
E_{\text{br}} = -m^2_z f_z \int d^3 r (\sigma - f_z) \tag{3.11d}
\]

In case of the proper-time regularization method, which we will restrict ourselves on for explicitness in the following, the regularization function \(R_1\) is given by (Meissner Th et al. 1988):

\[
R_1(\epsilon_\lambda, A) = (-) \frac{A}{\sqrt{4\pi}} \int_1^\infty ds s^{-3/2} e^{-s(\epsilon_\lambda/A)^2} \tag{3.12}
\]

A detailed and complete discussion of the various regularization schemes can be found in Blotz et al. (1990) and in Doering et al. (1992). Furthermore it should be noted, that because of \(\text{Tr} \hbar = 0\) in the unregularized case \((A \rightarrow \infty)\) eq. (3.11b) with eq. (3.12) can be written as sum over the negative (occupied) states

\[
E_{\text{sea}} = \sum_{\lambda, \epsilon_\lambda < 0} \epsilon_\lambda - \sum_{\lambda V, \epsilon_{\lambda V} < 0} \epsilon_{\lambda V} \tag{3.13}
\]

which corresponds to the naive picture of the occupied negative states (Dirac sea) (cf. Fig. 1.3). If a regularization is performed \(\text{tr} \ h = 0\) is only approximately true. Since its numerical value in
realistic cases of baryon number $B = 1$ is about 50 MeV it actually does not matter if one runs over the whole spectrum or only over the occupied states.

**Static Observables**

Finally we consider the expectation value of the static observable $O := \frac{1}{T} \int d^4 x \bar{q}(x) \beta O q(x)$, where $O$ represents an arbitrary time independent operator in Dirac- and isospin space. Because of $O = \frac{1}{2} \int d^4 x \bar{q}(x) \beta O q(x)$ we have after Wick rotation into Euclidean space:

$$\langle O^E \rangle_{sea} = \frac{1}{T} \int \frac{DqD\bar{q}}{Dq} \int d^4 x E \left[ \bar{q}(x) \beta O^E q(x) \right] e^{\frac{i}{T} \int \frac{d^4 x E}{d^4 x} (x)(-iD)q(x)}$$

$$= \frac{1}{T} \frac{\partial}{\partial \omega_E} \text{Sp} \left[ -iD - \omega_E \beta O^E \right]_{\omega=0} - \text{vac. contr.} \quad (3.14)$$

Using the relation

$$\lim_{k \to \infty} \int_{-k}^{+k} \frac{du}{2\pi i u - ix} = \frac{1}{2} \text{sign} x \quad (3.15)$$

one gets in the unregularized case:

$$\langle O^E \rangle_{sea} = \frac{1}{2} N_c \sum_{\lambda} (-\text{sign} \epsilon_\lambda) O^E_\lambda - \text{vac. contr.} \quad (3.16a)$$

with:

$$O^E_\lambda := \int d^3 r \bar{\phi}^\dagger_\lambda(\vec{r}) O^E(\vec{r}) = \int d^3 r \bar{\phi}_\lambda(\vec{r}) \beta \phi_\lambda(\vec{r}) \quad (3.16b)$$

In the regularized case one has to substitute $[\text{Sp} \ln] \to [\text{Sp} \ln]_{\text{reg}}$, which gives after rotating back into Minkowski space:

$$\langle O \rangle_{sea} = (+N_c) \sum_{\lambda} R_2(\epsilon_\lambda, \Lambda) \cdot O_\lambda - \text{vac. contr.} \quad (3.17)$$

where for the proper time method the explicit expression for the regularization function reads:

$$R_2(\epsilon_\lambda, \Lambda) = (-) \frac{1}{\sqrt{4\pi}} \int_1^\infty dss^{-1/2} \left( \frac{\epsilon_\lambda}{\Lambda} \right) e^{-s(\epsilon_\lambda/\Lambda)^2} \quad (3.18)$$
3.2. Baryon- and Topological Winding Number - Chemical Potential - Valence-Quarks

Baryon Number and Baryon Density

The baryon current $b_\mu$ is generally defined as:

$$b_\mu(x) := \frac{1}{N_c} \bar{q}(x) \gamma_\mu q(x)$$  \hspace{1cm} (3.19)

Its 4-divergence vanishes $\partial_\mu b^\mu = 0$ due to the $U(1)$-symmetry ($q \rightarrow e^{i\alpha} q$) of the Lagrangian giving rise to a conserved charge, the baryon number

$$B = \int d^3 r \rho_0(x)$$  \hspace{1cm} (3.20)

Applying eqs. (3.16, 3.17) for $\mathcal{O} = I$ we obtain for the expectation value of $B$ in case of time independent fields without regularization:

$$\langle B \rangle = \frac{1}{2} N_c \left[ \sum_\lambda (\text{sign} \epsilon_\lambda) - \sum_{\lambda_V} (\text{sign} \epsilon_{\lambda_V}) \right]$$  \hspace{1cm} (3.21)

For the baryonic density $\rho_0(\vec{r})$ we find correspondingly:

$$\langle \rho_0(\vec{r}) \rangle = \frac{1}{2} N_c \left[ \sum_\lambda (\text{sign} \epsilon_\lambda) \cdot \phi_\lambda(\vec{r}) - \sum_{\lambda_V} (\text{sign} \epsilon_{\lambda_V}) \cdot \phi_{\lambda_V}(\vec{r}) \right]$$  \hspace{1cm} (3.22)

Neither $\langle B \rangle$ nor $\langle \rho_0(\vec{r}) \rangle$ are UV divergent. For $\langle B \rangle$ this is clear from eq. (3.21) whereas for $\langle \rho_0(\vec{r}) \rangle$ it follows from the fact that the 2nd order gradient expansion, which gives rise to the logarithmic divergence, vanishes and the 4th order is already UV convergent (cf. app. B).

Therefore there is no need for a regularization of the baryon number. Furthermore as we see at the end of the next section, this is consistent with a more refined derivation of the baryon number, which in that case originates from the imaginary part of the Euclidean effective action. It is therefore consistent to start with a proper time regularized real part of the effective action as it is done in Chap. 2 and treat the baryon number unregularized, as long as one treats the imaginary part unregularized. However, one should mention that recently by Schlicz et al. (1993) also the imaginary part was regularized though this treatment does not preserve the anomaly structure of the theory.

Furthermore it should be noticed that due to the general method of calculating static expectation values of observables as it has been described in the last section, our expressions for $\langle B \rangle$ and $\langle \rho_0(\vec{r}) \rangle$ differ from the usual ones obtained in Minkowski space, where the sum $\sum_\lambda$ is performed only over the negative (occupied) states. For the global quantity $\langle B \rangle$ it is easy to see that both cases lead to the same result, whereas the local densities $\langle \rho_0(\vec{r}) \rangle$ may differ.

Thermochemical Potential $\mu$; Separation of the Valence Part

In order to constrain the baryon number of the system to a given value (e.g. $\langle B \rangle = 1$ for nucleons and hyperons) we proceed the way known from statistical mechanics (Huang 1987, Negele and
Orland (1987), and introduce the thermochemical potential $\mu$ as a Lagrange multiplier into the generating functional $\mathcal{Z}$ which becomes now the grand canonical sum of states $\mathcal{Z}(\mu)$ (Williams and Cahill 1983, Meissner Th et al 1990)

$$\mathcal{Z}(\mu) = \int \mathcal{D}q \mathcal{D}q^{-} e^{-\int d^4 x \bar{q}(iD - \mu)q}$$

(3.23)

After integrating over the fermion fields we arrive at the fermionic part of the grand canonical effective action

$$S_{\text{eff}}^{F}(\sigma, \bar{\sigma}, \mu) = (-)\text{Sp} \ln(-iD - \mu\beta)$$

(3.24)

The mesonic part remains unaffected. After subtracting the vacuum contribution the $S_{\text{eff}}^{F}$ can be split into a natural way into 2 parts:

$$S_{\text{eff}}^{F}(\sigma, \bar{\sigma}, \mu) - S_{\text{eff}}^{F}(\sigma_V, \bar{\sigma}_V, \mu_V = 0) =
\left[S_{\text{eff}}^{F}(\sigma, \bar{\sigma}, \mu) - S_{\text{eff}}^{F}(\sigma, \bar{\sigma}, \mu = 0)\right] + \left[S_{\text{eff}}^{F}(\sigma_V, \bar{\sigma}_V, \mu = 0) - S_{\text{eff}}^{F}(\sigma_V, \bar{\sigma}_V, \mu = 0)\right]
= S_{\text{val}}^{F}(\sigma, \bar{\sigma}, \mu) + S_{\text{sea}}^{F}(\sigma, \bar{\sigma})$$

(3.25)

where:

$$S_{\text{val}}^{F}(\sigma, \bar{\sigma}, \mu) = \left[S_{\text{eff}}^{F}(\sigma, \bar{\sigma}, \mu) - S_{\text{eff}}^{F}(\sigma, \bar{\sigma}, \mu = 0)\right]$$

(3.26)

and

$$S_{\text{sea}}^{F}(\sigma, \bar{\sigma}) = \left[S_{\text{eff}}^{F}(\sigma_V, \bar{\sigma}_V, \mu = 0) - S_{\text{eff}}^{F}(\sigma_V, \bar{\sigma}_V, \mu = 0)\right]$$

(3.27)

The $S_{\text{val}}^{F}(\sigma, \bar{\sigma})$ is of course nothing but the fermionic contribution (Dirac sea) to the effective chiral action $S_{\text{eff}}^{F}(\sigma, \bar{\sigma}) = T E_{\text{sea}}$ considered in the last section. In general it needs regularization. The valence contribution $S_{\text{val}}^{F}(\sigma, \bar{\sigma}, \mu)$ is finite and needs not to be regularized. It can be expressed in terms of the eigenvalues $\epsilon_{\lambda}$ by:

$$S_{\text{val}}^{F}(\sigma, \bar{\sigma}, \mu) = N_{c} T \sum_{0 < \epsilon_{\lambda} < \mu} (\epsilon_{\lambda} - \mu)$$

(3.28)

where we have used the relation:

$$\ln(-iu + \epsilon_{\lambda} - \mu) - \ln(-iu + \epsilon_{\lambda}) = \int_{\epsilon_{\lambda}}^{\epsilon_{\lambda} - \mu} dx \frac{1}{-iu + x}$$

(3.29)

as well as eq. (3.15). Similarly one can separate the grand canonical expectation value of any observable $O := \int d^3 r \bar{q}^{\dagger} O q$ into a valence and a sea part:

$$\langle O \rangle(\mu) = \langle O \rangle_{\text{val}}(\mu) + \langle O \rangle_{\text{sea}}$$

(3.30)

with

$$\langle O \rangle_{\text{val}}(\mu) = \langle O \rangle(\mu) - \langle O \rangle(\mu = 0) = N_{c} \sum_{0 < \epsilon_{\lambda} < \mu} O_{\lambda}$$

(3.31)
Especially for the baryon number $\langle B \rangle$, which is the integral of the time component of the baryon current $b^E_{\mu} = (1/N_c) \bar{q} \gamma^E_{\mu} q$ in Euclidean space, one obtains with setting $\beta E = \gamma_4$ in eq. (3.14) and using $k^E_0 = -i \epsilon^E_4$ (cf. App.A):

$$
\langle B \rangle (\mu) = \frac{-i}{N_c T} \int \frac{d}{d\omega_E} \text{Splog} \left( -iD - \omega_E \beta E - \mu \beta \right) \left( \sum_{\lambda} \frac{1}{u + i(\epsilon_\lambda - \mu)} \right) = \left( -\frac{1}{\pi} \right) \sum_{\lambda} \text{sign}(\epsilon_\lambda - \mu) = B_{\text{val}}(\mu) + B_{\text{sea}}
$$

with

$$
\langle B \rangle_{\text{val}}(\mu) = \sum_{0 < \epsilon_\lambda < \mu} 1 \quad (3.33)
$$

and

$$
\langle B \rangle_{\text{sea}} = \left( -\frac{1}{\pi} \right) \sum_{\lambda} \text{sign}\epsilon_\lambda \quad (3.34)
$$

Analogous expressions hold for the baryon density.

In any case one notices that the valence contribution is finite and that there is no need for a regularization of this part. It is also important to realize that the thermodynamical potential $\mu$ is a real number both in the Minkowski space and in the Euclidean space, which causes finally simply a shift in the eigenvalues $\epsilon_\lambda \rightarrow \epsilon_\lambda - \mu$. It is therefore different from the treatment of the time component of a four vector $\omega_\mu$, which has to be Wick rotated from Minkowski into Euclidean space like the real time: $\omega_0 \rightarrow i \omega_4$, where $\omega_0$ and $\omega_4$ are both real (cf. App.A). From that and from the antihermiticity of $\gamma_4$ it is clear from eq. (3.32) that the baryon number $B(\mu)$ originates from the imaginary part of the effective Euclidean action. Therefore the baryon number is finite and need no regularization. This philosophy will be prosecuted throughout this work. Actually the imaginary part is not zero in the present case due to the additional term: $\omega E \beta O E$.

Using eqs. (3.32) and (3.28) we can write the valence part of the grand canonical effective chiral action as:

$$
\frac{1}{T} S_{\text{eff}}^{\text{val}} = N_c \left[ \sum_{0 < \epsilon_\lambda < \mu} \epsilon_\lambda - \mu B_{\text{val}}(\mu) \right] \quad (3.35)
$$

As it is known from statistical mechanics the total energy of a system at zero temperature $T = \infty$ (ground state) is given by:

$$
E_{\text{tot}}(\mu) = \frac{1}{T} S_{\text{eff}}(\mu) + N_c \mu (B)(\mu) \quad (3.36)
$$

which leads finally to:

$$
E_{\text{tot}}(\mu) = E_{\text{sea}} + \epsilon_{\text{val}}(\mu) + E_{\text{mes}} + E_{\text{tr}} \quad (3.37a)
$$

with eq. (b) and

$$
\epsilon_{\text{val}}(\mu) = N_c \sum_{0 < \epsilon_\lambda < \mu} \epsilon_\lambda = N_c \epsilon_{\text{val}} \quad (3.37b)
$$
Valence Picture and Bosonized Picture

From Fig. 3.1 we can distinguish 3 different regions concerning the behavior of the valence level $O^+$ and, associated to it, the behavior of $E$ and $\langle B \rangle$ ($E_{mes} = E_{br} = 0$), which have been discussed already by Meissner Th et al. (1990a).

(A):

\[
\begin{align*}
\epsilon_{val} > \mu > 0 \\
\langle B \rangle(\mu) = 0 \\
\epsilon_{val}(\mu) = 0 \\
E_{tot} = E_{sea}
\end{align*}
\]  

(B):

\[
\begin{align*}
\mu > \epsilon_{val} > 0 \\
\langle B \rangle(\mu) = \langle B \rangle_{val} = 1 \\
\epsilon_{val}(\mu) = N_c \sum_{0<\epsilon_\lambda<\mu} \epsilon_\lambda = N_c \epsilon_{val} \\
E_{tot} = E_{sea} + N_c \sum_{0<\epsilon_\lambda<\mu} \epsilon_\lambda = E_{sea} + N_c \epsilon_{val}
\end{align*}
\]  

(C):

\[
\begin{align*}
\epsilon_{val} < 0 \\
\langle B \rangle(\mu) = \langle B \rangle_{sea} = 1 \\
\epsilon_{val}(\mu) = 0 \\
E_{tot} = E_{sea}
\end{align*}
\]

Case (A) is uninteresting for our purpose, because the valence particle is part of the positive continuum and $\langle B \rangle = 0$. The cases (B) and (C) give $\langle B \rangle = 1$ as desired. Hereby the valence orbit is counted explicitly for $E_{tot}(\mu)$ as well as for all other static observables if $\epsilon_{val} > 0$ (B). If $\epsilon_{val} < 0$ the valence particle gets part of the negative spectrum (Dirac sea) which now as itself carries baryon number $\langle B \rangle(\mu) = \langle B \rangle_{sea} = 1$ due to the fact that it contains now 1 orbital more.

For the following it is convenient to introduce the notation:

\[
\eta_{val} = \begin{cases} 
1 & \text{, if } \epsilon_{val} > 0 \\
0 & \text{, if } \epsilon_{val} < 0 
\end{cases}
\]

so that we can write for a system with $B = 1$:

\[
\begin{align*}
E_{B=1} &= N_c \eta_{val} \epsilon_{val} + E_{sea} + E_{mes} + E_{br} \\
\langle O \rangle_{B=1} &= N_c \eta_{val} \langle O \rangle_{val} + \langle O \rangle_{sea}
\end{align*}
\]

Fig. 3.2 shows $\epsilon_{val}, E_{sea}$ and $E_{tot}$ ($E_{mes} = E_{br} = 0$) for a meson profile with linear shape:

\[
\theta(r) = \begin{cases} 
-\pi(1 - \frac{r}{R}) & \text{, if } r < R \\
0 & \text{, if } r \geq R 
\end{cases}
\]
on the chiral circle with winding number \( n = 1 \), \( M = 465 \text{ MeV} \) and \( m_\pi = 0 \) in dependence of the size parameter \( R \). One clearly recognizes that \( E_{\text{tot}}(R) \) has a local minimum at \( R \approx 0.8 \text{ fm} \) corresponding to a solitonic solution with \( B = 1 \). In the next section we will construct these solutions selfconsistently by variation of \( E_{\text{tot}}[\theta(r)] \) with respect to all degrees of freedom \( \{\theta(r)\} \).

On a first glance to eqs. (3.41, 3.42) it appears as if observables change discontinuously their value if the valence single particle energy changes sign. In fact this is not true. The regularization function takes care of this automatically and indeed one can show analytically and numerically that no discontinuity and no kink appears (see Fig. 3.2. as examples).

Connection between Baryon Number \( B \) and Topological Winding Number \( n \):

**Gradient Expansion**

From Fig. 3.1 one can see that for very large values of the size parameter \( R \) the valence orbital approaches the states emerging from the negative continuum and at the end cannot be distinguished any more from them (Kahana et al. 1984, Kahana and Ripka 1984). On the other hand we know that for those large profile sizes \( R \) the gradients of the meson fields get small \( \partial_r \sigma \propto \partial_r \pi \propto \frac{1}{R} \) so that the gradient or heat kernel expansion, discussed in sect. 2. should be valid. The gradient expansion of the baryon current has been performed by Goldstone and Wilczek (1981) and is shown in appendix B. Indeed it turns out that the gradient expanded baryon number of the Dirac sea \( \langle B \rangle_{\text{sea}} \) is identical to the topological winding number \( n \) of the meson profile

\[
\lim_{R \to \infty} \langle B \rangle_{\text{sea}} = n
\]

which is \( n = 1 \) in the case considered above. This feature generally holds in the region (C). Therefore in the case for large \( R \) the valence particle has got part of the negative spectrum and the baryon number coincides with the topological winding number. Similar considerations hold for higher winding numbers (Kahana et al. 1984). For sufficiently large \( R \) one therefore gets close to the philosophy of the topological soliton models like e.g. the Skyrme model which contains no valence quarks but relates the baryon number of the soliton purely to the topological winding number of the Goldstone field. Furthermore it turns out, that for large \( R \) the energy of the Dirac sea \( E_{\text{sea}}(R) \) approaches the kinetic energy of the mesons, which is the leading order in the gradient expansion (Meissner Th et al. 1988):

\[
\lim_{R \to \infty} E_{\text{sea}}(R) = E_{\text{kin}}(R) = \frac{1}{2} \int d^3r \left[ (\nabla \sigma)^2 + (\nabla \pi)^2 \right] \propto R
\]

as long as this expansion is convergent or at least an asymptotic series in \( \frac{1}{R} \) (Zuk and Adjali 1992).
3.3. Solitonic Solutions of the Nonlinear Model

Mean Field Equations

The equations of motion for a system with baryon number \( B = 1 \) are given by the stationary points of the grand canonical effective chiral action \( S_{\text{eff}}(\mu, B = 1) \). In practice the hedgehog ansatz eq. (3.4) is used and the system is assumed to be constrained to the chiral circle \( (3.6) \). The restriction to the chiral circle is quite essential and its origin will be discussed in sect. 3.5 and 8.4 (Sieber et al. 1992; Meissner Th et al. 1993; Weiss et al. 1993). The stationary points of the grand canonical effective chiral action \( S_{\text{eff}}(\mu, B = 1) \) (see eq. (3.25)) with respect to the chiral angle \( \theta(r) \) are then given by:

\[
\delta(\theta(r)) S_{\text{eff}}(\mu, B = 1) = 0
\]

which reduces in case of time independent meson fields to the variation of the energy

\[
\delta(\theta(r)) E(\mu, B = 1) = 0
\]

In order to perform the variation we start from eqs. (3.42, 3.11) and use the spectral representation

\[
\epsilon_\lambda = \langle \lambda | \hat{h} | \lambda \rangle = \int d^3r \phi_\lambda^+(\vec{r}) \hat{h} \phi_\lambda(\vec{r})
\]

Because the variation \( \delta \langle \lambda | \lambda \rangle = \delta \int d^3r \phi_\lambda^+(\vec{r}) \phi_\lambda(\vec{r}) \) vanishes we get for the variation of the single particle energy \( \epsilon_\lambda \):

\[
\delta(\theta(r)) = g \int drr^2 [(-\sin \theta(r)) \cdot s_\lambda(r) + (\cos \theta(r)) \cdot p_\lambda(r)] \delta \theta(r)
\]

with

\[
s_\lambda(r) = \int d\Omega \bar{\phi}_\lambda(r, \Omega) \phi_\lambda(r, \Omega)
\]

\[
p_\lambda(r) = \int d\Omega \bar{\phi}_\lambda(r, \Omega) i\gamma_5(\vec{r}) \phi_\lambda(r, \Omega)
\]

which lead to the equations of motion:

\[
\sin \theta(r) \left[ N_c g \left( S_\theta(r) + \eta_{\text{tot}} \cdot v(r) \right) - 4\pi f_c m_c^2 \right] = 0
\]

\[
\cos \theta(r) \left[ N_c g \left( P_\theta(r) + \eta_{\text{tot}} P_{\text{tot}}(r) \right) \right] = 0
\]

where:

\[
S_\theta(r) = \sum_\lambda \mathcal{R}_2(\epsilon_\lambda, \Lambda) s_\lambda(r)
\]

\[
P_\theta(r) = \sum_\lambda \mathcal{R}_2(\epsilon_\lambda, \Lambda) p_\lambda(r)
\]

and the regularization function \( \mathcal{R}_2(\epsilon_\lambda, \Lambda) \) has been defined in eq. (3.18).

Numerical Treatment
Eq. (3.51) has been solved numerically by Reinhardt and Wünsch (1988) as well as Meissner Th et al. (1989), who used a standard selfconsistent procedure as it is known from Hartree and Hartree-Fock calculations in atomic and nuclear physics. One starts with a reasonably chosen profile $\theta_0(r)$ given e.g. by eq. (3.43), diagonalizes the $h$ as described in sect. 3.1, calculates $s_1, p_1, s_0, p_0$ from the eigenfunctions $\phi_j(r)$ and the eigenvalues $\lambda_j$ from eq. (3.52) and obtains a new profile function $\theta_1(r)$ from the equation of motion (3.51). The procedure is iterated until a desired degree of self-consistency is reached. An equivalent method for solving eq. (3.51) consists in parameterizing $\theta(r)$ in terms of a parameter set $\{\alpha_k, k = 0, 1, 2, \ldots\}$ covering the whole $r$-dependence of $\theta(r)$ as fully as possible and performing a minimization of the total energy in the $\{\alpha_n\}$-space. First calculations (Diakonov et al. 1988, Meissner Th et al. 1988) used a simple, one dimensional parameterization in terms of the size $R$ of an appropriate chosen meson profile $\theta \left( \frac{r}{R} \right)$ like it was done in eqs. (3.7, 3.43) and minimized just with respect to $R$ (cf. Fig. 3.2). The accuracy of this very rough ansatz was improved by increasing the number of parameters to $k = 3$ (Diakonov et al. 1989), which already gives results close to those of the selfconsistent method. In a very recent work $\theta(r)$ is calculated at a given number $k = 10 - 30$ meshpoints in a sufficiently large interval $[0, D]$ and a systematic search for the minimum of $E_{\text{tot}}$ in the $k-$dimensional space spanned by these meshpoints is performed using an elaborate minimization algorithm (Sieber et al. 1992). In other approaches (Diakonov et al. 1988, Adjali et al. 1991, 1992) the energy of the Dirac sea $E_{\text{sea}}[\theta]$ (eq. (3.116)) is approximated by expanding $\text{Sph} D^5 D \left( i D = i \vec{\gamma} - g(\sigma + i \vec{\tau} \vec{\gamma}_5) \right)$ in terms of the ‘perturbation’ $g \theta(\sigma + i \vec{\tau} \vec{\gamma}_5)$ like it was done in sect. 2.3. The corresponding expression is exact in the case of a very large meson profile size as well as in the case of small deviations from the vacuum configuration ($\sigma = \sigma_0 = f_0$, $\vec{\tau} = 0$). In doing so the numerical diagonalization of $h$ is avoided and the variational problem reduces to the solution of an integro-differential equation in $\theta$.

Generally the variation is performed for a given value of the constituent quark mass $M$ or equivalently the ratio $\lambda = \frac{A}{M}$, which is uniquely related to $M$ by (2.51) and (2.45). Furthermore finite pion masses ($m_\pi = 139 \text{MeV}$) as well as the chiral limit ($m_\pi = 0$) have been considered. It turns out, that solitonic solutions of eq. (3.51) exist if and only if $M$ exceeds an critical value $M > M_{c_\pi}$ or equivalently $\lambda < \lambda_{c_\pi}$. The corresponding values are shown in Tab. 3.1. This cusp behavior is typical for localized (solitonic solutions) of a system of coupled non linear equations (see e.g. Lee 1981 and ref. therein) and has been observed also in other chiral quark meson models (Birse 1990 and ref. therein).

**Selfconsistent Fields; Mean Field Quantities**

Fig. 3.3 and Fig. 3.4 show the total (mean field) energy $E_{\text{MF}} = E_{\text{tot}}$ as well as the isoscalar electric quadratic radius $\left< R^2 \right> = 4 \pi \int d^3 r \left< q_0(r) \right>$ which is nothing but the sum of corresponding proton and neutron contributions, in dependence of the constituent quark mass $M$ for $m_\pi = 0$ (Meissner Th et al. 1989, Meissner Th and Goeke 1990). Both quantities are divided into their valence and sea contributions, respectively. One should notice that the mesonic self energy $E_{\text{mes}}$ (eq. (3.11c)) vanishes due to the chiral circle constraint eq. (3.6). Using $m_\pi = 139 \text{MeV}$ one finds that the chiral breaking term $E_{\text{br}}$ (eq. (3.11d)) is small and its influence on the solitonic solutions is in general rather small of less than $10 - 15\%$ (Meissner Th et al. 1989, Meissner Th and Goeke 1991).
Apparently it affects the form of the asymptotic behavior of of the pion tail of the selfconsistent solution
\[
\lim_{r \to \infty} \pi(r) = A e^{-m_{\pi}r} \frac{1 + m_{\pi}r}{r^2}
\] (3.53)

Thus it has some influence on quantities which are sensitive to that form, especially if they diverge in the chiral limit \(m_{\pi} \to 0\), like e.g. the magnetic polarizability and the \(\Sigma\) commutator. Actually eq. (3.53) can be obtained by substituting the sea energy \(E_{\text{sea}}\) (eq. (3.116)) in the variational principle (eq. (3.47)) through the gradient expanded expression. One has then to solve the resulting differential equation in the coordinate space. It contains formally an infinite number of derivatives for which in the asymptotic region \((r \to \infty)\), only the contribution from the kinetic term \((\frac{\partial^2}{\partial r^2})^2\) survives.

In Fig. 3.5 and Fig. 3.6 the meson profiles \(\sigma(r)\) and \(\pi(r)\) of the selfconsistent solutions of eq. (3.51) for three representative values of the constituent quark mass \(M\) with \(m_{\pi} = 0\) are presented. The corresponding baryon densities are shown in Fig. 3.7. As one can see the actual form of \(\sigma(r)\) and \(\pi(r)\) as function of the physical distance \(r\) is quite independent of \(M\). If we denote with \(R\) some characteristic size of the profile, we find that the condition
\[
|\partial_r \sigma| \approx |\partial_r \pi| \ll M \Leftrightarrow 1 \ll M \cdot R
\] (3.54)

for the validity of the gradient expansion is fulfilled if \(M \geq 1000\,\text{MeV}\) and therefore in this case the baryon number \((B)_{\text{sea}}\) of the Dirac sea and the topological winding number \(n\) of the pion field coincide (cf. app. B). Indeed as one can see from Fig. 3.5 the valence energy becomes negative \(\epsilon_{\text{val}} < 0\) if \(M \geq 780\,\text{MeV}\). Due to the discussion in sect. 3.2 one therefore comes close to the philosophy of the topological soliton models if \(M\) gets large (cf. Fig. 1.4), whereas the valence quark picture holds for small \(M\) (cf. Fig. 1.3). The important point is the fact, that for a given \(M\) the model can decide between the two pictures dynamically by the equations of motion eq. (3.51) and therefore switch continuously between them just by the variation of \(M\). In order to make a definite decision which picture is favored one has to calculate baryonic observables and see for which values of \(M\) they are described properly. If one looks e.g. at the isoscalar charge radius \((R_i^2) = (R_i^2)_p + (R_i^2)_n\) (Fig. 3.6) one recognizes at once that one is restricted to rather small constituent quark masses \((M \approx 360\,\text{MeV})\) just above the critical cusp \(M_{\pi}\) in order to reproduce the experimental value of \(0.63\text{fm}^2\). The corresponding mean field energy lies at about \(E_{\text{MF}} \approx 1200\,\text{MeV}\). High values of \(M\) corresponding to the bosonized picture lead to a baryonic system whose extension is by far too small. The region \(M \approx 350\ldots450\,\text{MeV}\) is also consistent with that of various non relativistic constituent quark models (see e.g. Bhaduri 1988 and ref.therein). Furthermore one can see from Fig. 3.6 that the contribution of the valence quarks is clearly the dominating one. Counterexamples as e.g. the neutron squared charge radius or the nucleon polarizability are found as well. We will see in chapter 5 that this will be true for most of the nucleon observables. If and how this feature changes if vector mesons \((\omega, \phi, A_1)\) are explicitly coupled will be discussed in chapter 7.
3.4. Higher Winding and Baryon Numbers

For the construction of the solutions of the mean-field equation of motion for \( \theta(r) \), one has assumed that \( \theta(0) = -n\pi \), with the topological winding number \( n = 1 \). There is, however a priori no reason in this model, why \( n \) should be fixed to 1, because the baryon number \( B \) is carried by the valence quarks, if one chooses an appropriate thermochemical potential \( \mu \). This is different from the topological soliton models (e.g. the Skyrme model), where \( B = n \) from the very beginning. There is even no a priori reason why \( n \) should be an integer number, if one leaves mathematical arguments like continuity of the \( \pi \) field in the origin or the convergence of the gradient expansion for \( B \) (cf. app. B) aside. In contrast to the Skyrme model, where \( n \in \mathbb{Z} \) is required in order to obtain a finite energy, the total energy \( E_{\text{tot}} \) is finite for any value of \( n \) in the present approach. Diakonov et al. (1988,1989) and Berg et al. (1992) have investigated the behavior of \( E_{\text{tot}} \) as a function of \( n \) for a fixed profile form. It turned out, that the minimum of \( E_{\text{tot}} \) definitely lies at the point \( n = 1 \), which means in fact that this feature is also a dynamical consequence of the equations of motion. Unfortunately the whole analysis is based purely on numerical arguments and there is up to now no analytic proof of this fact.

In addition Berg et al. (1992) studied systems with higher baryon numbers \( B > 1 \). They found that local minima corresponding to solitonic solutions only exist if \( n \leq B \) and in any case these local minima appear for integer winding number \( n \). For example for the \( B = 2 \) system they found a soliton in the \( n = 1 \) sector with 2 valence orbitals \((0^+,0^-)\) occupied if \( M > 370 \text{MeV} \). In the \( n = 2 \) sector solitonic solutions exist if \( M > 560 \text{MeV} \). The main difference between the two cases is the fact, that for \( n = 2 \) the \( 0^+ \) as well as the \( 0^- \) orbital cross the zero line and get part of the negative spectrum whereas in the case of \( n = 1 \) only the \( 0^+ \) orbital comes down and the \( 0^- \) orbital remains in the positive spectrum. (cf. Fig. 3.1).

3.5. Collapse in the Linear Model

Up to now in any case the meson fields \( \sigma(r) \) and \( \pi(r) \) have been restricted to the chiral circle \( \sigma(r)^2 + \pi(r)^2 = f_0^2 \), known as nonlinear realization of chiral symmetry (Weinberg 1968). Due to this constraint the variation \( \delta E_{\text{tot}} = 0 \) is performed only with the chiral angle \( \theta(r) \) as degree of freedom. The motivation to consider the action as an effective chiral theory of the nucleon from an instanton liquid model of QCD (Diakonov et al.1986) even suggests that only these degrees of freedom, namely the Goldstone ones, should be taken into account. On the other side in spirit of the Nambu-Jona-Lasinio-model there is no reason at all why the meson fields should be restricted to the chiral circle and the constraint eq. (3.6) seems to be an artificial one. Sieber et al. (1992) performed a detailed analysis of the full linear model with both \( \sigma \) and \( \pi \)-degrees of freedom and showed that at least in case of the proper time regularization scheme no localized solitonic solution with finite energy exists in this case but the system collapses into a configuration with both size and energy being zero while \( B = 1 \) is maintained. To this end they choose a parameterization for
\[ \sigma(r) = f(r) \left[ 1 + Uf \left( \frac{r}{R} \right) \cos \theta \left( \frac{r}{R} \right) \right] \]
\[ \tau(r) = f(r)Uf \left( \frac{r}{R} \right) \sin \theta \left( \frac{r}{R} \right) \]

(3.55)

where \( f(r) \) is a strictly monotonously decreasing function with \( f(0) > 0 \), \( f(\infty) = 0 \) and \( \theta(0) = -n\pi \).

Taking the limit \( u \to \infty, R \to \infty \) but \( UR^2 = \text{const} \), the collapse described above occurs if \( 1 < \alpha < \frac{3}{2} \). This is due to the fact that in this limit for \( \alpha > 1 \) altogether \( n \) valence orbitals cross the zero line and get part of the negative spectrum which means that they are not taken into account in the expression for the total energy any more (\( \eta_{\text{val}} = 0 \) in eqs. (3.41, 3.42)) and the baryon number of the system gets \( \langle B \rangle = \langle B \rangle_{\text{sea}} = n \). Furthermore it turns out that the Dirac sea approaches its vacuum configuration, because any deviation from this is suppressed by the UV cutoff \( \Lambda \). The reason for the occurrence of this behavior is the fact that in solving the quark loop of NJL without vector mesons the sea energy \( E_{\text{sea}} \) gets regularized by \( \Lambda \) whereas the finite baryon number \( \langle B \rangle \) is not. On the other hand \( E_{\text{mes}} \) vanishes for \( R \to \infty \) as long as \( \alpha < \frac{3}{2} \), which proves the statement above.

Earlier works (Meissner Th et al.1989, Reinhardt and Wuensch 1989, Meissner Th and Goeke 1991), which claim the existence of solitonic solutions also in the linear model turned out to be premature in this respect since too small values of \( K_{\text{max}} \) (cf. sect. 3.1.), where chosen in the basis for the diagonalization of the single particle hamiltonian \( h \). This fact has been emphasized in the mean time by various authors (Watabe and Toki 1992, Sieber et al.1992, Kato et al.1993).

In sect.8.4. it will be shown that a modified version (Meissner Th et al. 1993) of the Nambu–Jona-Lasinio-model, which in addition to the spontaneously broken chiral symmetry also simulates the anomalous breaking of scale invariance in QCD shows perfectly stable solitonic solutions also without the non linear constraint, while the properties of the soliton as well as the relevant nucleon observables stay nearly unchanged. Thus the restriction of \( \sigma \) and \( \pi \) to the chiral circle is justified by a model which implements the concept of trace anomaly in QCD. Stability can also be achieved by adding the \( U_A(1) \) breaking t’Hooft term (Kato et al. 1993), at least for a certain range for t’Hooft coupling strength.
As in nonrelativistic many particle physics we face the problem that the hedgehog mean field solution of the classical equations of motion, which we have constructed in the last chapter, breaks the rotational, isorotational and the translational symmetry of the full theory. This means that the eigenstates $|\lambda\rangle$ of the 1-particle hamiltonian $h$ do not carry good spin, isospin or momentum quantum numbers. Because we want to describe nucleonic systems, which have these quantum numbers, those symmetries must be restored. This is done by coupling the corresponding expectation value $\langle \hat{J}, \langle \hat{T}, \langle \hat{P} \rangle \rangle$ through a Lagrange multiplier to the effective chiral action $S_{eff}$ like it was done in sect.3.2, in case of the baryon number $B$ with the thermochemical potential $\mu$. This turns out to be equivalent to consider the soliton in an (iso-)rotating or moving system, called cranking (sect.4.1) or pushing method (sect.4.3), respectively. Because the grand spin $\tilde{G} = \hat{J} + \hat{T}$ is a good quantum number of the hedgehog single particle state, spin and isospin degrees of freedom are coupled and it is enough to consider one of them, e.g. the isospin. Whereas the spectrum of the momentum operator $\hat{P}$ is continuous, the isospin degrees of freedom have to be quantized. For this we will use the semiclassical collective quantization method (sect.4.2). Zero point energies appear, because the expectation values of the 2-particle operators $\hat{T}^2$ and $\hat{P}^2$ do not vanish (sect.4.4). Finally we will be able to write down the expressions for the masses of nucleon $N$ and delta $\Delta$ in a system at rest and give numerical values (sect.4.4).

In this chapter we will throughout consider the non-linear version of the Nambu-Jona-Lasinio-model, i.e. restrict $\sigma$ and $\pi$ to the chiral circle $\sigma^2 + \pi^2 = \sigma_Y^2 = f^2_{\pi}$ and use the notation:

$$U = \sigma + i \pi \tilde{\pi}$$
$$U_5 = \sigma + i \gamma_5 \tilde{\pi}$$

(4.1)

### 4.1. Iso-Rotational Motion: Cranking

In order to restore the isorotational symmetry, i.e. to construct a system with good isospin quantum numbers (e.g. $N, \Delta$) we follow a procedure which was established in nonrelativistic many particle physics (see e.g. Ring and Schuck 1980) and is known as cranking approach. It has been used also quite successfully in case of the Skyrme model (Adkins et al.1983) as well as in the chiral sigma model with valence quarks (Cohen and Broniowski 1986). Furthermore one can generalize the method to $SU(3)$-flavor and calculate hyperon properties, which will be done in chapter 6.

**Adiabatic Isorotation**

The main idea is to perform an adiabatic isorotation of the hedgehog meson fields with the angular frequency $\Omega$:

$$\sigma(\vec{x}) \rightarrow \tilde{\sigma}(t, \vec{x}) = \sigma(\vec{x})$$
$$\pi^a(\vec{x}) \rightarrow \tilde{\pi}^a(t, \vec{x}) = D^{a\beta}(t)\pi^\beta(\vec{x})$$
$$U_5(\vec{x}) \rightarrow \tilde{U}_5(t, \vec{x}) = R^A(t)U_5(\vec{x})R(t)$$

(4.2)

with the $SU(2)$-rotation matrix

$$R(t) = e^{i \frac{\tilde{\Omega} t}{2}}$$

(4.3)
and its $SO(3)$-representation $D$, defined by:

$$\mathcal{R}(t)\tau^a\mathcal{R}^\dagger(t) = \tau^b D^{ba}(t) \tag{4.4}$$

For simplicity we will first stay in Minkowski space without regularization and valence quarks, i.e. we set the thermochemical potential $\mu$ to zero, and comment on the general case later on.

The effective action in the rotating system is given by:

$$e^{i S_{eff}[\tilde{U}_z]} = \int D\tilde{q} D\tilde{q} e^{i \int d^4x [\tilde{q}\overleftarrow{\partial} D(\tilde{U}_z)]\tilde{q}} \tag{4.5}$$

with: $\tilde{q} = R^\dagger(t)q$, and therefore:

$$S_{eff}[\tilde{U}_z(\Omega)] = \text{Sp} \left[ iD + \beta \overrightarrow{\Omega} \frac{\tau^a}{2} \right] = \text{Sp} \left[ i\tilde{\partial}_t - \left( h - \overrightarrow{\Omega} \frac{\tau^a}{2} \right) \right] \tag{4.6}$$

From this we see, that $\overrightarrow{\Omega}$ acts as a Lagrange multiplier constraining the isospin

$$T^a = \int d^4x \tilde{q}(\vec{x}) \beta \frac{\tau^a}{2} q(\vec{x}) \tag{4.7}$$

similar to the thermochemical potential $\mu$ in case of the baryon number (sect.3.2). The 1-particle hamiltonian $h(\overrightarrow{\Omega})$ in the rotating system can be read off from eq. (4.6) to:

$$h(\overrightarrow{\Omega}) = h - \overrightarrow{\Omega} \frac{\tau^a}{2} \tag{4.8}$$

Perturbative Cranking. Moment of Inertia

We assume $\overrightarrow{\Omega}$ to be small, so that the problem can be treated perturbatively. We will comment on the validity of this assumption later on.

The 1st order in $\Omega$ vanishes

$$\frac{\delta S_{eff}(\Omega)}{\delta \Omega^a} \bigg|_{\Omega=0} = \langle T^a \rangle = 0 \tag{4.9}$$

because of hedgehog symmetry:

$$\sum_{\lambda(G,G_z)} \langle \Lambda(G,G_z) | \frac{\tau^a}{2} \Lambda(G,G_z) \rangle = 0 \tag{4.10}$$

In second order we get:

$$S_{eff}(\tilde{U}_z) = T E(\tilde{U}_z) = T \left[ E_{MF} + \frac{1}{2} \Omega^a \Theta^{ab} \Omega^b \right] \tag{4.11}$$

where the moment of inertia tensor $\Theta^{ab}$ is defined by:

$$\Theta^{ab} = \frac{1}{T} \left. \frac{\delta^2 S_{eff}(\Omega)}{\delta \Omega^a \delta \Omega^b} \right|_{\Omega=0} = \frac{1}{T} \text{Sp} \left[ (i\tilde{\partial}_t - h)^{-1} \frac{\tau^a}{2} (i\tilde{\partial}_t - h)^{-1} \frac{\tau^b}{2} \right] \tag{4.12}$$
In terms of the 1-particle eigenstates $|\lambda\rangle$ (cf. eq. (3.5)) $\Theta^{ab}$ is given as a sum over all particle $(\epsilon_\nu > 0)$ - hole $(\epsilon_\lambda < 0)$ matrix elements of the 'perturbation' $\vec{F}$:

$$\Theta^{ab} = \frac{N_c}{2} \sum_{\epsilon_\lambda < 0, \epsilon_\nu > 0} \frac{\langle \lambda | \vec{r}^a | \nu \rangle \langle \nu | \vec{r}^b | \lambda \rangle}{\epsilon_\nu - \epsilon_\lambda}$$

(4.13)

which is known as Inglis formula (Ring and Schuck 1980)

Of course it is necessary to regularize $S_{eff}(\Omega)$ and $\Theta$. In case of the proper time regularization, which works in Euclidean space, the expression for $\Theta^{ab}$ was derived by Reinhardt (1989):

$$\Theta_0^{ab} = \frac{N_c}{4} \frac{1}{\sqrt{4\pi}} \int_0^\infty ds \frac{1}{\lambda^2} \sum_{\lambda \nu} \langle \lambda | \vec{r}^a | \nu \rangle \langle \nu | \vec{r}^b | \lambda \rangle \mathcal{R}_\Theta(\epsilon_\lambda, \epsilon_\nu, s)$$

$$\mathcal{R}_\Theta(\epsilon_\lambda, \epsilon_\nu, s) = \left[ \frac{e^{-s \epsilon_\lambda^2} - e^{-s \epsilon_\nu^2}}{\epsilon_\lambda^2 - \epsilon_\nu^2} - s \frac{\epsilon_\lambda e^{-s \epsilon_\nu^2} + \epsilon_\nu e^{-s \epsilon_\lambda^2}}{\epsilon_\lambda^2 + \epsilon_\nu^2} \right]$$

(4.14)

Hereby it is essential to perform the continuation of the rotation frequency $\vec{\Omega}$ into the Euclidean space ($\vec{\Omega} \rightarrow \vec{\Omega}_E = i\vec{\Omega}$) like it is done for the time component of a physical 4-vector, e.g. the time as itself $(t \rightarrow \tau = it)$ or the time component of a $\rho$-meson. This means that both $\vec{\Omega}$ and $\vec{\Omega}_E$ are real numbers and the fermion determinant in Euclidean space is: $\text{SpIn}(-iD - i\beta \vec{\Omega}_E \vec{x})$. Otherwise it turns out that the regularized moment of inertia for the vacuum configuration $U_5 = 1$ does not vanish, which has to be considered as unphysical (Reinhardt 1989).

The separation of the valence quark contribution by using the grand canonical effective action in the rotating system $S_{eff}(\mu, \Omega)$ with the thermochemical potential $\mu$ (cf. sect. 3.2.) is straightforward yielding:

$$\Theta^{ab}_{\text{val}} = \frac{N_c}{2} \sum_{\epsilon_\nu < \epsilon_{\text{val}}} \frac{\langle \text{val} | \vec{r}^a | \nu \rangle \langle \nu | \vec{r}^b | \text{val} \rangle}{\epsilon_\nu - \epsilon_{\text{val}}}$$

(4.15)

which is nothing but the standard 2nd order perturbation theory result for the orbital $|\text{val}\rangle$ with the perturbed hamiltonian eq. (4.8).

On simple symmetry reasons $\Theta^{ab}$ is diagonal and can be written as $\Theta^{ab} = \delta^{ab} \Theta$. By using the 1-particle eigenstates $|\lambda\rangle$ and $|\nu\rangle$ of the selfconsistent solitonic solution as they were obtained in sect.3.3, Goek et al. (1991) and Wakamatsu and Yoshiki (1991) calculated numerically the moment of inertia, which will serve as the essential quantity for obtaining the masses of $N$ and $\Delta$ from the soliton mean field energy $E_{MF}$ (see Fig. 4.1).

Observables in the Rotating System

Generally the expectation value of an observable $O = \int d^3x q^\dagger(\vec{r}) O q(\vec{r})$ in the rotating system reads (in Minkowski space):

$$\langle \hat{O} \rangle(\Omega) = \frac{1}{T} \int \frac{DqD\bar{q}}{\mathcal{Z}} \int d^4x [\bar{q} \beta \lambda O q] e^{i \int d^4x \frac{1}{2} [\bar{q} D(\vec{U}_5)] [\bar{q} D(\vec{U}_5)]]$$

$$\mathcal{Z} = \int \frac{DqD\bar{q}}{\mathcal{Z}} \int d^4x [\bar{q} \beta \lambda O q] e^{i \int d^4x \frac{1}{2} [\bar{q} D(\vec{U}_5)] [\bar{q} D(\vec{U}_5)]]$$

$$= \frac{1}{T} \delta \mathcal{S} \text{SpIn} \left[ i \bar{D}(\vec{U}_5) + \beta \vec{\Omega}_2 \vec{\tau} + \omega \mathcal{R} \mathcal{O} \mathcal{R} \right]_{\omega = 0}$$

(4.16)
One should be aware that this expression differs from the simpler one

$$\langle \mathcal{O} \rangle (\Omega) = \frac{1}{T} \frac{\delta}{\delta \omega} \text{SpIn} \left[ iD(U_\omega) + \beta \Omega \frac{\mathcal{F}}{2} + \omega \mathcal{O} \right]_{\omega=0} \quad (4.17)$$

if $\mathcal{O}$ is an isovector by:

$$\langle \tilde{\mathcal{O}} \rangle (\Omega) = D^{ja} \langle \mathcal{O}^b \rangle (\Omega) \quad (4.18)$$

In the perturbative cranking approach any $\langle \mathcal{O} \rangle (\Omega)$ is expanded up to 1st order in $\Omega$. If this order vanishes generally only the 0th order is taken into account. For the case of $\beta_A$ and the isovector magnetic moment we will extend this method to higher orders and discuss the results shortly in sect. 5.1. and 5.2.

In the special case of the isospin operator eq. (4.7) we have:

$$\langle \mathcal{T} \rangle (\Omega) = D^{ja} \langle \mathcal{T}^b \rangle (\Omega) = D^{ja} \Theta^{bc} \Omega^c = \Theta \bar{D}^{ja} \Omega^b \quad (4.19)$$

On the other side for the total spin: $J^a = \int d^3x \vec{r}(\vec{x}) \beta \left( \frac{\vec{a}}{2} + \vec{J} \times \vec{S} \right) q(\vec{x}) = G^a - T^a$ one finds:

$$\langle \mathcal{J} \rangle (\Omega) = \langle \mathcal{J}^a \rangle (\Omega) = \langle G^a \rangle (\Omega) - \langle T^a \rangle (\Omega) = -\langle T^a \rangle (\Omega) = -\Theta \Omega^a \quad (4.20)$$

which is true because $[\hbar, G^a] = 0$ and therefore $\langle G^a \rangle$ vanishes up to 1st order in $\Omega$ as it is known from standard quantum mechanics perturbation theory. From eq. (4.19) and eq. (4.20) we read off the relation between spin and isospin expectation values:

$$D^{ja} \langle \mathcal{J}^b \rangle + \langle \mathcal{T} \rangle = 0 \quad (4.21)$$

and furthermore

$$\langle \mathcal{J} \rangle^2 = \langle \mathcal{T} \rangle^2 \quad (4.22)$$

4.2. Collective Quantization of the Iso-Rotational Degrees of Freedom

Up to now the expectation values $\langle \mathcal{T} \rangle (\Omega)$ and $\langle \mathcal{J} \rangle (\Omega)$ are continuous quantities. The *semiclassical collective quantization* method, which was established by Adkins et al. (1983) in the Skyrme model, can be used for quantizing the rotational and isorotational degrees of freedom. The main idea is to express $\langle \mathcal{T} \rangle (\Omega)$, $\langle \mathcal{J} \rangle (\Omega)$ as well as the rotational contribution of $S_{ij}(\Omega)$, which can be written as $\int dt L_{rot}(\Omega) = \frac{1}{2} \Omega^2 \Theta$, in terms of the components of the $SU(2)$ matrix $\mathcal{R}(t)$:

$$\mathcal{R} = \theta^i + i\hat{b}^a \tau^a \quad \text{with} \quad \sum_{a=0}^4 \beta^a = 1 \quad (4.23)$$

and in terms of its time derivatives $i_\mathcal{R} \hat{b}^a$:

$$\langle \mathcal{T} \rangle (\Omega) = \Theta D^{ja}(\Omega) \hat{b}^b = (-i) \Theta \text{Tr}_{\tau} \left[ \mathcal{R}^\dagger \hat{R} \tau^a \right]$$

$$\langle \mathcal{J} \rangle (\Omega) = -\Theta(\Omega) \hat{b}^a = (-i) \Theta \text{Tr}_{\tau} \left[ \mathcal{R} \hat{R}^\dagger \tau^a \right]$$

$$L_{rot}(B) = \Theta \text{Tr}_{\tau} \left[ \mathcal{R} \hat{R}^\dagger \right] = 2 \Theta \sum_a \beta^a \quad (4.24)$$

41
and to consider $b^a$ as space components on the 3-dimensional sphere $S^3$. The corresponding canonical conjugate momenta are given by:

$$\pi^a(b^a) = \frac{\partial L_{rot}(B)}{\partial \dot{b}^a} = 4 \Theta \dot{b}^a$$  \hspace{1cm} (4.25)$$

The quantization is performed in the standard way by substituting the coordinates $b^a$ and the conjugate momenta $\pi^a$ through the operators

$$b^a \rightarrow \hat{b}^a$$

$$\pi^a \rightarrow \hat{\pi}^a = (-i) \frac{\partial}{\partial \hat{b}^a}$$  \hspace{1cm} (4.26)$$

on which the following commutator rules are imposed

$$[\hat{b}^a, \hat{\pi}^\beta] = (+i)\delta^a_{\beta}$$  \hspace{1cm} (4.27)$$

The $\hat{b}^a$ and $\hat{\pi}^a$ are considered to act on a collective wave function $|\Psi(b)\rangle$ which is normalized on the 3-dimensional unit sphere $S^3$ with respect to the surface integration measure $d\mu(b)$:

$$\langle \Psi(b)|\Psi(b)\rangle = \int d\mu(b)\langle \Psi(b)|b\rangle\langle b|\Psi(b)\rangle = 1$$  \hspace{1cm} (4.28)$$

The $d\mu(b)$ can be expressed by the angular representation of $S^3$ by:

$$d\mu(b) = d\psi d\phi d\theta \sin \phi d\theta \sin^2 \theta$$  \hspace{1cm} (4.29)$$

with

$$b_0 = \cos \theta$$

$$b_1 = \sin \theta \cos \phi$$

$$b_2 = \sin \theta \sin \phi \cos \psi$$

$$b_3 = \sin \theta \sin \phi \sin \psi$$  \hspace{1cm} (4.30)$$

and $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$, $\psi \in [0, \pi]$. The $\langle \hat{T}\rangle(\Omega)$ and $\langle \hat{J}\rangle(\Omega)$ now go over into the isospin and spin operators $\hat{t}^a$ and $\hat{j}^a$, respectively, which are given by:

$$\hat{t}^a = \frac{i}{2} \left[ b^a \frac{\partial}{\partial b^b} - b^b \frac{\partial}{\partial b^a} - \epsilon^{abc} b^c \frac{\partial}{\partial b^a} \right]$$

$$\hat{j}^k = \frac{i}{2} \left[ b^a \frac{\partial}{\partial b^l} - b^l \frac{\partial}{\partial b^a} - \epsilon^{klm} b^l \frac{\partial}{\partial b^m} \right]$$  \hspace{1cm} (4.31)$$

and fulfill the desired commutation relations:

$$[\hat{t}^a, \hat{j}^b] = i \epsilon^{abc} \hat{j}^c$$

$$[\hat{j}^k, \hat{j}^l] = i \epsilon^{klm} \hat{j}^m$$  \hspace{1cm} (4.32)$$
The collective wave functions for proton (p) and neutron (n) with spin \( \uparrow \) and \( \downarrow \) read:

\[
\begin{align*}
\langle \ell | p \uparrow \rangle &= \frac{i}{\pi}(b^1 + ib^2) \\
\langle \ell | n \uparrow \rangle &= \frac{1}{\pi}(b^0 - ib^3) \\
\langle \ell | p \downarrow \rangle &= \frac{1}{\pi}(b^0 + ib^3) \\
\langle \ell | n \downarrow \rangle &= -\frac{i}{\pi}(b^1 - ib^2)
\end{align*}
\]

(4.33)

whereas for the \( \Delta \) we have e.g.:

\[
\langle \ell | \Delta^{++}, s_z = \frac{3}{2} \rangle = \frac{\sqrt{2}}{\pi} (b^1 + ib^2)^3
\]

(4.34)

from which the other \( \Delta \)-states can be constructed by applying the spin- and isospin lowering operators \( j^- \) and \( t^- \) respectively.

Finally by knowing those collective wave functions it is possible to calculate any matrix element of an observable acting in the collective space (\( \ell \)-space) between two collective wave functions. The general recipe is the following: One uses eq. (4.16) and expands the expressions up to 1st non-vanishing order in the rotational frequency \( \Omega^a \). Then one replaces \( \Omega^a \) by the operator \( \frac{\ell^a}{\sqrt{\ell}} \). This expression can now be sandwiched between the collective wave functions. Actually if terms higher than of first non-vanishing order in \( \Omega \) are taken into account this simple recipe must be modified due to some time-ordering problems (see for details sect. 5.1. and 5.2.).

Important matrix elements, which we will need later on are e.g.:

\[
\langle p \uparrow | \ell^{[3]} | p \uparrow \rangle = \left(-\frac{1}{3}\right) \delta^{[3]}
\]

(4.35)

and of course:

\[
\langle p \uparrow | \ell^{[2]} | p \uparrow \rangle = \frac{1}{2}
\]

(4.36)

whereas for any collective state with quantum numbers \( T,J \) we have:

\[
\begin{align*}
\langle \Psi_T J | \ell^2 | \Psi_T J \rangle &= T \cdot (T + 1) \\
\langle \Psi_T J | J^2 | \Psi_T J \rangle &= J \cdot (J + 1)
\end{align*}
\]

(4.37)

Therefore we see with eq. (4.22) that for the collective quantized eigenstates the general constraint

\[ J = T \]

(4.38)

holds.
4.3. Translational Motion: Pushing

As it does for the (iso-)rotational symmetry the soliton breaks also the translational invariance of the effective action. Because we want to consider nucleonic systems with good linear momentum, especially for momentum zero, i.e. at rest, this symmetry has to be restored as well. The basic idea is the same as in the case of the isorotational motion: In order to avoid approximately the complicated boosting procedure one couples the corresponding operator, the linear momentum $\vec{P} = \sum_i \vec{x}_i$ through a Lagrange multiplier $\vec{v}$ to the effective action $S_{\text{eff}}[U_5(\vec{x})]$. By applying the generators of the translation $e^{i\vec{v}\vec{P}_t}$ it can be seen to be equivalent to calculate the action for a field $U_5(\vec{x}) = U_5(\vec{x} - \vec{v}t)$ which moves with velocity $\vec{v}$ (pushing model):

$$S_{\text{eff}}[U_5(\vec{v})] = \text{Spln} \left( iD[U_5(\vec{v})] \right) = \text{Spln} \left\{ e^{-i\vec{v}\vec{P}_t} (iD[U_5]) e^{i\vec{v}\vec{P}_t} \right\}$$

(4.39)

Assuming that $v$ is small, which we will also comment later on, we can treat the problem perturbatively:

$$S_{\text{eff}}(\vec{v}) = \mathcal{T} E(\vec{v}) = \mathcal{T} \left( E_{\text{MF}} + \frac{1}{2} v^2 M^* + \mathcal{O}(v^3) \right)$$

(4.40)

Hereby we made use of the fact, that the 1st order in $\vec{v}$, which is nothing but the expectation value of the linear momentum $\langle \vec{P} \rangle$ in the static configuration $U_5(\vec{x})$, vanishes due to Ehrenfest’s theorem already at the 1-particle level:

$$\langle \lambda | \vec{P} | \lambda \rangle = \frac{i}{2} \langle \lambda | [P^2, \vec{x}] | \lambda \rangle = \frac{i}{2} \langle \lambda | [\vec{h}, \vec{x}] | \lambda \rangle = 0$$

(4.41)

and because of eqs. (3.16, 3.17) also in general. The inertial mass $M^*$ defined by:

$$M^* = \left. \frac{\delta^2 S_{\text{eff}}[U_5]}{\delta v^2} \right|_{v=0}$$

(4.42)

is generally given by the particle-hole matrix elements of $\vec{P}$ similar to the Inglis formula eq. (4.13). Other than for the isotrotational motion in case of the translational motion Lorentz invariance of the NJL Lagrangian guarantees that the inertial mass $M^*$ is equal to the soliton mean field energy $E_{\text{MF}}$: For small $v$, $S_{\text{eff}}[U_5(v)]$ in eq. (4.39) is the nonrelativistic Galilei limit of the effective action, which arises by boosting the soliton field in $v = \frac{v}{v^2}$ direction with the boost-velocity $\omega = \arctanh(v)$ (cf. app. C). Indeed the soliton energy $E$ transforms under this boost like the time component of a Lorentz 4-vector (Betz and Goldflam 1983), i.e. the energy $E(\omega)$ in the boosted system is given by:

$$E(\omega) = \cosh(\omega) E_{\text{MF}} = E_{\text{MF}} + \frac{1}{2} v^2 E_{\text{MF}} + \mathcal{O}(v^3)$$

(4.43)

which after comparing with eq. (4.40) shows the desired identity:

$$M^* = E_{\text{MF}}$$

(4.44)

An explicit proof of eq. (4.43) for the unregularized action Minkowski space using Poincare algebra as well as the selfconsistent mean field equation of motion (3.47) is given in appendix C. Pobylitsa
et al. (1992) have shown, that this conclusion also holds, if one considers a regularized theory with a finite cutoff in Euclidean space as long as the regularization scheme is gauge-invariant. This means, in other words, that the central transformation identity

\[
\text{Sp} \left\{ B^1(\omega) \right\} A = \text{Indet} \left\{ B^1(\omega) \right\} A = \text{Sp} \left\{ A \right\} = \text{Sp} \left\{ A \right\} \quad (4.45)
\]

where \( A \) denotes an arbitrary operator and \( B(\omega) \) the generator of the boost, has still to be valid, if \( \text{Sp} \left\{ A \right\} = \text{Indet} \) gets regularized.

4.4. Spurious Zero Point Energies - Masses of Nucleon \( N \) and \( \Delta \)

Spurious zero point energies for the translational as well as the (iso-)rotational motion, which have to be subtracted from the total energy, if a semiclassical quantization is performed, arise due to the fact, that even at mean field level the expectation values for the operators \( \mathcal{B}^2 \) and \( \mathcal{T}^2 \) are finite.

**Expectation Values of 1- and 2-Particle Operators**

Let us first have a look at the expectation values of 1- and 2-particle operators in general. For this we consider the pure 1-particle operators \([O^1(1)]\) and the pure 2-particle operator \([O^2(2)]\) defined by (Negele and Orland 1987):

\[
[O^1(1)] = \int d^3 x \hat{q}_\alpha(\vec{x}) \hat{q}_\beta(\vec{x}) \\
[O^2(1)] = \int d^3 x \hat{q}_\alpha(\vec{x}) \hat{O}^{(2)}_{\alpha\beta}(\vec{x}) \\
[O^2(2)] = \int d^3 x_1 d^3 x_2 \hat{q}_{\alpha_1}(\vec{x}_1) \hat{q}_{\alpha_2}(\vec{x}_2) \hat{O}_{\alpha_1\beta_1} \hat{O}_{\alpha_2\beta_2}(\vec{x}_1) \hat{q}_{\beta_1}(\vec{x}_1)
\]

where the \( \alpha \)’s and \( \beta \)’s denote some Dirac or isospin indices. Using the anticommutator relations for the fermion operators \( \hat{q} \) and \( \hat{q} \dagger \)

\[
\{ \hat{q}_\alpha(\vec{x}), \hat{q}_\beta^\dagger(\vec{y}) \} = \delta^3(\vec{x} - \vec{y}) \delta_{\alpha\beta} \\
\{ \hat{q}_\alpha(\vec{x}), \hat{q}_\beta(\vec{y}) \} = 0 \\
\{ \hat{q}_\alpha^\dagger(\vec{x}), \hat{q}_\beta^\dagger(\vec{y}) \} = 0 \quad (4.47)
\]

we can decompose the squared 1-particle operator \([O^1(1)]^2\) into:

\[
[|O^1(1)|]^2 = [O^2(1)]^2 + [O^2(2)]
\]

Using the notation:

\[
e^{i\mathcal{S}(\alpha)} := Z(\alpha) = \int \mathcal{D} q \mathcal{D} \hat{q} e^{i\int d^3 x \hat{q} (i\mathcal{D} + i\alpha \mathcal{O})(\hat{q})} \quad (4.49)
\]

the expectation value of the last summand in eq. (4.48) can be written as:

\[
\langle |O^2(1)| \rangle \rangle = \frac{1}{(iT)^2} \left[ \frac{1}{Z(\alpha)} \frac{\delta^2}{\delta \alpha^2} Z(\alpha) \right]_{\alpha = 0} = \langle |O^1(1)| \rangle^2 + \frac{1}{iT} \frac{\delta^2 S(\alpha)}{\delta \alpha^2} \bigg|_{\alpha = 0} \quad (4.50)
\]
For static field configurations $S(\alpha)$ is proportional to $T$ so that the last term in eq. (4.50) vanishes in the ground state (zero temperature limit) $T \to \infty$. Therefore we finally obtain:

$$\lim_{T \to \infty} \langle \{[O^1]\}^2 \rangle = \langle [O^2]_1 \rangle + \langle [O^1] \rangle^2$$  (4.51)

**Zero Modes of the (Iso-)Rotational Motion, Band-Head-Energy**

From eq. (4.51) we obtain for the expectation value of the operator $\hat{T}^2$ in the rotating system (cf. eqs. 4.16, 4.19):

$$\langle \{[\hat{T}^2]_1\}(\Omega) \rangle = \langle [\hat{T}^2]_1 \rangle + \left( \langle [\hat{T}^2]_1 \rangle(\Omega) \right)^2 = \Theta^2 \Omega^2 + \langle [\hat{T}^2]_1 \rangle(\Omega)$$  (4.52)

The expectation value of the 1-particle operator $\hat{T}^2_1(\Omega)$ is independent of $\Omega$, because $[\hat{T}^2]_1(\Omega)$ commutes both with $\hbar$ and $\hbar(\Omega)$. Then the expectation value can be shown to get no contribution from the Dirac sea and hence simply reads:

$$\langle [\hat{T}^2]_1(\Omega) \rangle = \langle [\hat{T}^2]_1 \rangle = N_c \langle B \rangle \frac{1}{2}(\frac{1}{2} + 1) = N_c \langle B \rangle \text{val} \frac{1}{2}(\frac{1}{2} + 1) = \frac{9}{4}$$  (4.53)

If we consider $\Omega^2$ as a Lagrange multiplier for adjusting the quantum number $T$, we have the condition:

$$\langle \{[\hat{T}^2]_1\}(\Omega) \rangle = T(T + 1)$$  (4.54)

and therefore find:

$$\Theta^2 \Omega^2 = T(T + 1) - \langle [\hat{T}^2]_1 \rangle$$  (4.55)

Inserting this equation into the expression for the energy in the rotating system (cf. eq. 4.11):

$$E(\Omega) = E_{MF} + \frac{1}{2} \Theta \Omega^2$$  (4.56)

one obtains for the energy of a system with isospin $T$ and spin $J$:

$$E_{JT} = E_{MF} + \frac{T(T + 1)}{2\Theta} - \frac{\langle [\hat{T}^2]_1 \rangle}{2\Theta}$$  (4.57)

which means that the band head term $\frac{\langle [\hat{T}^2]_1 \rangle}{2\Theta}$ has to be subtracted from the cranked energy $\frac{E_{MF} + \frac{T(T + 1)}{2\Theta}}{2\Theta}$. The expression eq. (4.57) is familiar from nuclear many body physics, see e.g. Ring and Schuck (1980) and Blaizot and Ripka (1988).

**Zero Modes of the Translational Motion, Center-of-Mass Energy**

Instead of a proper boosting for the translational degrees of freedom we adopt a purely non-relativistic treatment, which consists in separating the whole motion into an collective as well as an intrinsic one (Ring and Schuck 1980). Thereby we hope that at least the magnitude of the effect is described reasonably.
For this let us look at a $N$-particle system interacting by a purely local time- and velocity independent one-body force $V(\vec{x})$. We consider the static and the pushed total Hamiltonians:

$$
H = \sum_{k=1}^{N} \frac{\vec{p}_k^2}{2m_k} + V(\vec{x}_k) \tag{4.58}
$$

$$
H_\nu = \sum_{k=1}^{N} \frac{\vec{p}_k^2}{2m_k} + V(\vec{x}_k - \vec{v}t)
$$

respectively. The corresponding solutions of the time-dependent Schrödinger equations

$$
\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle
$$

$$
\partial_t |\Psi_\nu(t)\rangle = H_\nu |\Psi_\nu(t)\rangle \tag{4.59}
$$

are connected by the unitary generators of the Galilei transformation (see e.g. Fonda and Ghirardi 1972):

$$
|\Psi_\nu(t)\rangle = e^{i\frac{1}{2}Mv^2t} e^{-i\vec{p}\vec{v}t} e^{i\vec{M}\vec{\mathcal{R}}\vec{v}} |\Psi(t)\rangle \tag{4.60}
$$

with:

$$
\vec{p} = \sum_{k=1}^{N} \vec{p}_k
$$

$$
M = \sum_{k=1}^{N} m_k \tag{4.61}
$$

$$
\vec{R} = \frac{1}{M} \sum_{k=1}^{N} m_k \vec{x}_k
$$

Applying this transformation one can easily show that the expectation values of the operator $\langle [\vec{P}_{(1)}]_1 \rangle^2 = \sum_{i,j} \vec{p}_i \vec{p}_j$, in the moving and in the rest system are related by:

$$
\left\langle \sum_{i,j} \vec{P}_i \vec{P}_j \right\rangle (v) = \left\langle \sum_{i,j} \vec{P}_i \vec{P}_j \right\rangle + M^2 v^2 \tag{4.62}
$$

$\sum_{i,j} \vec{p}_i \vec{p}_j$ describes the center of mass motion and is also called the square of the collective momentum $\vec{p}^2_{\text{coll}}(v) = \sum_{i,j} \vec{p}_i \vec{p}_j$. Using eq. (4.52) as well as the fact, that $\langle [\vec{P}_{(1)}]_1 \rangle = 0$ in a static system (Ehrenfest’s theorem eq. (4.41)) we obtain a relation, which fixes $v^2$:

$$
\vec{p}^2_{\text{coll}}(v) = M^2 v^2 + \langle [\vec{P}_{(1)}]_1^2 \rangle \tag{4.63}
$$

Substituting eq. (4.63) into the expression for the energy of the soliton in a moving system eq. (4.43) one recognizes that the total energy can be separated into a collective and an intrinsic part:

$$
E[\vec{p}^2_{\text{coll}}(v)] = \frac{\vec{p}^2_{\text{coll}}}{2E_{\text{MF}}} + \left[ \frac{E_{\text{MF}} - \langle [\vec{P}_{(1)}]_1^2 \rangle}{2E_{\text{MF}}} \right] \tag{4.64}
$$
where we have set $E_{MF}$ for the total mass $M$. The collective term $\frac{1}{2}\frac{\Pi_{coll}^2}{E_{MF}}$ is the kinetic center of mass energy and vanishes, if we transform to a system with resting c.o.m. The relevant part is therefore the intrinsic energy $E_{MF} - \frac{1}{2}\frac{\Pi_{coll}^2}{E_{MF}}$, in which analogous to eq. (4.57) the spurious zero point energy $\frac{1}{2}\frac{\Pi_{coll}^2}{E_{MF}}$ has got subtracted from the classical mean field energy $E_{MF}$.

Summarizing we find finally for the total intrinsic energy for a system with spin $J$ and isospin $T$ whose structure is known from e.g. Ring and Schuck (1980) or Blaizot and Ripka (1988):

$$E_{[JT,\Pi_{coll}^2=0]} = E_{MF} + \frac{T(T+1)}{2\Theta} - \frac{\langle \hat{T}^2 \rangle (T+1)}{2\Theta} - \frac{\langle \hat{P}_{coll}^2 \rangle}{2E_{MF}}$$

(4.65)

General Aspects of the Zero Mode Treatment

Aside from the fact, that we have handled the center of mass motion purely nonrelativistically, our treatment of the spurious zero modes goes clearly beyond the semiclassical approach, which can be seen e.g. from the condition eq. (4.55) for the Lagrange multiplier $\Omega^2$, which is incompatible with the one for $\Omega^a$ in eq. (4.19) on a classical level since eq. (4.55) allows imaginary omega. One should also note that the 'correction' terms to $E_{MF}$, which as itself is of $\mathcal{O}(N_c)$ are of different order in $N_c$, namely $\mathcal{O} \left( \frac{1}{N_c} \right)$ for the centrifugal term $\frac{T(T+1)}{2\Theta}$ and $\mathcal{O}(N^0)$ for the band-head term $\frac{\langle \hat{P}_{coll}^2 \rangle}{2\Theta}$ and the center of mass term $\frac{\langle \hat{P}_{coll}^2 \rangle}{2M}$, respectively.

In nonrelativistic many particle physics the form of eq. (4.65) can be obtained within certain approximation by using Peierls-Yoccoz projection techniques (Ring and Schuck 1980) for the angular and linear momentum. These techniques have been successfully applied in soliton models with valence quarks (Birse and Banerjee 1984, Birse 1985, Lübeck et al. 1986, Fiolhais et al. 1988, Fiolhais et al. 1991, Neuber and Goexc 1992, for a review cf. Birse 1990). Unfortunately it is up to now not clear how to establish them in case of the NJL soliton, because the regularized Dirac sea does not allow the definition of a Fock state, which is necessary for applying projection methods.

In case of the Skyrme model it turned out that considering RPA fluctuations around the mean field solution, which breaks the (iso-)rotational and translational symmetry of the full theory, allows a treatment of the corresponding eigen modes (Moussallam and Kalafatis 1991, Holzwarth 1992), analogous to the case of two dimensional soliton models (Rajaraman 1982). Altogether these modes lower the classical soliton energy and about 80% of the lowering originates from the rotational and translational zero modes. Hence it is probably a good approximation in (4.65) to concentrate on the zero-modes of the baryon.

A full treatment of the RPA-modes in the present NJL model has not been done by now since the Dirac sea complicates the formalism tremendously. On the other hand, as long as there exists no consistent treatment of the zero modes in our model, the method described above is as good as the semiclassical quantization by itself and one can hope that eq. (4.65) describes at least roughly the mass of a particle at rest ($\langle \Pi_{coll} \rangle = 0$) with spin $J$ and isospin $T$.

Masses of $N$ and $\Delta$, Numerical Results

From eq. (4.65) we can read off the expressions for the total masses of a nucleon $N$ ($J = T = \frac{1}{2}$)
as well as a Δ (J = T = 3/2), respectively (Polyvitsa et al.1992):

\[
M_N = E_{MF} + \frac{3}{8\Theta} - \frac{9}{8\Theta} - \frac{\langle \mathcal{H}^2 \rangle_{(11)}}{2E_{MF}}
\]

\[
M_\Delta = E_{MF} + \frac{15}{8\Theta} - \frac{9}{8\Theta} - \frac{\langle \mathcal{H}^2 \rangle_{(11)}}{2E_{MF}}
\]  

(4.66)

For the numerical calculation we take the mean field energy of the selfconsistent solutions (cf. sect. 3.3). The Θ as well as the expectation value \(\langle \mathcal{H}^2 \rangle_{(11)}\) can be obtained from the 1-particle eigenstates \(|\lambda\rangle\) of this solution using eqs. (4.15, 3.17). The numerical results in dependence of the constituent quark mass \(M\) are shown in Fig. 4.1, 4.2 (Polyvitsa et al.1992). In the relevant region \(M \approx 400\text{MeV}\) it turns out that the rotational zero point energy \(\frac{9}{8\Theta}\) lies around \(100\text{MeV}\) whereas the translational zero point energy amounts to about \(300\text{MeV}\). This is the order of magnitude obtained also in nonrelativistic quark models (see e.g. Bhaduri 1988 and ref. therein) as well as in relativistic soliton models using Peierls-Yoccoz projection techniques (Fiolhais et al.1991, Neuber and Goeke 1992). On the other hand especially the value for the translational zero point is quite high in comparison with the soliton mean field energy \(E_{MF} \approx 1200\text{MeV}\), so that the perturbative treatment in \(\Omega\) and \(v\) appears more than questionable. It is also interesting to note that the "cranking" term \(\frac{T(T+1)}{2\Theta}\) can be numerically of the same order of magnitude as the band-head energy \(\frac{\langle \mathcal{H}^2 \rangle_{(11)}}{2\Theta}\), although they are of different order in \(N_c\).

Finally \(M_N\) and \(M_\Delta\) (Fig. 4.2) are at \(M \approx 400\text{MeV}\) with \(M_N \approx 900\text{MeV}\) and \(M_\Delta \approx 1100\text{MeV}\), respectively, somewhat close to their experimental values (\(M_N = 938\text{MeV}, M_\Delta = 1230\text{MeV}\)). Especially one notes that the nucleon becomes stable against decay into \(N_c = 3\) free quarks as consequence of the subtraction of the spurious zero point energies.

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5. Nucleon Form Factors and Observables

We are now supplied with the basic techniques to calculate various nucleon observables from the mean field solutions obtained in chapter 3. We will consider especially the electromagnetic form factors of proton and neutron including radii and magnetic moments (sect.5.1) as well as the axial vector coupling constant $g_A$ and the axial form factor $g_A(q^2)$ (sect.5.2). Furthermore the pion nucleon coupling $g_{\pi NN}$ and the form factor $g_{\pi NN}(q^2)$ are discussed and the Goldberger-Treiman relation between $g_A$ and $g_{\pi NN}$ is exhibited (sect.5.3). Finally we deal with the nucleon $\Sigma$-term (sect.5.4). The spin of the proton that is carried by the quarks will be discussed in the context of the $SU(3)$ soliton in chapter 6.

5.1. Electromagnetic Form Factors, Moments and Radii

Electromagnetic Current

On a classical level the electromagnetic current $j_{\mu}^{em}$ arises as Noether current of the electromagnetic $U(1)$ phase transformation:

$$ q \rightarrow e^{iQ \cdot q} $$

with the charge matrix

$$ Q = \left( \begin{array}{c} \frac{2}{3} \\
-\frac{1}{3} \end{array} \right) = \frac{1}{6} I + \frac{\tau^3}{2} $$

and is therefore given by:

$$ j_{\mu}^{em} = \bar{q} \gamma_{\mu} Q q $$

It is the sum of an isoscalar and an isovector part. Applying the general method described in sect.4.2, we find for the sea part of the expectation value of $j_{\mu}^{em}$ in the rotating (cranked) soliton in Minkowski space (cf.eq. (4.16)):

$$ \langle j_{\mu}^{em}(x) \rangle = \frac{1}{T} \frac{\delta}{\delta A_\mu(x)} \text{Spn} \left[ i D + J + \frac{\alpha_s^2}{2} + A_\mu(x) R(\beta \gamma^\mu Q) R^\dagger \right]_{A_\mu(x)=0} $$

If one performs a Wick rotation to Euclidean space the time component $A_0$ has to be handled in the same way as the cranking frequency $\Omega$ (cf. sect.4.2), i.e. the time component of a physical 4-vector: $A_0 = A_4 = -iA_0$, where $A_4$ and $A_0$ are both real. The valence part arises in the standard way by introducing a chemical potential (cf. sect.3.2).

One should mention that $\partial^\mu \langle j_{\mu}^{em}(x) \rangle$ does not necessarily vanish on the fermion 1-loop level, because of the regularization involved. The same is true also for the divergence of the axial current (cf.eq. (2.7)). Apparently the result of the divergence depends on the regularization scheme used and indeed, for momentum cutoff schemes the Noether theorems are violated in the solitonic sector (Döring et al.1992). On the other hand the proper-time method preserves those invariances. This has been checked for $U(1)$, $SU(N_f)_{\gamma}$ and $SU(N_f)_{\gamma}$ transformations.

Electric Form Factors of Proton and Neutron
The electric form factor $G_E(q^2)$ is defined by the matrixelement of the time component $\hat{A}_0^{cm}$ between nucleon states $|N_i(p_1, s_1, t_1)|$ with 4-momentum $p_1$, spin z-component $s_1$ and isospin z-component $t_1$ ($i=1,2$) (Bernstein 1968):

$$\langle N_2(p_2, s_2, t_2)|\hat{A}_0^{cm}(0)|N_1(p_1, s_1, t_1)\rangle = \bar{u}(p_2, s_2, t_2)\gamma^\mu u(p_1, s_1, t_1)G_E(q^2)$$

(5.5)

In eq. (5.5) $q = p_2 - p_1$ denotes the 4-momentum transfer and

$$u(p_i, s_i, t_i) = \sqrt{E + M_N} \left( \frac{1}{E + M_N} \right)^{1/2} \chi_{s_i} \chi_{t_i}$$

(5.6)

the spinor of a free pointlike nucleon with mass $M_N$. It is convenient to use the Breit system defined by:

$$p_1 = (E, -\vec{q}/2), \quad p_2 = (E, +\vec{q}/2), \quad q = (0, \vec{q})$$

$$E = \sqrt{M_N^2 + \vec{q}^2/4}, \quad Q^2 = -q^2 = \vec{q}^2, \quad Q = |\vec{q}|$$

(5.7)

There eq. (5.5) simplifies to:

$$\langle N_2(p_2, s_2, t_2)|\hat{A}_0^{cm}(0)|N_1(p_1, s_1, t_1)\rangle = G_E(q^2)\chi_{s_2}^\dagger \chi_{t_2}^\dagger Q\chi_{s_1} \chi_{t_1}$$

(5.8)

The nucleon state $|N_i(p_1, s_1, t_1)\rangle$ is treated in the static approximation and center of mass corrections are neglected. This means:

$$|N(i, s_i, t_i)\rangle = \sqrt{(2\pi)^3}\delta(0) s_i t_i$$

(5.9)

In our approach we take for $|s_i t_i\rangle$ the collective quantized nucleon states $|p \uparrow\rangle, |p \downarrow\rangle, |n \uparrow\rangle$ or $|n \downarrow\rangle$.

Like the electromagnetic current $j_{\mu}^{em}$ itself $G_E(q^2)$ separates into an isoscalar and an isovector part

$$G_E(q^2) = \frac{1}{2}G_{E}^{T=0}(q^2) + t_3 G_{E}^{T=1}(q^2)$$

(5.10)

It is easy to see that for the isoscalar part the first nonvanishing term is of zeroth order in $\Omega$, whereas for the isovector part is the term of first order in $\Omega$. These two quantities are then related to proton and neutron experimental currents by $G_{E}^{T=0}(q^2) = G_{E}(q^2)_p + G_{E}(q^2)_n$ and $G_{E}^{T=1}(q^2) = G_{E}(q^2)_p - G_{E}(q^2)_n$.

From eq. (5.5) one can extract $G_{E}^{T=0}(q^2)$ and $G_{E}^{T=1}(q^2)$ after some algebra to (Gorski et al.1992):

$$G_E^{T=0}(q^2) = \int d^3x e^{-i\vec{q}\vec{x}} \frac{N_c}{3} \left\{ |\phi_{\nu,\lambda}(\vec{x})|^2 - \frac{1}{2} \sum_\lambda \text{sign}\epsilon_\lambda |\phi_\lambda(\vec{x})|^2 \right\}$$

$$G_E^{T=1}(q^2) = \int d^3x e^{-i\vec{q}\vec{x}} \frac{N_c}{12} \int d^3y \left\{ - \sum_{\lambda \neq \nu, 1} \frac{[\phi_{\nu,\lambda}^\dagger(\vec{x})\bar{\phi}_\lambda(\vec{y})][\phi_{\nu,\lambda}^\dagger(\vec{y})\bar{\phi}_\lambda(\vec{x})]}{\epsilon_{\nu,\lambda} - \epsilon_\lambda} - \frac{1}{4\sqrt{\pi}} \int \frac{ds}{s^2} \sum_{\lambda \neq \nu} \left[ \frac{\epsilon_\nu e^{-s\epsilon_\nu^2} + \epsilon_\lambda e^{-s\epsilon_\lambda^2}}{\epsilon_\lambda - \epsilon_\nu} + \frac{1}{s} \frac{e^{-s\epsilon_\nu^2} - e^{-s\epsilon_\lambda^2}}{s \epsilon_\lambda - s \epsilon_\nu} \right] \cdot [\phi_\lambda^\dagger(\vec{x})\bar{\phi}_\nu(\vec{y})][\phi_\nu^\dagger(\vec{y})\bar{\phi}_\lambda(\vec{x})] \right\}$$

(5.11)
where \( \Theta \) denotes the moment of inertia (cf. sect. 4.1) and \( \phi_\lambda(\vec{x}) = \langle \vec{x} | \lambda \rangle \) the 1-particle eigenfunctions in \( \vec{x} \)-representation.

The results for the proton and neutron form factor as well as the corresponding charge densities are shown in Fig. 5.2-5.5 (Wakamatsu 1991, Gorski et al. 1992). As one can see, the proton form factor \( G^p_E(q^2) \) is described very well in contrast to the neutron form factor \( G^n_E(q^2) \). Furthermore we recognize that the neutron charge density is dominated by the sea quarks at large distances \( r \), which confirms the popular picture that the long negative tail of the neutron charge distribution is made by the pion cloud (Thomas 1983), which is connected in our model with the polarization of the Dirac sea by gradient or heat kernel expansion of the fermion determinant.

The corresponding quadratic radii

\[
\langle R^2 \rangle_E = -6 \left. \frac{dG_E(q^2)}{dq^2} \right|_{q^2=0}
\]

are shown in Fig. 5.1. The fact that the \( \langle R^2 \rangle_E \) is negative is due to the long negative tail in the corresponding charge distribution. It originates in the present model from the polarization of the Dirac sea, which in other models corresponds to the pion cloud. This negative tail is obviously overemphasized in the present model \( \langle R^2 \rangle_E = -0.21 \text{fm}^2 \) compared to the experimental value of \( \langle R^2 \rangle_E = -0.12 \text{fm}^2 \). One should compare these values with the Skyrme model, which gives \(-0.36 \text{fm}^2\) in the scalar version (Braaten et al. 1986a, 1986b) and \(-0.24 \text{fm}^2\) with vector mesons (Kaiser et al. 1987).

Magnetic Moments and Form Factors

The same considerations can be applied in case of the magnetic form factor which is related to the space components of the electromagnetic current \( j^\text{em}_i \) in the Breit frame through

\[
\langle N_2(p_2, s_2, t_2) \big| J^\text{em}_i(0) \big| N_1(p_1, s_1, t_1) \rangle = i \frac{G_M(q^2)}{2M_N} \chi^\dagger_x \chi^\dagger_y (\vec{\sigma} \times \vec{q}) \chi_x \chi_y 
\]

The main difference compared to \( G_E(q^2) \) is the fact, that in case of \( G_M(q^2) \) the leading order of the isovector part is \( \mathcal{O}(\Omega^0) \) whereas the leading order of the isoscalar part is \( \mathcal{O}(\Omega^1) \).

At \( q^2 = 0 \) one obtains the magnetic moments:

\[
\mu^{T=0} = \mu^p + \mu^n \\
\mu^{T=1} = \mu^p - \mu^n
\]

which have been calculated in the present model by Wakamatsu and Yoshiki (1990). The results are shown in Tab. 5.1 and compared with the experimental values (\( \mu^p = 2.79, \mu^n = -1.91, \mu^{T=1} = 4.70 \) and \( \mu^{T=0} = 0.88 \)). One recognizes that for the relevant values of the constituent quark mass \( M \approx 400 \text{MeV} \) the isovector part \( \mu^{T=1} \) has a relatively large sea quark contribution and its total value comes out too small. The isoscalar part \( \mu^{T=0} \) is clearly dominated by the valence quarks and can be reproduced quite reasonable.
A calculation of the full $q^2$ dependence of $G_M(q^2)$ has been performed recently (Gorski et al. 1993) and shows a good agreement with the experimental results as far as the $q^2$ dependence is concerned (Fig. 5.6, 5.7).

Similar to the axial vector coupling constant, to be discussed in sect. 5.2, actually the first non-vanishing term for the isovector magnetic moment is of zeroth order in the rotational frequency $\Omega$. However there are important corrections in linear order of $\Omega$, which are entirely due to the time-ordering of operators in non-local theories (Christov et al. 1993). Without going into mathematical details the zeroth order results are presented in Fig. 5.8 and compared with the experimental data (Christov et al. 1993b). Apparently the corrections are quite important (Compare with the axial vector coupling constant in Fig. 5.9).

Furthermore one has to keep in mind that all calculations have been performed in zeroth order of the pushing velocity $v$, which means, that any center of mass corrections to nucleon observables other than the nucleon mass (cf. sect. 4.4) have been completely neglected. On the other hand one knows from nonrelativistic quark models (Bhaduri 1988 and ref. therein) as well as other effective quark models (Betz and Gold 1988, Dethier et al. 1983, Fiebig and Hadjimichael 1984a, 1984b, Braaten et al. 1986a, 1986b, Luebeck et al. 1988, Leech and Birse 1989, Stern and Clement 1989, Fiolhais et al. 1991, Neuber and Goeke 1992) that those corrections might be noticeable.

5.2. Axial Vector Coupling Constant and Axial Form Factor

The general form of the matrix element of the axial current (cf. eq. (2.7))

$$ A^a_{\mu}(x) = \bar{u}(x) \gamma_{\mu} \gamma_5 T^a q(x) $$

between nucleon states (eq. (5.9)) can be written as:

$$ \langle N_1(p_1, s_1, t_1) | A^a_{\mu}(0) | N_2(p_2, s_2, t_2) \rangle = \bar{u}(p_1, s_1, t_1) \frac{\gamma_{\mu} \gamma_5}{2} [ g_A(q^2) + g_A(q^2) h_A(q^2) ] u(p_2, s_2, t_2) $$

where $u(p_i, s_i, t_i), i = 1, 2$ denotes the free nucleon spinor (cf. eq. (5.6)). $g_A(q^2)$ is the axial and $h_A(q^2)$ the polar form factor of the nucleon. At zero momentum transfer $g_A(q^2)$ reduces to the axial vector coupling constant $g_A = g_A(q^2 = 0)$ given by:

$$ g_A = 2 \langle p \mid 1 \int d^3 x (\tilde{A}^a_{\mu = 3}(x))|(\Omega)\rangle \mid p \rangle $$

5.3 The Axial Vector Current in Zeroth Rotational Order

Using the method of collective quantization described in sects. 4.1 and 4.2 we have in zeroth order $\Omega$ (cf. eqs. (4.16, 4.35)):

$$ g_A = 2 \frac{1}{3} \int d^3 x \left\langle q^i (\vec{x}) \frac{\sigma^3_3 \gamma_3}{2} q(\vec{x}) \right\rangle $$

Furthermore the sea contribution of $g_A$ gets regularized due to eqs. (3.17, 3.18) with $\mathcal{O} = \frac{\sigma^3_3 \gamma_3}{2}$. 

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Using the mean field equations of motion eqs. (3.47, 3.51) it can be shown that the \( \text{PCAC} \) relation eq. (2.8) is maintained for the expectation value of the axial current, i.e. it is valid also at the 1-quark loop level to order \( \sim \Omega^{(0)} \) as long as the regularization scheme applied respects chiral invariance (Meissner Th and Goek e 1991).

\[
\partial^\mu \langle A_\mu^a(x) \rangle = -m_\pi^2 f_\pi \pi^a(x)
\]  

(5.19)

As we have mentioned in the last section this is true, if one uses e.g. the proper time regularization, but violated in case of momentum cutoffs.

Fig. 5.9 shows the numerical results for various constituent masses \( M \) calculated from eq. (5.18) with the selfconsistent profiles. In the relevant region \( M \approx 400 \text{MeV} \) one obtains in zeroth order of \( \Omega \) an axial vector coupling constant \( g_A \approx 0.8 \), which is too small compared to the experimental value of \( g_A = 1.23 \) (Wakamatsu and Yoshiki 1990, Meissner Th and Goek e 1991). The valence quark contribution is clearly dominating.

Applying eq. (5.19) to eq. (5.17) \( g_A \) including the valence part can be written as radial integral over the pion field:

\[
g_A = -\frac{8\pi}{3} m_\pi^2 f_\pi \int dr r^2 \pi(r)
\]  

(5.20)

This sum rule turns out to be fulfilled numerically as well (Meissner Th and Goek e 1991a).

The momentum dependence of the form factor \( g_A(q^2) \) is usually parameterized by a dipole form:

\[
\frac{g_A(q^2)}{g_A(0)} = \left(1 - \frac{q^2}{M_A^2}\right)^{-2}
\]  

(5.21)

where the most recent experiments determine the dipole mass to \( M_A = 1.09 \pm 0.03 \text{GeV} \) (Ahrens et al. 1988). Meissner Th and Goek e (1991a) obtained in the present model a value of \( M_A = 0.95 \text{GeV} \), which is almost compatible with the experimental result. Hence the \( q^2 \) dependence of the axial form factor is reproduced.

The axial vector current with first-order rotational corrections

Recently it has been shown, that due to the non-commutativity of some collective operators (Wakamatsu and Watabe 1993) and due to the explicit time-ordering of time-dependent operators (Christov et al. 1993b, Blotz et al. 1993c), that there are important contributions from the rotational corrections in the first order of the frequency \( \Omega \sim 1/N_c \). Their origin lies in the fact, that the generators of the \( SU(2)_{\text{spin}} \)-group \( J^i \) do not commute with the D-functions \( D^{ik} \). Furthermore one has to take care of the time-dependence of the operators, because one has to refer to the operator formalism (and not to the functional integral) when dealing with the rotation matrices \( \mathcal{R}(t) \) (Christov et al. 1993b). Therefore in the expansion of (4.16), there emerges a term of the form \( \sim [J^a, D^{ik}] \), which is non-vanishing.

Actual calculations (For details see Christov et al. 1993b) including these corrections improve the value for \( g_A \), such that the total value of \( g_A \) for \( M \approx 400 \text{MeV} \) in NJL comes out to be \( g_A = 1.15 - 1.30 \), which is quite close to experiment. The formulas for these additional terms
are presented together with the corresponding terms in SU(3) in chap. 6. Detailed values for $g_A$ in SU(2) including these corrections are presented in Fig. 5.9 and the importance of the $\Omega^1$ corrections to the isovector magnetic moment has been shown in Fig. 5.8.

There are two philosophies (Blotz et al. 1994). First: One evaluates the pion and sigma field from the equation of motion eq. (3.51), which are given in $O(\Omega^0)$. Then one evaluates with this the axial vector current and a new pion field, both in order $O(\Omega^0 + \Omega^1)$. This treatment is strictly consistent with an expansion in the $1/N_c$ spirit as far as rotations are concerned and it is consistent with the perturbative cranking procedure. On the other hand PCAC is violated. If one uses the numbers of Alkofer and Weigel (1993), one may obtain a rough feeling how far PCAC is violated and for reasonable values of the constituent mass this violation seems to be small. Second: One derives equations of motion for the pion and also for the axial vector current, both up to $O(1/N_c)$ expansion and the perturbative expansion of the cranking procedure. On the present level of investigation one cannot prefer one procedure to the other.

### 5.3. Pion Nucleon Coupling and Form Factor

The pion nucleon form factor $g_{\pi NN}(q^2)$ is defined through the nucleon matrix element of the pionic current: $J^{a}_{\pi}(x)$

$$\langle N(p) | J^{a}_{\pi}(0) | N(p') \rangle = (\Omega + m^2) \pi^a(x) = J^{a}_{\pi}(x)$$

(5.22)

as:

$$\langle N_2(p_2, s_2, t_2) | J^{a}_{\pi}(0) | N_1(p_1, s_1, t_1) \rangle$$

$$= (-i) \delta(p_2, s_2, t_2) \pi^a \gamma_5 \gamma_\mu(p_1, s_1, t_1) g_{\pi NN}(q^2)$$

(5.23)

For static pion fields one has: $J^{a}_{\pi}(\vec{r}) = (-\nabla^2 + m^2) \pi^a(\vec{r})$, which remains also valid in the cranking approximation because the 2nd time derivative in eq. (5.22) is of 2nd order in the cranking frequency $\Omega$ and therefore neglected. From eq. (5.23) by using eq. (5.22) the $g_{\pi NN}(Q)$ ($Q^2 = -q^2$) can be finally be extracted to:

$$g_{\pi NN}(Q^2) = (m_N^2 + Q^2) \frac{4\pi}{3} \int d\tau r^3 \frac{J_1(Q\tau)}{Q\tau} \pi(r)$$

(5.24)

Especially at $Q^2 = -q^2 = 0$ one has:

$$g_{\pi NN}(0) = m_N^2 \frac{4\pi}{9} \int d\tau r^3 \pi(r)$$

(5.25)

Comparing eq. (5.25) with eq. (5.20) we recognize the famous Goldberger-Treiman relation (Goldberger and Treiman 1958, Cheng and Li 1984):

$$g_{\pi NN}(0) = \frac{M_N}{f_\pi} g_A$$

(5.26)

which arises as a consequence of the PCAC relation eq. (5.19) and holds as long as the regularization scheme respects chiral symmetry.
The above formula deals with spacelike $q^2 = -Q^2 < 0$. However, direct experimental data for the $\pi - N$ interaction are only available for timelike $q^2$, even on shell $q^2 = -Q^2 \geq m^2_{\pi}$. The eq. (5.24) can be analytically continued to the full timelike region $q^2 = -Q^2 > 0$ (Cohen 1986):

$$
\frac{g_{\pi NN}(Q^2 < 0)}{2M_N} = \left[ \frac{-(m^2_{\pi} + Q^2)}{2M_N} \right] \cdot \frac{4\pi}{3} \int_0^R dr r^3 \left( \frac{j_1(Qr)}{Qr} \right) \pi(r)
+ \left[ \frac{-(m^2_{\pi} + Q^2)}{2M_N} \right] \cdot \frac{4\pi}{3} A e^{-m_{\pi}R} \left( \frac{j_1(Qr)}{Qr} \right)
+ \left[ \frac{4\pi}{3} A e^{-m_{\pi}R} \right] \frac{1}{|Q|} \left( m_{\pi} \sinh(|Q| R) + |Q| \cosh(|Q| R) \right)
$$

(5.27)

The A is given by the asymptotic form of the pion field in eq. (3.53). Especially for on shell pions $q^2 = -Q^2 = m^2_{\pi}$ one finds

$$
\frac{g_{\pi NN}(q^2 = m^2_{\pi})}{2M_N} = -\frac{4\pi}{3} A
$$

(5.28)

with the experimental value $g_{\pi NN}(q^2 = m^2_{\pi}) = 13.6$. Tab. 5.2 shows the values for $g_{\pi NN}(q^2 = m^2_{\pi})$ and $g_{\pi NN}(0)$ in our model, where the nucleon mass from eq. (4.66) and Fig. 4.2 have been used. Because of the Goldberger-Treiman relation eq. (5.26) it is clear that the discrepancy between the theoretical and the experimental value of $g_{\pi NN} = 13.6$ is the same as in case of $g_A$ evaluated to $\Omega^{(0)}$ considered in the last section.

In the spacelike region the $q^2$ dependence of the pion nucleon form factor $g_{\pi NN}(q^2 < 0)$ can be fitted to $N N$ scattering data by using one boson exchange potentials (OBEP). In a monopole parameterization one obtains in such an approach

$$
\frac{g_{\pi NN}(q^2 < 0)}{g_{\pi NN}(q^2 = m^2_{\pi})} = \frac{\Lambda_{\pi NN} - m^2_{\pi}}{\Lambda_{\pi NN} - q^2}
$$

(5.29)

with a monopole mass of $\Lambda_{\pi NN} = 1530\, \text{MeV}$ (Machleidt et al.1987). In our model we obtain a monopole cutoff of $\Lambda_{\pi NN} = 790\, \text{MeV}$. Also all other chiral models predict a much lower value than the OBEP, e.g. the Skyrme model without vector mesons $580\, \text{MeV}$ (Cohen 1986), the Skyrme model with vector mesons $850\, \text{MeV}$ (Kaiser et al.1987), the projected linear chiral sigma model $690\, \text{MeV}$ (Alberto et al.1988). The same order of magnitude for $\Lambda_{\pi NN}$ is needed in charge exchange reactions (Esbensen and Lee 1985), in more recent estimates within meson exchange models (Deister et al.1990, Janssen et al. 1993). A similar value of $\Lambda$ is also obtained by substituting the experimental values for $g_{\pi NN}(q^2 = m^2_{\pi}) = 13.6$ and $g_{\pi NN}(0) = \frac{M_N}{f_{\pi}} g_A = 12.4$ from the Goldberger-Treiman relation (5.26) into eq. (5.29) which then yields $\Lambda_{\pi NN} = 468\, \text{MeV}$. The discrepancy between the OBEP calculation and all the other approaches might lie in the fact that for a correct description of the $N N$ interaction other, more complicated processes than the $\pi NN$-vertex have to be taken into account. A refitting of the $N N$-phase shifts by means of OBEP models with different exchange processes yields also lower values for $\Lambda_{\pi NN}$ (Holinde 1992).
5.4. Nucleon Sigma Term and Form Factor

The nucleon sigma term $\Sigma_N$ is defined as analogon to the quark condensate $\langle \bar{q}q \rangle_V$ (2.22) in the nucleon sector

$$\Sigma_N = m_0 \int d^4x \langle N|\hat{\sigma}(\vec{x})q(\vec{x})|N \rangle$$

(5.30)

As an isoscalar quantity the $\Sigma_N$ stays uninfluenced by the cranking procedure in SU(2). Because the current mass $m_0$ couples to the Dirac operator like a Lagrange multiplier for the condition $m_0\bar{q}q$ (cf. eq. (3.14)) one can write $\Sigma_N$ in the convenient form:

$$\Sigma_N = m_0 \frac{\partial E_{MF}(\hat{\mu})}{\partial \hat{\mu}} \bigg|_{\hat{\mu}=0}$$

(5.31)

The corresponding form factor is defined by:

$$\sigma(q^2)\tilde{u}(p_1, s_1, t_1)u(p_2, s_2, t_2) = m_0\langle N|\hat{\sigma}(p_2, s_2, t_2)|\tilde{u}(0)q(0)|N(p_1, s_1, t_1)\rangle$$

(5.32)

which gives with eqs. (5.6, 5.9):

$$\sigma(Q^2 = -q^2) = \int d^3r_{\hat{\mu}}(Qr)\langle \tilde{u}(\vec{r})q(\vec{r}) \rangle$$

(5.33)

At the Cheng-Dashen point $q^2 = 2m^2_\pi$, the $\sigma(q^2)$ can be determined from $\pi - N$-scattering data to $\sigma(2m^2_\pi) \approx 60\text{MeV}$ (Gasser and Leutwyler 1982). From dispersion relations an extrapolation to $q^2 = 0$ has been performed, which gives $\Delta_\sigma = \sigma(2m^2_\pi) - \sigma(0) = 15\text{MeV}$ (Gasser and Leutwyler 1991). Hence the present NJL model should evaluate the sigma term $\Sigma_N$ of eq. (5.30) to $45 \pm 5\text{MeV}$. One should note however that from chiral perturbation theory a much smaller value $\Delta_\sigma \approx 5\text{MeV}$ is obtained (Gasser et al. 1988a, 1988b). In contrast to all other observables in our model $\Sigma_N$ turns out to be somewhat sensitive to the regularization scheme applied. For the proper time scheme one gets a value of $\Sigma_N = \sigma(0) \approx 35\ldots40\text{MeV}$ (Meissner Th and Goeke 1991, Wakamatsu 1993), which is also obtained in the Pauli-Villars scheme (Schueren et al. 1992), whereas with an extension of the proper time method

$$\int_{1/\Lambda^2} d\tau F(\tau) - \int_0^\infty \phi(\tau)F(\tau)$$

(5.34)

also higher values for $\Sigma_N$ can be obtained for an appropriate chosen cutoff function $\phi(\tau)$ (Blotz et al. 1993b). Details for SU(3) will be given in sect. 6.1.

The eq. (5.33) can be analytically continued to the time like region $q^2 = -Q^2 > 0$. The scalar form factor has been calculated by Schueren et al. (1992). One finds from the deviation from the Cheng-Dashen point (Schueren et al. 1992)

$$\Delta_\sigma = \sigma(2m^2_\pi) - \sigma(0) = 7\text{MeV}$$

(5.35)

similar to the value obtained from chiral perturbation theory ($\Delta_\sigma \approx 5\text{MeV}$, Gasser et al. 1988a, 1988b) but only half as large as the one by means of dispersion relations ($\Delta_\sigma \approx 15\text{MeV}$, Gasser et al. 1991) and by a Bethe-Salpeter approach in the meson exchange picture (Pearce et al. 1992).

Wakamatsu (1992b) has also calculated the isospin violation of the $\bar{q}q$ content in the nucleon (Gottfried sum rule) and found a reasonable agreement with the experimental result from NMC data.
6. The SU(3)-flavour NJL-model

In the following chapter we will investigate the extension of the SU(2)-NJL model to the larger symmetry group of SU(3) (Blotz et al. 1992, 1993b, 1993c, 1993d; Weigel et al. 1992a, 1992b). In the vacuum sector, we extend in this way the number of Goldstone bosons to the kaons and the eta. Especially these lightest mesons are often considered as the dominating degrees of freedom for the low energy regime of the strong interactions. Because the masses of these particles are still much lower than the nucleon mass, the extension to this larger symmetry group is a priori sensible.

Historically the success of hadronic isospin symmetry SU(2) (Heisenberg 1932), which is based on the almost mass-equality between the up and down quarks and the flavour independence of QCD, leads to the discovery of flavour SU(3) (Gell-Mann and Ne’eman 1964). It was found that, assuming that u, d and s quarks transform as the fundamental representation of SU(3), the spin 1/2 nucleons belong to the 8-dimensional representation and the spin 3/2 particles to the 10-dimensional representation of the group. From the experimental observation of these symmetrically arranged particles in the multiplets one is usually inclined to consider SU(3) as the more ’physical’ symmetry. That this can be supported within a selfconsistent chiral model for mesons and baryons, is shown below. After considering the modifications of the vacuum sector due to SU(3), which include the mixing of the \( \eta,\eta' \)-system, the \( U_A(1) \)-anomaly and the Gell-Mann Okubo mass relation, the baryon number one sector is described.

To this aim we will concentrate here on the quantization of rotational zero modes (Adkins et al. 1983), which rests on the assumption that SU(3) is indeed a good symmetry of the strong interaction. An alternative treatment considers the baryon as a bound state of a heavy meson and the background field of the SU(2) soliton (Callan and Klebanov 1985, 1988; Callan et al. 1988; Weigel et al. 1993). It is believed that this treatment gives an exact answer for extreme heavy mesons, such as those from a SU(4) representation, but it is still not clear whether it is more appropriate for the intermediate energy scale of the kaon system of SU(3).

Therefore special attention is drawn here to the quantization of the generators of the full SU(3) group and to second order corrections in the strange current quark mass for the splitting of the multiplets. Finally the axial vector coupling constants \( g_A^0, g_A^3 = g_A \) and \( g_A^8 \), currently under refined experimental investigation, are presented with some recently found corrections arising from some non-commutativity of generators (Wakamatsu and Watabe 1993) and explicit time-reordering (Christov et al. 1993, Blotz et al. 1993d).

6.1. The NJL with SU(3)-flavours: The vacuum sector

The vacuum or mesonic sector of the SU(3)-NJL model is intensively discussed in the review articles of Klevansky (1992) and Vogl and Weise (1991). However they mainly used an operator formalism based on the pure quark NJL Lagrangian. So we will summarize here the most important parts within the functional formalism. This has advantages in the baryonic sector, which is our main concern.

SU(3) Symmetry Breaking and the Redundant \( U_A(1) \)
The Nambu-Jona-Lasinio Lagrangian with scalar and pseudoscalar couplings on the level of the four-fermion interaction and with a SU(3) symmetry is conveniently written as

\[ \mathcal{L}_{NJL} = \bar{q}(i\gamma - m)q - \frac{G}{2} \left[ (\bar{q}(x)\lambda^{0}q(x))^{2} + (\bar{q}(x)\gamma_{5}\lambda^{a}q(x))^{2} \right] \]

where \( m = \text{diag}(m_{u}, m_{d}, m_{s}) = m_{1}\mathbf{1} + m_{2}\lambda_{3} + m_{3}\lambda_{8} \) is the current quark mass matrix. The \( \lambda_{a}, a = 0, \ldots, 8 \) are the usual Gell-Mann matrices with \( \lambda^{0} = \sqrt{\frac{1}{2}}\mathbf{1} \). Under infinitesimal chiral SU(3) transformations the quark fields transform as

\[ q \rightarrow (1 - i\frac{1}{2}\lambda^{a}\alpha^{a} - i\frac{1}{2}\lambda^{a}\beta^{a}\gamma_{5})q \]

\[ \bar{q} \rightarrow \bar{q}(1 + i\frac{1}{2}\lambda^{a}\alpha^{a} - i\frac{1}{2}\lambda^{a}\beta^{a}\gamma_{5}) \]

From the eqs. (6.2, 6.3) it is clear that the singlets \((\bar{q}\lambda^{0}q)^{2}\) and \((\bar{q}\gamma_{5}\lambda^{0}q)^{2}\) corresponding to U(3) chiral meson fields have to be included in the Lagrangian in order to be invariant under the chiral SU(3)\(_{R} \otimes SU(3)_{L}\) transformation.

This is in contrast to the SU(2) case, where we had the freedom to choose the chiral fields from a SU(2) or U(2) representation. In addition, this classical Lagrangian (6.1) is invariant under \( U_{A}(1) \otimes U_{V}(1) \) transformations, where the singlet and octet parts of the quark bilinears \( \bar{q}\lambda^{a}q \) and \( \bar{q}\gamma_{5}\lambda^{a}q \) transform independently.

**Currents and Divergences - PCAC**

The Noether currents of the classical Lagrangian from the infinitesimal transformations (6.2, 6.3) are given by

\[ V_{\mu}^{a} = -i\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}q)}\delta q = -\bar{q}\gamma_{\mu}\frac{1}{2}\lambda^{a}q \]

\[ A_{\mu}^{a} = -i\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}q)}\delta Aq = -\bar{q}\gamma_{\mu}\gamma_{5}\frac{1}{2}\lambda^{a}q \]

and their divergences from the Gell-Mann Levi equations are given by

\[ \partial_{\mu}V_{\mu}^{a} = \frac{\partial\mathcal{L}}{\partial\alpha^{a}} = \bar{q}\gamma_{5}\frac{1}{2}\{\lambda^{a}, m\}q \]

\[ \partial_{\mu}A_{\mu}^{a} = \frac{\partial\mathcal{L}}{\partial\beta^{a}} = \bar{q}\gamma_{5}\frac{1}{2}\{\lambda^{a}, m\}q \]

Assuming isospin symmetry, i.e. \( m_{u} = m_{d} = \tilde{m} \), one observes that

\[ \partial_{\mu}V_{\mu}^{a} = 0, \quad \text{for} \quad a = 1, 2, 3, 8 \]

(6.8)

from which follows consistently the conservation of isospin and hypercharge. Furthermore the divergences of the axial currents are proportional to the corresponding pseudoscalar quark bilinears, which will be identified later with the corresponding composite meson fields. We have

\[ \partial_{\mu}A_{\mu}^{a=1,2,3} = -\bar{q}\gamma_{5}\lambda^{a}q\tilde{m} \]

(6.9)
\[ \partial_\mu A^{\alpha \mu}_{\mu} = -\bar{q}i\gamma_5 \lambda^\alpha \frac{1}{2} (\bar{n} + m_s) \quad (6.10) \]

In matrix form the remaining two currents are given by

\[ \partial_\mu \begin{pmatrix} A^0_{\mu} \\ A^8_{\mu} \end{pmatrix} = \mathcal{D} \begin{pmatrix} v^0 \\ v^8 \end{pmatrix} \quad (6.11) \]

with

\[ \mathcal{D} = \begin{pmatrix} \frac{2\bar{n} + m_s}{3} & \frac{\sqrt{3}}{3} (\bar{n} - m_s) \\ \frac{\sqrt{3}}{3} (\bar{n} - m_s) & \frac{2\bar{n} + 2m_s}{3} \end{pmatrix} \quad (6.12) \]

and \( v^a = -\bar{q}i\gamma_5 \lambda^a q \). So we observe that the singlet and octet currents are mixed. However, we can disentangle these terms by means of an orthogonal transformation according to

\[ \partial_\mu \begin{pmatrix} \tilde{A}^0_{\mu} \\ \tilde{A}^8_{\mu} \end{pmatrix} = \tilde{\mathcal{D}} \begin{pmatrix} \tilde{v}^0 \\ \tilde{v}^8 \end{pmatrix} \quad (6.13) \]

with

\[ \begin{pmatrix} \tilde{A}^0_{\mu} \\ \tilde{A}^8_{\mu} \end{pmatrix} = R \begin{pmatrix} A^0_{\mu} \\ A^8_{\mu} \end{pmatrix}, \quad \begin{pmatrix} \tilde{v}^0 \\ \tilde{v}^8 \end{pmatrix} = R \begin{pmatrix} v^0 \\ v^8 \end{pmatrix} \quad (6.14) \]

Here \( \tilde{\mathcal{D}} = R \mathcal{D} R^{-1} \) is a diagonal matrix and the rotation matrix is given by

\[ R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (6.15) \]

The result of this diagonalization is an angle \( \theta \), \( \tan 2\theta = -2\sqrt{2} \) i.e. \( \theta \approx -35.26 \), which is called the ideal mixing angle. We will come back to this point, when we consider the mesonic mass spectrum.

The eigenvalues of \( \tilde{\mathcal{D}} \) are simply \( \bar{n} \) and \( m_s \). This can be understood in the following way. Whereas the former states, which we denote by \( \eta_0 \) and \( \eta_8 \) are group theoretically given by \( \eta_0 \sim \bar{u}u + \bar{d}d + \bar{s}s \) and \( \eta_8 \sim \bar{u}u + \bar{d}d - 2\bar{s}s \), the rotated states are given by \( \tilde{\eta}_0 \sim \bar{u}u + \bar{d}d \) or \( \tilde{\eta}_8 \sim \eta' \sim \bar{s}s \). This is reflected by the two eigenvalues of \( \mathcal{D} \), which are either strange or non-strange masses. This will become clearer when we consider the mesonic two point functions.

**U(1) Symmetry Breaking**

As a consequence of the \( SU(3)_R \otimes SU(3)_L \) invariance in the chiral limit and assuming a spontaneously broken vacuum by a non-vanishing expectation value of some scalar composite field, we immediately get from Goldstone’s theorem that according to the 9 broken generators of the symmetry group there will be 9 massless Goldstone bosons. In nature however, the \( \eta' \) with \( m_{\eta'} = 957\text{MeV} \) cannot be regarded as a Goldstone boson, because its mass is similar to two twice the constituent quark mass. ’t Hooft (1976b) proposed a mechanism supported in QCD, which breaks the redundant \( U(1) \) symmetry, but conserves the \( SU(3) \) symmetry. Such a term is given by

\[ L_{\text{det}} = \kappa \left[ \det(\bar{q}_i P_R q_j) + \det(\bar{q}_i P_L q_j) \right] \quad (6.16) \]
where \( i = u, d, s \) and \( P_{R/L} = \frac{1}{2}(1 \pm \gamma_5) \) are the right and left helicity projection operators\(^2\). Under an infinitesimal chiral transformation it transforms into

\[
\mathcal{L}_{\text{det}} = \kappa \left\{ \det(1 - i\lambda^a\beta^a) \det(\bar{q}_h P_R q_h) + \det(1 + i\lambda^a\beta^a) \det(\bar{q}_L P_L q_L) \right\}
\]

This expression is obviously invariant if and only if the matrices \( \lambda_a \) are traceless and therefore \( \det(1 + i\lambda^a\beta^a) = 1 \). This is the case if the transformation corresponding to \( \beta_0 \) is excluded from the full axial group \( U_A(3) \). So one is left with a \( SU(3)_V \otimes SU(3)_A \otimes U(1)_V \) symmetric expression.

**Bosonization**

In order to bosonize expressions like \( \mathcal{L}_{\text{det}} \), which are not only quadratic in the fields but also of higher order, one has to modify the Gaussian bosonization procedure of sect. 2.1. Therefore we introduce according to Zahed and Brown (1986), Reinhardt and Alkofer (1988)

\[
1 = \int D\sigma_a D\sigma_b \delta(S_a - \tilde{S}^{1/2} \lambda^a q) \delta(P_a - \tilde{S}^{1/2} \gamma_5 \lambda^a q)
\]

\[
= \int D\sigma_a D\sigma_b \int D\sigma_a \sigma_b \exp i \int d^4x \sigma_a(S_a - \tilde{S}^{1/2} \lambda^a q) + \sigma_a(P_a - \tilde{S}^{1/2} \gamma_5 \lambda^a q) \]

In this way, any quark bilinear in \( \mathcal{L}_{\text{det}} \) can be replaced by the fields \( S_a, P_a \). A stationary phase approximation for \( S_a, P_a \) and \( \sigma_a, \pi_a \) immediately relate \( S_a, P_a \) with \( \sigma_a \) and \( \pi_a \), according to

\[
\frac{\partial S}{\partial S_a} = -4GS_a + \sigma_a + \frac{\partial \mathcal{L}_{\text{det}}}{\partial S_a} = 0 \quad (6.19)
\]

\[
\frac{\partial S}{\partial P_a} = -4GP_a + \pi_a + \frac{\partial \mathcal{L}_{\text{det}}}{\partial P_a} = 0 \quad (6.20)
\]

Flavour mixing in this context means that in general \( S_a = S_a[\sigma] \). In the case of \( U(2) \) chiral fields the presence of the \( 't \) Hooft term it holds that \( S_a = \sigma_a \) whereas for \( U(N) \), \( N \geq 3 \), chiral fields the \( 't \) Hooft term leads to a mixing, such that the \( S_a \) are in general non-linear functions of the \( \sigma_a \).

Because we consider here the case of \( SU(3) \) symmetry and no \( \mathcal{L}_{\text{det}} \)-term we can use immediately the stationary phase equation (6.19, 6.20) in order to eliminate the \( S_a, P_a \). We obtain for the Lagrangian after a rescaling of the meson fields \( \phi_a \rightarrow \phi \phi_a \) the form

\[
\mathcal{L}_{\text{NJL}} = \frac{1}{2} \left( -i \bar{\phi} \gamma^\mu + g(\sigma_a \lambda_a + \gamma_5 \pi_a \lambda_a) \phi + \frac{1}{2} \mu^2 (\sigma_a \sigma_a + \pi_a \pi_a) \right)
\]

as one would obtain also from a Gaussian integral multiplicator (Eguchi 1976). After Grassmann integration over the quark fields one obtains finally the effective action in terms of classical meson fields, corresponding to a one-fermion loop approximation:

\[
S_{efl} = \text{Splog}( -i \bar{\phi} \gamma^\mu + g(\sigma_a \lambda_a + \gamma_5 \pi_a \lambda_a) \phi + \frac{1}{2} \mu^2 (\sigma_a \sigma_a + \pi_a \pi_a) )
\]

This expression is in a sense formal, because we have to apply a specific regularization scheme. We will choose here the double step proper time scheme, given by eq. (2.30) where \( \phi(\tau) = e^{\theta(1 - 1/\Lambda_1^2) + (1 - c) \theta(1 - 1/\Lambda_2^2)} \) contains now two more parameters. This additional freedom will be used later in the mesonic sector to fix the non-strange current quark mass to the preferred value.

\(^2\)See Witten (1979\(a\)), Veneziano (1979) and Alkofer and Zahed (1990) for a discussion of a minimal \( U_A(1) \) breaking expression that is free of flavour mixing.
6.2. Fixing of the parameters

As we have said already the expression (6.22) has to be regularized and as we know from sect.
2.2. it should be done from the very beginning in order to avoid discrepancies. For pedagogic
reasons the regularized form will not be written out explicitly. From the action eq. (6.22), which
always can be cast into the form

\[ S_{\text{eff}}[\phi] = \int d^4x V_{\text{eff}}[\phi] + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \ldots, \]  
(6.23)

one obtains for the non-derivative part of eq. (6.23) the effective potential \( V_{\text{eff}} \)

\[ V_{\text{eff}}(\sigma_0, \pi_0) = -\text{Tr} \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \left( k + m + g(\sigma_0 \lambda_0 + i\gamma_5 \pi_0 \lambda_0) \right) + \frac{1}{2} \mu^2 \left( \sigma_0 \sigma_0 + \pi_0 \pi_0 \right) \]  
(6.24)

which differs for homogeneous fields from the effective action only by the four dimensional space-
time volume.\(^3\) Actually, spontaneous breaking of chiral symmetry takes place and the \( \sigma_0 \) and
\( \sigma_S \) are the only candidates of fields, which can acquire a non-vanishing vacuum expectation value.
This follows from conservation of parity, strangeness and isospin.

However it has been shown (Pagels 1975) that the vacuum is \( SU(2)_R \otimes SU(2)_L \) invariant in
the chiral limit, if \( \sigma_0 \) and \( \sigma_S \) have non-zero vacuum expectation values. From Goldstone’s theorem
(Goldstone 1961) we know that the number of broken symmetries of the Lagrangian is related
to the number of massless Goldstone bosons. Actually the physical world knows 8 pseudoscalar
light particles, namely \( \pi, K \) and \( \eta \). If we neglect the \( U_A(1) \) breaking by instantons, even the
\( \eta' \) can be regarded as Goldstone boson. Since other would-be Goldstone bosons are not known
experimentally, the true vacuum state of the NJL model, if it is physical, has to be \( SU(3) \) in
variant in the chiral limit in accordance with Goldstone’s theorem. Fortunately the NJL vacuum is indeed
characterized by a large vacuum expectation value of the \( \sigma_0 \) and a vanishing one for \( \sigma_S \). This
situation is changed for explicit symmetry breaking, like the presence of the ’t Hooft determinant,
where the \( U_A(1) \) symmetry is broken, or current quark masses, where the \( SU(3)_V \) symmetry is
broken.

Now from the actual form of the potential (6.24) the non-trivial stationary phase conditions
are given by

\[ \left. \frac{dV_{\text{eff}}}{d\sigma_0} \right|_{\text{vac}} = \left. \frac{dV_{\text{eff}}}{d\sigma_S} \right|_{\text{vac}} = 0. \]  
(6.25)

After some algebraic manipulations, one is left with

\[ \mu^2 \left( 1 - \frac{m_0}{M_u} \right) = 8 N_c g^2 \hat{I}_1(M_u) \]  
(6.26)

\[ \mu^2 \left( 1 - \frac{m_s}{M_s} \right) = 8 N_c g^2 \hat{I}_1(M_s) \]  
(6.27)

\(^3\)Compare with sect. 2.6., where the parameter fixing is done with using the effective action
only.
where we have set \( M_u = \sqrt{\frac{2}{3} g \sigma_0 + m_1 + \frac{1}{\sqrt{3}} (g \sigma_0 + m_2)} \) and \( M_s = \sqrt{\frac{2}{3} g \sigma_0 + m_1 - \frac{2}{\sqrt{3}} (g \sigma_0 + m_3)} \) and 

\[
I_1(M_i) = \int_{\mathcal{E}} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + M_i^2}
\]  

(6.28)

Mesonic Spectra

The masses of the pseudoscalar and scalar meson nonets follow directly from the second functional derivative of the effective action. This gives an expression, which is up to a normalization factor \( Z_{ab} \) proportional to the inverse of a bosonic propagator. So it follows from general arguments that the analogue of eq. (6.28) is

\[
\frac{1}{Z_{ab}} \frac{1}{\delta^4(p_1 - p_2) \delta \pi_s(p_1) \delta \pi_s(p_2)} \bigg|_{p_1 = p_2, q^2 = -q^2} = (p^2 + m_{ab}^2)
\]  

(6.29)

As described in sect. 2.5, one can now either use the normalization point \( q^2 = 0 \), which correspond to a derivation of the masses from a gradient expansion, or at the on-shell point \( q^2 = m_{ab}^2 \), so that the free propagator is defined with its physical mass.

Defining 

\[
I_2(M_i, M_j, q^2) = \int_{\mathcal{E}} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + M_i^2} \frac{1}{(k + q)^2 + M_j^2}
\]  

(6.30)

we obtain for the normalization factor

\[
Z_{ab} = 4 N_c g^2 I_2(M_a, M_b, q^2 = -m_{ab}^2)
\]  

(6.31)

depending on the flavour content of the given meson. Therefore the corresponding fields have to be rescaled according to \( \phi^{ab} = Z_{ab}^{1/2} \phi^a \). This gives

\[
m_{\pi}^2 = \frac{1}{Z_{\pi}} \mu_{\pi}^2 m_0 M_u, \quad m_K^2 = \frac{1}{Z_K} \mu_K^2 \frac{m_0}{M_u} (m_s + (M_u - M_s)^2)
\]  

(6.32)

For the \( \eta_0 \) and \( \eta_8 \), the situation is more involved, since we have again mixing between them. However, similar to the PCAC relations (6.9, 6.10) above, we can remove the mixing by the same ideal mixing angle \( \theta = -35.26^\circ \), if \( q^2 = 0 \). Then the masses of the new \( \eta \) and \( \eta' \) are given by

\[
m_{\eta}^2 = \frac{1}{Z_{\eta}} \mu_{\eta}^2 m_0 \approx m_{\pi}^2
\]  

(6.33)

\[
m_{\eta'}^2 = \frac{1}{Z_{\eta'}} \mu_{\eta'}^2 m_s \approx 2 m_K^2 - m_{\pi}^2
\]  

(6.34)

One can formally assign masses to the mixed \( \eta_0 \) and \( \eta_8 \) states and gets

\[
m_{\eta_0}^2 \approx \frac{1}{3} m_{\pi}^2 + \frac{2}{3} m_K^2
\]  

(6.35)
\[ m_{\eta_8}^2 \simeq -\frac{1}{3}m_\pi^2 + \frac{4}{3}m_K^2 \]  

\[ m_{\pi}^2 + 3m_{\eta_8}^2 - 4m_K^2 = 0 \]  

(6.36)  

(6.37)

Within the NJL model it is clear that it is the \( \eta_8 \) particle, which fulfills the Gell-Mann (1962) and Okubo (1962) (GMO) relation

In nature however it is the \( \eta \), which should fulfill the GMO relation. This problem, however, can indeed be solved if one includes the 't Hooft interaction, which has the main effect of pushing the \( \eta_0 \) mass, such that the \( \eta \) and \( \eta' \) are driven back to their group theoretical states \( \eta_0 \) and \( \eta_8 \) (Hatsuda and Kunihiro 1991).

Decay Constants

From the general form of axial vector matrix elements between pseudoscalar mesons and the vacuum \(< 0 \mid A^a_{\mu} \mid \pi^a(p) \rangle = -i\gamma_{\mu}f_\pi \pi^a(p)\rangle\), we immediately deduce the pion and kaon decay constants as

\[ f_\pi = \frac{M_\pi}{g}Z_{\pi}^{1/2} \], \[ f_K = \frac{M_s + M_u}{2g}Z_K^{1/2} \]  

(6.38)

Chiral Perturbation for \( m_s \)

Because we will treat the baryonic sector perturbatively in the current quark masses, we will give here also the mass ratio for kaons and pions perturbatively in first order in \( m_s \)

\[ \frac{m_K^2}{m_\pi^2} = \frac{m_s + m_0}{2m_0} + \mathcal{O}\left(\frac{m_s}{M_u}\right) + \mathcal{O}\left(\frac{m_s}{M_s}\right) \].  

(6.39)

It was already noted by Hatsuda (1990) and Hatsuda and Kunihiro (1991), that this relation gets large corrections order by order in perturbation theory. However there are cancellation effects of the non-linear terms in the denominator and numerator of the exact expression for \( m_K^2/m_\pi^2 \), so that the exact result almost coincides with the approximate relation (6.39) (Schneider 1994). That means that although perturbation theory for the whole vacuum does not work, we are not \textit{a priori} going into severe troubles if we make use of eq. (6.39), which coincides with the exact answer. Later in the baryon sector we will examine the validity of perturbation theory for \( m_s \) again. So we summarize that by this equation the current quark mass ratio is fixed to \( m_s/m_0 \sim 24.5 \) for given experimental mesonic data, \( m_\pi = 139 \text{MeV} \) and \( m_K = 496 \text{MeV} \). Chosing our regularization function \( \phi(\tau) \) such as to reproduce the most reasonable value of \( m_0 \sim 6 \text{MeV} \), this corresponds immediately to \( m_s \sim 150 \text{MeV} \), which is also a reasonable value infered from examinations of hyperon spectra. So the quark masses, that will be needed to fit the hyperon splittings in Sect. 6.3, will be compared with the ratio given by eq. (6.39). Furthermore the kaon decay constant \( f_K \) equals \( f_\pi \) in this approximation and for the constituent quark masses one obtains \( M_u = M_d = M_s \).
6.3. Collective Quantization of the SU(3)-Soliton

In sect. 4.2, we have dealt already the quantization of rotational modes in the case of SU(2) symmetry. Now in the case of the SU(3) group, the procedure is a little bit more involved. This is a reflection of the fact that SU(3) is now a rank 2 group, whereas SU(2) has rank 1 and that the configuration space for the SU(3) rotations is now restricted due to some trivial embedding of the SU(2) isospin subgroup into SU(3) (Mazur et al. 1984). Especially this embedding, which was proposed first by Witten (1983b), ensures in the end that only the physical representations emerge as the lowest possible ones (Balachandran et al. 1985, Chemtob 1985, Mazur et al. 1984).

But first we concentrate on deriving the left and right generators of the group. According to Witten, we make for the SU(3) chiral field the following ansatz:

$$ U(x) = \begin{pmatrix} U_2(x) & 0 \\ 0 & 1 \end{pmatrix} $$  \hspace{1cm} (6.40)

where $U_2(x) = (\sigma(2) + i\gamma_5 \tilde{\sigma})/f_x$ and the SU(2) sigma field $\sigma(2)$ is defined according to $\sigma(2) = \sigma_0 \sqrt{\frac{2}{3}} + \frac{1}{\sqrt{3}} \sigma_8$. Then the $\sigma_0$ and $\sigma_8$ fields themselves are constrained by $\sigma_0 \sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}} \sigma_8 = f_x$, as long as there is no flavour mixing interaction present. This gives the 1 in the lower right corner of $U(x)$. However this constraint is not really necessary and recently solitons were found (Kato et al. 1993), where it is released. In order to define quantization rules, we again introduce the time dependence by writing

$$ U(x, t) = A(t)U(x)A^\dagger(t) $$  \hspace{1cm} (6.41)

where $A(t)$ is now a time dependent matrix of SU(3). The theory remains invariant if one performs on $A(t)$ the symmetry operations

$$ A(t) \to A(t)G_{\text{spat}}, \quad G_{\text{spat}} \in SU(2) $$  \hspace{1cm} (6.42)

and

$$ A(t) \to A(t)G_{\text{flav}}, \quad G_{\text{flav}} \in SU(3). $$  \hspace{1cm} (6.43)

The right multiplication of $A(t)$ can be identified as spatial rotation because any isospin rotation of the chiral field can be undone by a suitable space rotation. This is due to the symmetric hedgehog ansatz. Note that because $U(x)$ commutes with the hypercharge group $U_Y(1)$, with generators $\exp i\gamma_8 \lambda_8$, the configuration space for the generalized coordinates of the SU(3) matrix $A(t)$ is restricted to $SU(2)_{\text{I}} \oplus U_Y(1)$ (Balachandran et al. 1985). However, following the work of Balachandran et al. (1985), Chemtob (1985) and Praszalowicz (1985), one can regard the symmetry of the collective coordinates as SU(3) and treat the freedom for right multiplication $U_R(t)$ as a constraint for the states.

Then one can perform the time dependent rotation of the effective action as described in sect. 4.2 and obtain

$$ S_{\text{eff}}^{\gamma_5} = -\text{Splog} \left( \partial_x + H + A^\dagger(t)A(t) - i\gamma_5 A^\dagger(t)mA(t) \right) $$  \hspace{1cm} (6.44)
with

\[ H = -i \gamma_4 \left( i \partial_4 \gamma_i^i - mU(x) \right) \]  

(6.45)

Because of the antihermiticity of \( A^\dagger(t) \dot{A}(t) \), any expansion of \( S_{eff} \) in odd powers of this quantity constitutes a contribution of the imaginary part of the effective Euclidean action and is therefore a finite quantity. As in the case of the baryon number in sect. 3.2, it is reasonable not to regularize these quantities. This is because they are connected with topological indices, which otherwise are not exact integers. We will see this, when we consider e.g. the right hypercharge in this model.

To this aim, we write the Maurer Cartan form \( A^\dagger(t) \dot{A}(t) \) as

\[ A^\dagger(t) \dot{A}(t) = q_a A^\dagger \partial_a A = i \frac{1}{2} \dot{q}_a C_{\alpha}^A \lambda_A = i \frac{1}{2} \Omega_A \lambda_A \]  

(6.46)

where the \( q_a \) are the coordinates of SU(3) and the \( C_{\alpha}^A \) are the vielbeins, which fulfill

\[ C_{\alpha}^A C_{\beta}^A = \delta_{\alpha}^\beta, \quad \left( C_{\alpha}^A \right)^{-1} = C_{\alpha}^A \]  

(6.47)

and the important Maurer Cartan identity, following from the definition of the structure constant \( f_{ABC} \) of the \( \lambda \)-matrices,

\[ C_{\alpha}^B \partial_\alpha C_{B}^A = C_{\alpha}^A \partial_\alpha C_{A}^B = -f_{BAE} C_{E}^C \]  

(6.48)

The effective Lagrangian after this SU(3) rotation is given by (in the chiral limit):

\[ L_{eff} = \frac{1}{2} \Omega_A I_{AB} \Omega_B - \frac{N_c}{2 \sqrt{3}} B(U) \Omega_8 = \frac{1}{2} q_a g_{a\beta} \dot{q}_\beta + Z^a \dot{q}_a \]  

(6.49)

where \( B(U) \) is the baryon number of the system with chiral field \( U(x) \) and the metric

\[ g_{a\beta} = C_{\alpha}^A I_{AB} C_{B}^\beta, \quad Z^a = -\frac{N_c}{2 \sqrt{3}} C_{a8} \]  

(6.50)

Note that in contrast to SU(2), there is now a term linear in the angular frequency \( \Omega_8 \). This is due to the imaginary part of the effective Euclidean action, which is vanishing for SU(2). As mentioned above such a term poses a problem for the quantization, because the corresponding generators are constraint in this case (see e.g. Balachandran et al. 1985 for a discussion of this).

Note that this Lagrangian also describes a particle moving in a monopole gauge field (Jackiw 1983), given by the gauge-variant \( Z^a \). As a result, it leads to a quantization condition for electric and magnetic charge. In the present model it is the Wess-Zumino term, which is the leading order term of the gradient expanded baryon current \( B(U) \) (2.38), that plays the role of the monopole. The consequence of this gauge-variant term in the present quark model is a quantization condition for the number of colors (see App. D for details).

But let us proceed step by step. From eq. (6.49) we can define canonical momenta by

\[ \pi_a = \frac{\partial L}{\partial \dot{q}_a} \]  

(6.51)
and right generators by the symmetrized form (Tayota 1987)

\[ R_a = -\frac{1}{2} \{ \pi_a, C^\alpha_a \} . \] (6.52)

In the same way one can define left generators by using the form \( A(t)A^1(t) \) with some vielbeins \( E^\alpha_a \) and

\[ L_a = -\frac{1}{2} \{ \pi_a, E^\alpha_a \} . \] (6.53)

From the Maurer Cartan identity and imposing the canonical quantization by

\[ [\pi_\alpha, q^\beta] = -i\delta^\beta_\alpha, \quad [\pi_\alpha, \pi_\beta] = [q_\alpha, q_\beta] = 0 \] (6.54)

one can derive

\[ [R_a, R_b] = -if_{abc} R_c, \quad [L_a, L_b] = if_{abc} L_c \] (6.55)

and the action on \( SU(3) \) transformation matrices as

\[ [R_a, A] = -A \frac{1}{2} \lambda_a, \quad [L_a, A] = -L \frac{1}{2} \lambda_a A \] (6.56)

Now we see from eq. (6.55) that the left generators \( L_a \) obviously obey correct commutation relations for \( SU(3) \) and can be identified with left flavour rotations, because of the remarks at the beginning of this paragraph and eq. (6.56). The generators \( R_a \) can be related to some right transformations according to eq. (6.56) and act on the space of spatial rotations. Because the eighth component is connected with the baryon number, the space is also called the spin-baryon number space.

We set the baryon wave functions according to Salam and Strathdee (1982) as

\[ \Psi(A) = \eta \sqrt{n} \langle i | D^{(n)}(A) | j > \] (6.57)

where \( n \) is the dimension of the representation and \( \eta \) is a suitable phase factor. The basis states \( \langle i | = \langle I, I_3, Y | \) and \( | j > = | J, J_3, Y_R > \) carry the flavour and spin quantum numbers. The Wigner function \( D^{(n)}(A) \) is a matrix of the adjoint representation of \( SU(3) \) and is given by

\[ D^{(n)}_{AB}(A) = \frac{1}{2} \text{tr} A^1 A \lambda_A \lambda_B \] (6.58)

Using the commutation relations eq. (6.56) one obtains the action of the generators on the wave function

\[ [R_C, D_{AB}] = i f_{CDB} D_{AD}, \quad [L_C, D_{AB}] = i f_{CAD} D_{DB} \] (6.59)

From these one can deduce the action of the generators \( R_A, L_A \) on the wave functions. Furthermore, one can identify left indices of \( D_{AB} \) as flavour quantum numbers, chosen as hypercharge \( Y \) and isospin \( I, I_3 \). The right indices therefore correspond to the right hypercharge \( Y_R \) and spin \( J, J_3 \).
Now we come back to our rotated Lagrangian \( L_M^\text{rot} \). Using the properties of the hedgehog ansatz, we find

\[
\begin{cases} 
I_{AB} = \frac{1}{2} \delta_{AB} & \text{for } A, B = 1, 2, 3 \\
I_{2}\delta_{AB} & \text{for } A, B = 4, 5, 6, 7 \\
0 & \text{for } A, B = 8 
\end{cases}
\]  

(6.60)

where \( I_{AB} \) is given by

\[
I_{AB} = -\frac{N_c}{4} \int \frac{d\omega}{2\pi} \text{tr} \left[ \frac{1}{i\omega + H} \lambda^A \frac{1}{i\omega + H} \lambda^B \right].
\]  

(6.61)

and the separation into valence and sea-part is analogue to Sect. 4.2. The right generators get explicitly

\[
R_A = -\frac{\partial L_M}{\partial \Omega_A} = \begin{cases} 
-I_1 \Omega_A, & A = 1, 2, 3 \\
-I_2 \Omega_A, & A = 4, \ldots, 7 \\
\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{2} Y_R & A = 8.
\end{cases}
\]  

(6.62)

so that we can go from \( L \) to the hamiltonian \( H \) by

\[
H = -\sum_A R_A \Omega_A - L
\]  

(6.63)

and obtain

\[
H^{\text{sym cell}} = M_{cl} + \frac{1}{2 I_2} \sum_{A=1}^7 R_A^2 + \left( \frac{1}{2 I_1} - \frac{1}{2 I_2} \right) \sum_{A=1}^3 R_A^2
\]  

\[
= M_{cl} + \frac{1}{2 I_2} C_2(SU(3)_{R/L}) + \left( \frac{1}{2 I_1} - \frac{1}{2 I_2} \right) C_1(SU(2)_R)
\]  

(6.64)

where the \( C_2(SU(3)_{R/L}) \) is the Casimir operator of SU(3) and \( C_1(SU(2)_R) \) is the corresponding one for the right SU(2). The eigenstates of these operators are the irreducible representations, which are usually labelled by the quantum numbers \( p \) and \( q \), according to the rank 2 of SU(3). From the construction of states, which was done e.g. by de Swart, it is clear that \( Y_R = 1 \) restricts the possible representations to those, which contain a basis state with \( Y = 1 \). Therefore only triality zero states survive, for which \((p - q) \mod 3 \equiv 0\). As lowest possible one these are the octet \( \{8\} = \{p = 1, q = 1\} \) and the decuplet \( \{10\} = \{p = 3, q = 0\} \). With formula eq. (6.64) at hand, we can determine the masses for the center of the octet and decuplet states. In order to remove the degeneracy of the states, within the multiplet, we have to switch on a finite symmetry breaking via a finite current quark mass. In this way, we obtain

\[
L_M^{\text{mu}} = -\frac{2}{3} \frac{m_u}{m_u + m_d} \Sigma \left( 1 - D_{8S}^{(8)} \right) - \frac{2m_s}{\sqrt{3}} K_{AB} D_{8A}^{(8)} \Omega_B
\]  

(6.65)

where \( \Sigma \) is the SU(2) sigma commutator, defined in sect. 2.1 and \( K_{AB} \) are the so called anomalous moments of inertia, given by (compare with Park and Rho, 1988, in chiral bag models)

\[
K_{AB} = -\frac{N_c}{4} \int \frac{d\omega}{2\pi} \text{tr} \left[ \frac{1}{i\omega + H} \lambda^A \frac{1}{i\omega + H} \gamma^4 \lambda^B \right].
\]  

(6.66)
and obey a similar structure like the $I_{AB}$:

$$K_{AB} = \begin{cases} K_1 \delta_{AB} & \text{for } A, B = 1, 2, 3 \\ K_2 \delta_{AB} & \text{for } A, B = 4, 5, 6, 7 \\ 0 & \text{for } A, B = 8 \end{cases} \quad (6.67)$$

Because they originate from the imaginary part of the effective Euclidean action, they need no regularization. The quantization condition changes to

$$R_A = -\frac{\partial L_M}{\partial \Omega_A} = \begin{cases} -(I_1 \Omega_A - \frac{2m_s}{\sqrt{3}} K_1 D_{8A}), & A = 1, 2, 3 \\ -(I_2 \Omega_A - \frac{2m_s}{\sqrt{3}} K_2 D_{8A}), & A = 4, \ldots, 7 \\ \frac{1}{3} \sqrt{3}, & A = 8 \end{cases} \quad (6.68)$$

so that one obtains for the symmetry breaking part of the collective hamiltonian:

$$H_{\text{coll}}^{sb} = -m_s \frac{K_2}{I_2} Y - \frac{2m_s}{\sqrt{3}} \left( \frac{K_1}{I_1} - \frac{K_2}{I_2} \right) \sum_{A=1}^{3} D_{8A}^{(8)}(A) R_A + \frac{2}{3} \frac{m_s}{m_u + m_d} \sum \left( 1 - D_{88}^{(8)}(A) \right) + \frac{N_c m_s}{3} \frac{K_2}{I_2} D_{88}^{(8)}(A). \quad (6.69)$$

where we have used the relation $\sum_A D_{8A} R_A = L_8 = \frac{\sqrt{3}}{2} Y$. The different values for the moments of inertia $I_1$, $I_2$, $K_1$ and $K_2$ from the selfconsistent solitonic solutions can be found in Tab. 6.1. for two values of the constituent quark mass $M$.

Now we are in the position to evaluate the mass splittings within the hyperon multiplets. This will be done in the next section by evaluating the expectation value of the collective hamiltonian in a given baryonic state.

6.4. Static properties of SU(3) Baryons

After writing down the collective hamiltonian, we can evaluate it in different ways. First we can set baryon wave functions as described in eq. (6.57), which are eigenfunctions of the symmetric hamiltonian $H_{\text{symm}}$. All we have to do is then to sandwich the D-functions of the symmetry breaking part of the collective hamiltonian $H_{\text{coll}}^{sb}$ between baryon wavefunctions (Adkins et al. 1983). On the other hand, one can parametrize the right generators by differential operators of the Euler angles of the given representation (Yabu and Ando 1988, Park et al. 1991, Park and Weigel 1992). Then it is possible to evaluate the hamiltonian between the exact wave functions, which can be perturbatively expressed as a series in the symmetry breaking quark mass $m_s$. After we have clarified the different procedures, we will comment on the numerical evidence of the exact treatment.

The Perturbative Treatment of the Collective Hamiltonian

Having collected the leading terms in view of a $1/N_c$ expansion for the collective hamiltonian, we have to sandwich this operator between the wave-functions of the baryons. Because of the symmetry breaking terms in $H_{\text{coll}}^{sb}$, the functions $D_{AB}^{(n)}(A)$ in octet and decuplet representation are
not exact eigenstates of the full Hamiltonian. In principle, one has to write down a perturbative expansion

$$| B > = | B, R > + \sum_{N \neq R} \frac{< B, N | H_{coll}^{b} | B, R >}{E_{R}^{b} - E_{N}^{b}} | B, N > + \ldots$$  \hspace{1cm} (6.70)$$

where $R$ is the lowest representation in which the baryon $B$ can be found and the summation goes over all higher dimensional representations $N$, which have non-vanishing overlap with the ground state in $R$. So it is clear that the proton, e.g., is a sum of a proton in an octet representation and corresponding states in $[10]$ and $[27]$. Furthermore these states enters in eq. (6.70) linearly with the symmetry breaking parameter $m_s$. Because only the symmetry breaking terms in the collective Hamiltonian have a non-vanishing overlap with these higher representations, they appear in the mass for the first time in the second order of $m_s$. So one can define a perturbative treatment of $m_s$ in first order, in which the first order $m_s$-correction of the wave-function (6.70) do not appear.

The effects of these terms will be discussed in the next section.

In order to evaluate $H_{coll}^{b}$ in the baryon states $| B, R >$, we need the integral over 3D-functions. These integrals are well known (Blotz et al. 1993b) and are summarized in Tab. 6.2 (cf. de Swart, 1963, for a general discussion). As a result, we can express the mass splittings in terms of the 2 quantities

$$\Delta = \frac{2}{3} \frac{m_s}{m_u + m_d} \Sigma + m_s \left( \frac{2}{I_2} - \frac{3}{I_1} \right)$$  \hspace{1cm} (6.71)$$

and

$$\delta = m_s \frac{K_1}{I_1}$$  \hspace{1cm} (6.72)$$

This is little bit astonishing, since we have three operators $Y, D_{88}$ and $\sum_{A=1,2,3} D_{8A} R_A$. However due to a non-trivial group theoretical property (Blotz et al. 1993b), the two quantities $\Delta$ and $\delta$ are sufficient. Then we obtain for the splitting between the center of the multiplets (experimentally $\approx 230$ MeV):

$$\Delta_{8-10} = \frac{3}{2 I_1}$$  \hspace{1cm} (6.73)$$

which is solely given by the difference of the Casimir operators $C_2(SU(3)_{R,L})$ and $C_2(SU(2)_{R})$ in octet and decuplet representation. One should note that this formula coincides with $N - \Delta$ splitting in $SU(2)$ theory. This means that the same expression has to be considered as a different physical quantity depending on the use of the $SU(2)$ or $SU(3)$ collective quantization. Then we can write for the masses of the strange baryons relative to the mass of the $\Sigma^*$, which we fix for the moment to the experimental value, as

$$\Delta m_N = - \frac{3}{10} \Delta - \delta - \Delta_{8-10}$$  \hspace{1cm} (6.74a)$$

[4]Semiclassical quantization is known to yield always values for the masses which are by several hundred MeV too high. Remedies to this by means of band-head and centre-of-mass corrections will be used below.
The reason that we fixed the $\Sigma^*$ to the experimental value is that the quantization always yields too large values for the masses if one does not employ band head and center of mass corrections as discussed below.

From the formulas eqs. (6.74) above follow the mass relation of Guadagnini (1984)

$$m_{\Xi^*} - m_{\Sigma^*} + m_N = \frac{1}{8}(11m_A - 3m_{\Sigma^*}).$$

(6.75)

It is interesting to note that it is obtained in the pseudoscalar Skyrme model only by introducing the hypercharge operator and several coefficients (Guadagnini 1984) by hand. In our approach this term arise naturally in the theory and the coefficients in front of all of them are completely determined by the selfconsistent soliton solution.

Furthermore one obtains the Gell-Mann Okubo relations (Gell-Mann 1962, Okubo 1962)

$$2(m_N + m_{\Xi^*}) = 3m_A + m_{\Sigma^*}$$

(6.76)

and

$$m_\Omega - m_{\Xi^*} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Sigma^*} - m_\Delta$$

(6.77)

automatically in this approach by means of eq. (6.74). One should note that these relations are mass sum rules which rely basically on the fact, that the SU(3) flavour symmetry breaking part of the strong interaction can be treated in 1st order perturbation theory (Cheng and Li 1984). Their validity shows therefore the consistency of the whole quantization procedure. On the other hand the values of the hyperon masses themselves depend on the dynamics of the model which is considered. In Tab. 6.1 we have collected the difference of the theoretical mass from the experimental one for two values of the constituent quark mass $M$. As one can see from the table for the value $m_s = 150\ MeV$, which correspond to eq. (6.39) for $m_0 = 6.1\ MeV$, the splitting prediction is better than 50MeV, except for the $\Omega$, which drops out with 80MeV. However it is instructive to increase $m_s$ to a value $m_s = 200\ MeV$, i.e. this corresponds to enlarge $m_K$ to $\simeq 570\ MeV$ (exp. $m_K = 496\ MeV$), which gives a very good agreement with experiment up to 20MeV for all baryons under consideration.

Comment on Validity of Chiral Expansion
In Sect. 6.1, following (6.39), we mentioned the failure of chiral perturbation theory for some vacuum parameters. Now we are in a position to answer this question for the baryonic sector and it will be done here for the case of the total masses as an example. Other observables are considered in the literature (Blotz et al. 1993a, 1993d). These two sectors need not to have the same behaviour, because the vacuum sector is dominated by different integral equations compared to the soliton. Actually we can write the classical soliton mass in perturbation theory for \( m_s \) as

\[
M_{cl} = M_0 + m_s M_1 + m_s^2 M_2 + O(m_s^3) \tag{6.78}
\]

with the result within the present model:

\[
M_{cl} = M_0 \left( 1 + \frac{m_s}{4M_0} \Sigma - \frac{2}{9} \frac{m_s^2 N_0}{M_0} \right) = 1250 + 426.9 - 26.0 \text{MeV} \tag{6.79}
\]

for the preferred values of \( M = 418 \text{MeV}, \Sigma = 56 \text{MeV}, N_0 = 0.668 \text{fm} \) and \( m_s = 186 \text{MeV} \). So one can say that at least for the action the validity of the chiral expansion seems to be more reliable for the baryon sector than for the vacuum (Hatsuda 1990).

The Yabu-Ando Diagonalization Method

In contrast to the former method of evaluating the collective hamiltonian between the symmetric eigenfunctions of the Casimir operators, Yabu and Ando (1988) developed a method for treating the baryonic wave functions to all orders in \( m_s \). The idea is to express the rotation matrix \( A \) by its eight Euler angles and derive also for the right generators \( R_A \) an explicit differential operator form in terms of these Euler angles.

If we adopt the same parametrization of the rotation matrix \( A \) as discussed by Yabu and Ando (1988), we can write

\[
A = R(\alpha, \beta, \gamma) \exp(-i\nu \lambda_4) R(\alpha', \beta', \gamma') \exp(-i\nu \lambda_4/\sqrt{3}) \tag{6.80}
\]

where the \( R(\alpha, \beta, \gamma) \) is the Euler-angle rotation matrices of SU(2)-isospin. Because of the hedgehog ansatz for the chiral field, isospin rotations and space rotations are intimately connected in a way, that any isospin rotation can be undone by a corresponding rotation in coordinate space. Therefore the \( R(\alpha', \beta', \gamma') \) is the SU(2) spin rotation matrix. Now we would like to obtain an expression for the right generators \( R_A \) in terms of these eight Euler angles, conveniently written as \( \alpha_a \). Following the work of Nelson (1967) and Park et al. (1991), we make the ansatz

\[
R_a = i d_{ab}(\alpha) \frac{\partial}{\partial \alpha_b}, \tag{6.81}
\]

such that \( R_a \) are a linear differential operator of the \( \alpha_a \). The unknown matrix \( d_{ab} \) can be determined by inserting this ansatz into the definition of the \( R_a \)

\[
AR_b A^\dagger = A \lambda_b A^\dagger \tag{6.82}
\]

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Now both sides of (6.82) can be explicitly worked out and give with eqs. (6.80, 6.81)

\[ \lambda_a E_{ac}(\alpha) d_{cb}(\alpha) = C_{ab}(\alpha) \lambda_b \]

with the some known matrices \( E_{ac}(\alpha) \) and \( C_{ab}(\alpha) \). Comparison yields \( d_{ab}(\alpha) = E_{ac}^{-1}(\alpha) C_{cb}(\alpha) \). Their rather lengthy forms are summarized by Park and Weigel (1992). Then one can directly act with these operators on the wave functions, which are themselves D-functions and therefore also given in terms of \( A \).

The result of this method can be seen in Fig. 6.1, where the deviation of the theoretical prediction from the experimental one is shown in dependence of the strange current quark mass. Although \( m_s \) is fixed to be \( \simeq 150 \text{MeV} \) from the mesonic sector, it is instructive to see that although the predictions for \( m_s = 150 \text{MeV} \) are already good, a slight increase to \( m_s = 187 \text{MeV} \) leads to an almost perfect agreement (see also Tab. 6.3). The difference to the former perturbative treatment is in addition illustrated in Fig. 6.2, where the Yabu-Ando method is compared to the perturbative treatment for two particles of the multiplets. As it is clear from these curves, the Yabu-Ando method gives a significant contribution to the masses for current masses above 100 MeV. Similar calculations have been performed by Weigel et al. (1992b). The formalism of these authors differs from the present one in the fact that they do not follow strictly a perturbative expansion in \( \frac{1}{N_c} \) and \( m_s \). Hence they also have different constituent quark masses for the strange and non-strange quarks. The resulting numbers are quite close to those presented here.

Furthermore Praszalowicz et al. (1993a) made a refined calculation in the lines of the preceding sections with the inclusion of isospin breaking current quark masses. As a result they found that one can make a prediction for the hadronic part of the isospin splitting within the multiplets, which agrees with the experimental data (Gasser and Leutwyler 1982) with a high accuracy. In this work it turned out that the anomalous moments of inertia play again a crucial role in order to predict the splittings such as the neutron proton mass difference, which could be explained neither in the former SU(2) pseudoscalar Skyrme model, where it vanishes, nor in a U(2) or SU(3) extension (Jain et al. 1989) of the model. In Fig. 6.3 the hadronic part of the isospin splitting within the octet is shown and compared with the experimental predictions from Gasser and Leutwyler (1982). In addition it could be shown by Praszalowicz et al. (1993) that the complete agreement between theory and experiment within the experimental error bars is achieved for that \( m_d - m_u \) difference, which follows from the mesonic sector from the masses of the charged kaons and pions. This can be viewed as a reflection of the fact, that perturbation theory works quite well for the small isospin breaking parts. In addition it was shown by Blotz et al. (1993e), that the recently measured Gottfried sum, which was considered in the present model for the case of SU(2) by Wakamatsu (1992b), comes out in the SU(3) version quite close to the value of the NMC measurements (NMC, Amaudruz et al. 1991). In addition the \( \Sigma \)-term comes out to be \( \Sigma \simeq 45 \text{MeV} \), which is again close to the experimental value of Gasser et al. (1991)

**Total Masses and Zero-Point Energy Corrections**

By now we have discussed the mass splittings within the baryon octet and decuplet and the splitting between the octet and decuplet. As mentioned already the absolute energies of particles
come out by several hundred MeV too high in the semiclassical approximation no matter which chiral model is used. Subtraction mechanism, as they are discussed by Polytitsa et al. (1992) in SU(2) and by Blotz et al. (1993b) in SU(3) (compare with Jain et al. 1988b for a discussion in the Skyrme model) can in principle cure the situation. In order to obtain reasonable values for the absolute masses of the hyperons one has to subtract the spurious zero mode energies similar to the way described in sect. 4.4.. The corresponding terms of the rotational zero modes in SU(3) and of the translational one read

\[ \Delta M^{\text{rot}} = \frac{1}{2I_2} \langle C_2(\text{SU}(3)) \rangle + \frac{1}{2} \left( \frac{1}{I_1} - \frac{1}{I_2} \right) \langle C_2(\text{SU}(2)) \rangle \]

\[ = \frac{1}{I_2} \frac{7}{8} + \frac{1}{I_1} \frac{9}{8} \]

\[ \Delta M^{\text{transl}} = \frac{<[B_i^2](1)>}{2M_{cl}} \]

(6.83)

where we used the values of the Casimir operators for the fundamental representation. These are \( \langle C_2(\text{SU}(3)) \rangle = 4N_c/3 \) and \( \langle C_2(\text{SU}(2)) \rangle = 3N_c/4 \). The translational zero mode subtraction coincide for SU(3) with the SU(2) result \( \Delta M^{\text{transl}} \) of (4.66). Subtracting these terms from the classical soliton mass \( M_{cl} \) gives for the center of the octet and decuplet

\[ M_{8} = M_{cl} + \frac{1}{I_2} \frac{3}{4} + \frac{1}{I_1} \frac{3}{8} - \Delta M^{\text{rot}} - \Delta M^{\text{transl}} \]

(6.84)

\[ M_{10} = M_{cl} + \frac{1}{I_2} \frac{3}{4} + \frac{1}{I_1} \frac{15}{8} - \Delta M^{\text{rot}} - \Delta M^{\text{transl}} \]

(6.85)

If one performs these corrections to the classical energy analogous to (4.65) in the present case of SU(3) (Blotz et al. 1993b) the absolute mass of the \( \Sigma^* \)-particle changes from 2262 MeV to 1494 MeV (experimentally \( m_{\Sigma^*} = 1385.5 \text{ MeV} \)). Thus with these corrections the masses of all the hyperons of the octet and decuplet are reproduced with a constant shift of \( \simeq 100 \text{ MeV} \), being in fact a quite impressive result.

### Expectation Value of Axial Currents

In a similar way to chap. 5 one can evaluate the expectation value of the SU(3) axial vector current operator. However, as it was mentioned in the SU(2) case already, there are important \( 1/N_c \) corrections to vector and axial vector currents. These are entirely due to the fact, that the collective operators \( \Omega \) and \( A_\lambda^+ \lambda^A \) in general do not commute and have to be explicitly time-ordered (Christov et al. 1993b, Blotz et al. 1993d). Without going into details we will nevertheless give here a shortcut derivation. From the path-integral one obtains for \( A^\xi_\mu(x), \alpha = 0, 3, 8; \)

\[ < A^\xi_\mu(x) > = \frac{\delta}{\delta s(x)} \left[ \text{Sp}_{(s)} \log (\delta_4 + H + i\Omega_E - i\gamma_4 A^+ \lambda^A mA + i s(x) \gamma_4 \gamma_5 A^+ \lambda^A) \right] \]

(6.86)

where \( \text{Sp}_{(s)} \) means now that the time dependent operators \( (A(t), \Omega) \) within the trace have to be time-ordered, before performing the trace in time direction. This however disagrees with the usual definition of the trace mapping and its origin is the fact, that we are not going to perform the
path-integral over the rotation matrices $A(t)$ in the functional sense (Blotz et al. 1993d, Christov et al. 1993b). Instead one usually switches to the operator formalism, as it was done throughout this work (cf. Dyakonov et al. 1988), which involves the evaluation of time-ordered products of operators. Because of this, the trace in eq. (6.86) has to be modified in a way, that respects an explicit time-ordering of the operators (cf. Blotz et al. 1993d). To obtain a c-number value for the current, finally one has to sandwich the operator $<\hat{A}_\mu^a(x)>$ between suitable baryon wave-functions.

In addition to the SU(2) case, eq. (6.86) contains the symmetry breaking term from the strange current quark mass. Therefore the expression for axial current up to the first order in the rotational matrices $i\Omega_E$ and the strange current quark mass $m$ is given by

$$<\hat{A}_\mu^a(x)> = \text{Sp(to)} \left\{ \frac{1}{\partial_4 + H} \gamma_\mu \gamma_5 A^\dagger A \right\} - \text{Sp(to)} \left\{ \frac{1}{\partial_4 + H} \gamma_\mu \gamma_5 A^\dagger A \right\} \frac{1}{\partial_4 + H} \delta H + O(\delta H^2)$$

(6.87)

where the perturbation $\delta H$ is now given by

$$\delta H = i\Omega_E - \gamma_4 A^+ mA$$

(6.88)

Evaluating this yields for the $a = 3, 8$ part of the axial vector coupling constant (Blotz et al. 1993d)

$$g_A^3 = M_3 D_{a3} + \frac{4 M_{A4} d_{3i4}}{I_2} d_{a4} R_6 -$$

$$\frac{2iQ_{12}}{I_1} D_{a3} - \frac{2iQ_{41}}{I_2} D_{a3}$$

$$+ \frac{2 M_{83}}{I_1} R_3 D_{a8}(1 + \frac{4m_s}{\sqrt{3}} \frac{K_1}{I_1} D_{83})$$

$$+ \frac{4 m_s}{\sqrt{3}} N_{83} D_{a3} D_{83} + \frac{8 m_s}{\sqrt{3}} \left( N_{A4} - M_{A4} \frac{K_2}{I_2} \right) d_{3i4} d_{a4} D_{86}$$

$$- \frac{4 m_s}{\sqrt{3}} N_{83} D_{a3}(1 - D_{88})$$

(6.89)

and for the singlet part (Blotz et al. 1993a)

$$g_A^0 = \frac{2\sqrt{3} M_{S3}}{I_1} R_3 - 4 m_s D_{S8} \left( \frac{K_1}{I_1} M_{S3} - N_{S3} \right)$$

(6.90)

There we used the following definitions for the various moments of inertia. For the anomalous moment $M_{bc}$, we find

$$M_{bc} = \frac{N}{4} \sum_{n,m} <n|\sigma_3 \lambda_b|m> <m|\lambda_c|n> R_M(E_n, E_m)$$

(6.91)

with

$$R_M(E_n, E_m) = \frac{1}{2} \frac{\text{sign}(E_n - \mu) - \text{sign}(E_m - \mu)}{E_n - E_m}$$

(6.92)
and where the chemical potential $\mu$ lies always between the valence level and positive continuum of states, in order to describe a baryon number $B = 1$ system. For the proper time regularized normal moments $N_{bc} = N_{bc,\text{val}} + N_{bc,\text{sea}}$ from the symmetry breaking, we find

$$N_{bc,\text{val}} = \frac{N_c}{2} \sum_n < n | \sigma_3 \lambda_b | n > \frac{E_n - E_v}{E_n - E_u}$$  \hspace{1cm} (6.93)$$

and

$$N_{bc,\text{sea}} = \frac{N_c}{4} \sum_{n,m} < n | \sigma_3 \lambda_b | m > < n | \lambda_c \gamma_0 | n > R_\beta(E_n, E_m)$$  \hspace{1cm} (6.94)$$

with

$$R_\beta(E_n, E_m) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{dt}{\sqrt{t}} \phi(t) \left[ \frac{E_n e^{-tE_n^2} - E_m e^{-tE_m^2}}{E_n - E_m} \right]$$  \hspace{1cm} (6.95)$$

and for the *antisymmetric* moments $Q_{bc}$ from the explicit time-reordering we define $Q_{bc} = Q_{bc,\text{val}} + Q_{bc,\text{sea}}$:

$$Q_{bc,\text{val}} = \frac{N_c}{2} \sum_n < m | \sigma_3 \lambda_b | n > < n | \lambda_c | m > \text{sign} E_n$$  \hspace{1cm} (6.96)$$

and

$$Q_{bc,\text{sea}} = \frac{N_c}{4} \sum_{n,m} < m | \sigma_3 \lambda_b | n > < n | \lambda_c | m > R_Q(E_n, E_m)$$  \hspace{1cm} (6.97)$$

In the case that the regularization function $\phi(t)$ is given by $\phi(t) = c_\mu \theta(1-1/\Lambda^2)$ the $R_Q(E_n, E_m)$ takes the simple form

$$R_Q(E_n, E_m) = c_\mu \int_0^1 \frac{d\alpha}{2\pi} \frac{\alpha(E_n + E_m) - E_m \exp \left(-[\alpha E_n^2 + (1-\alpha)E_m^2]/\Lambda^2\right)}{\sqrt{\alpha(1-\alpha)} \alpha E_n^2 + (1-\alpha)E_m^2}$$  \hspace{1cm} (6.98)$$

In the infinite cutoff limit it again reduces to

$$R_Q(E_n, E_m) = \frac{1}{2} \frac{\text{sign}(E_m - \mu) - \text{sign}(E_n - \mu)}{E_m - E_n}$$  \hspace{1cm} (6.99)$$

Note that the regularization function $R_Q(E_n, E_m)$ in eq. (6.98) is now antisymmetric with respect to the states $m$ and $n$ in contrast to $R_M(E_n, E_m)$ in eq. (6.92). This is a reflection of the fact that the matrix elements of eq. (6.97) are now antisymmetric with respect to $m$ and $n$.

**Numerical Results.**

Similar to the case of SU(2) in Sect. 5.2., the terms with the *antisymmetric* $Q_{bc}$ expressions also serve here as large corrections to the lowest order terms ($\Omega^0$) and pushes especially the value of $g_A^3$, which was 50% too low without these terms, a little bit beyond its experimental value (see Tab. 6.4). Together with the values of $g_A^3$ and $g_A^4$ which has the interpretation of being the spin of the proton that is carried by the quarks, the contributions from the lowest order $\Omega^0$ and from the symmetric and antisymmetric rotational corrections are presented in Tab. 6.4 and compared with the experimental numbers from recent EMC and SMC experiments. One should note that $g_A^0$ should be interpreted as the spin of the proton (normalized to unity), which is carried by the
quark. As it is clear from these numbers, the antisymmetric contributions from the time-reordering play a significant role. At the present stage of the work the NJL numbers are very good. The corresponding expressions emerge from the real part of the Euclidean effective action, though they turn out to be finite and need no regularization. Furthermore, and this is most important, they are not present in the usual Skyrme model approach. So this is in a sense another reflection of the explicit quark degrees of freedom, which obviously cannot be described perturbatively in the chiral fields and which provides therefore a clear distinction between Skyrme type models and Nambu-Jona-Lasinio type models.
7. The NJL-model with vector mesons

In this chapter we consider the influence of vector couplings on the NJL model. After giving some motivation, the bosonization and regularization of the extended model are presented. The vacuum and mesonic sectors are analyzed both assuming the mesons to be on-shell or off-shell. Finally a system with baryon number one is constructed and its solitonic solutions are analyzed.

7.1. The Effective NJL Action with Vector Mesons

Why Vector Mesons?

Since their discovery (Nambu 1957, Frazer and Fulco 1959, 1960), vector mesons have been intimately linked to the internal structure of the nucleon. In fact, the success of the vector dominance hypothesis (Sakurai 1960; Gell-Mann and Zachariasen 1961) and its later field theoretical realization through current-field identities (Kroll et al. 1967) in many hadronic reactions is obvious (for a review see e.g. Gourdin 1974). In addition, vector mesons play a crucial role in low energy nuclear physics, since they are responsible for the medium and short range NN interaction (Machleidt et al. 1987). It seems natural to ask how vector mesons may be implemented in a NJL type model.

Up to now, we have referred to the NJL model with scalar-isoscalar and pseudoscalar-isovector couplings (σπ-version of NJL). The introduction of other couplings makes possible to describe a wider meson spectroscopy, since it is known that the σπ version of the model does not account for vector or axial vector degrees of freedom, i.e. the corresponding correlation functions do not possess poles.

If additional vector couplings are included, the NJL-model indeed incorporates in a natural way important phenomenological principles found long before the advent of QCD: Sakurai’s universality (Sakurai 1966, 1969) and vector meson dominance realized by means of current-field identities (Kroll et a. 1967). It also provides a relationship to effective low energy Lagrangians with vector mesons (Kleinert 1978, Dhar et al. 1984, 1985; Ebert and Reinhardt 1986; Wakamatsu and Weise 1988) both in massive Yang-Mills (Lee and Nieh 1967; Gasiorowicz and Geffen 1969; Meissner UG 1988) or in hidden symmetry (Bando et al. 1984, 1988) representation. Furthermore, it includes the gauged Wess-Zumino term (Witten 1983) with the vector mesons interpreted as internal gauge fields (Dhar et al. 1984, 1985; Ebert and Reinhardt 1986; Wakamatsu 1989). In addition, almost all attempts claiming any parentage of the NJL model to QCD require the explicit inclusion of vector mesons (Dhar et al. 1984, 1985; Cahill and Roberts 1985; Schaden et al. 1990; Chanfray et al. 1991). An exception to this is the instanton liquid model (Diakonov and Petrov 1986).

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The extended $SU(2)_R \otimes SU(2)_L \otimes U(1)_B$ classically invariant NJL-model including scalar, pseudoscalar, vector and axial-vector couplings reads
\begin{equation}
\mathcal{L}_{NJL} = \bar{q}(i\gamma^\mu - m_0)q + \frac{G_1}{2}[(\bar{q}q)^3 + (\bar{q}\gamma_5 q)^3]
- \frac{G_2}{2}[(\bar{q}\gamma^\mu q)^3 + (\bar{q}\gamma_5 \gamma^\mu q)^3] - \frac{G_3}{2}(\bar{q}\gamma^\mu q)^2
\end{equation}

It is important to mention that the terms multiplying the coupling constants $G_1$, $G_2$ and $G_3$ are themselves chirally invariant. Thus at the classical level the conservation laws given in chapter 2 remain valid. Furthermore, since the coupling constants have dimensions of inverse mass squared it is convenient to choose them as follows
\begin{equation}
G_1 = \frac{g_\pi^2}{\mu^2}, \quad G_2 = \frac{g_\rho^2}{m_\rho^2}, \quad G_3 = \frac{g_\omega^2}{m_\omega^2}
\end{equation}

where $g_\pi$, $g_\rho$ and $g_\omega$ are dimensionless coupling constants and $m_\rho$ and $m_\omega$ are the vector meson masses. The parameter $\mu$ has mass dimension and it will turn out to be identical to the $\mu$ introduced in chap. 2.

Similarly to the $\sigma\pi$ case the corresponding generating functional is made well defined by performing a Wick rotation. In order to keep track of Lorentz invariance the time components of the vector fields are rotated as well (cf. app. A)
\begin{equation}
\begin{align*}
x^0 &= ix^4, \quad \omega^0(x_0, \vec{x}) \rightarrow i\omega^4(x_4, \vec{x}), \quad \vec{\omega}(x_0, \vec{x}) \rightarrow \vec{\omega}(x_4, \vec{x})
\end{align*}
\end{equation}

Following the steps of chapter 2 we multiply in addition by the Gaussian factor
\begin{equation}
\int D\omega_D\bar{p}D\bar{a}\exp\left\{-\frac{1}{2}\int dx \left[m_\omega^2 \omega^2 + m_\rho^2 (\bar{p}^2 + \bar{a}^2)\right]\right\}
\end{equation}

An important point is that the Wick rotation makes the exponent in the Gaussian factor to have a well defined negative sign. Similarly to chapter 2 Euclidean indices will be understood unless otherwise stated. The Gaussian factor is chirally symmetric if the fields are assumed to transform as
\begin{equation}
\begin{align*}
\omega &\rightarrow \omega, \\
\bar{p}_\mu &\rightarrow \bar{p}_\mu + \vec{\sigma} \times \bar{p}_\mu + \vec{\beta} \times \bar{a}_\mu, \\
\bar{a}_\mu &\rightarrow \bar{a}_\mu + \vec{\sigma} \times \bar{a}_\mu + \vec{\beta} \times \bar{p}_\mu
\end{align*}
\end{equation}

under the global chiral $SU(2)_R \otimes SU(2)_L$ group. Performing the shifts
\begin{equation}
\begin{align*}
\omega_{\mu} &\rightarrow \omega_{\mu} - \frac{g_\omega}{m_\omega^2} \bar{q}\gamma_5 q \\
\bar{p}_\mu &\rightarrow \bar{p}_\mu - \frac{g_\rho}{m_\rho^2} \bar{q}\gamma_5 q \frac{2}{2} q \\
\bar{a}_\mu &\rightarrow \bar{a}_\mu - \frac{g_\omega}{m_\omega^2} \bar{q}\gamma_5 q \frac{2}{2} q
\end{align*}
\end{equation}
one gets the semibosonized Lagrangian
\[
\mathcal{L}^{eff}(x) = \bar{q} \left(-i\gamma_\mu \left(\sigma + i\gamma_5 \vec{\pi}\right) + g_\rho \left(\vec{p} + i\gamma_5 \vec{\gamma}_5\right) \frac{\tau_2}{2} - g_\omega \vec{\psi} + \vec{m}_0\right) q + \frac{\mu^2}{2} \left(\sigma^2 + \vec{\pi}^2\right) + \frac{m_\rho^2}{2} \left(p_\mu^2 + \vec{d}_\mu^2\right) + \frac{m_\omega^2}{2} \omega_\mu^2.
\]
(7.7)

At the classical level, the quark part of this Lagrangian is formally invariant under local chiral transformations. We will see later that this local symmetry is broken by the regularization thus leading to a chiral anomaly. After integration of the quarks the extended action reads
\[
S_{eff}[\sigma, \pi, \omega, \rho, a] = -\text{Splog}\left(-i\gamma_\mu \left(\sigma + i\gamma_5 \vec{\pi}\right) + g_\rho \left(\vec{p} + i\gamma_5 \vec{\gamma}_5\right) \frac{\tau_2}{2} - g_\omega \vec{\psi} + \vec{m}_0\right) + \frac{\mu^2}{2} \int d^4x \left(\sigma^2 + \vec{\pi}^2\right) + \frac{m_\rho^2}{2} \int d^4x \left(p_\mu^2 + \vec{d}_\mu^2\right) + \frac{m_\omega^2}{2} \int d^4x \omega_\mu^2.
\]
(7.8)

Again we treat the theory in the stationary phase approximation. The corresponding classical equations of motion give the current field identities:
\[
J_B^\mu = \frac{m_\rho^2}{g_\rho} \omega_\mu, \quad J_V^\mu = \frac{m_\omega^2}{g_\omega} \vec{d}_\mu, \quad J_A^\mu = \frac{m_\omega^2}{g_\rho} d_\mu
\]
(7.9)

with \( J_B^\mu \), \( J_V^\mu \) and \( J_A^\mu \) being the baryon-, vector- and axial vector currents
\[
J_B^\mu = \bar{q} \gamma_\mu \frac{1}{2} \vec{q}, \quad J_V^\mu = \bar{q} \gamma_\mu \gamma_5 \frac{1}{2} \vec{q}, \quad J_A^\mu = \bar{q} \gamma_\mu \gamma_5 \frac{1}{2} \vec{q}, \quad J_B^\mu = \bar{q} \gamma_\mu \gamma_5 \frac{1}{2} \vec{q}
\]
(7.10)

The parameters \( m_\rho, g_\rho, m_\omega, g_\omega \) will be considered later. Due to the Wick rotation the effective action is a complex number which can be separated into real and imaginary parts
\[
S = \text{Re}S + i\text{Im}S, \quad \text{Re}S = \frac{1}{2} \left(S + S^\dagger\right), \quad \text{Im}S = \frac{1}{2i} \left(S - S^\dagger\right)
\]
(7.11)

On the basis of the combined symmetry operation \( Q = G\gamma_5 \) with \( G \) the usual G-parity (charge conjugation plus rotation of 180 degrees around the y isospin axis) it has been shown (Doering et al. 1992) that the real part of the action is an even function of the \( \omega \) field whereas the imaginary part is an odd function. In particular, if the \( \omega \) field vanishes the Euclidean action becomes a real number. Notice that this argument does not apply anymore if one considers SU(3) flavour.

Regularization of the Effective Action - Chiral Anomaly

As we have said the quark contribution to the semibosonized Lagrangian is formally invariant under chiral local transformations. On the other hand, a regularization has to be introduced to make the effective action finite. However, there is no regularization which preserves both vector and axial gauge symmetries simultaneously. Thus there appears a chiral anomaly (Adler 1969; Bell and Jackiw 1969). The subject of chiral anomalies is rather involved and we refer the reader to the recent review of Ball (1989) for a detailed discussion. (For a more elementary level see e.g. Petersen 1985). We just mention here that there is some mathematical ambiguity as to which
symmetry should be conserved and which should be destroyed. This freedom depends on the particular physical application. The fact that QCD is the fundamental theory underlying the NJL model suggests shifting the anomaly to the pure axial sector, i.e. to make use of a vector gauge invariant regularization.

Dhar et al. (1984, 1985) have proven that the proper prescription to achieve vector current conservation is to regularize the real part and not to regularize the imaginary part, so that the effective action becomes

\[ S_{\text{eff}} = \frac{1}{2} \int_{0}^{\infty} d\tau \phi(\tau, \Lambda) S_{\text{e}} - D_{I}^{J} + \frac{1}{2} \left( \text{Splog}(iD) - \text{Splog}(-iD^{I}) \right) \]  \( \text{mass terms} \) (7.12)

Then the real part of the one loop contribution is both vector and axial gauge invariant while the imaginary part exhibits an axial anomaly. This prescription leads to the original Bardeen's form of the chiral anomaly (Bardeen 1969), however with the vector and axial fields interpreted as dynamical degrees of freedom.

In the case of the NJL model with dynamical vector mesons one gets the following anomalous Ward identities (Wess and Zumino 1971, Wess 1972), which are generalizations of the Noether theorem in quantum field theory, in Minkowski space

\[ \partial_{\mu} j_{\mu}^{B} = 0 \]

\[ \partial_{\mu} j_{\mu}^{V} = 0 \]

\[ \partial_{\mu} j_{\mu}^{A} = 2i m_{0} \Phi \frac{\pi}{2} + \frac{N_{c}}{4\pi} \tau g_{\omega g_{\mu}\rho_{\alpha} g_{\rho_{\beta}}} \partial_{\mu} \omega^{\rho} \left( \frac{1}{2} \delta^{\rho}_{\beta} \rho^{\beta} + g_{\rho} \rho^{\rho} \times \rho^{\beta} - \rho^{\rho} \times \rho^{\beta} \right) \]  \( \text{(7.13)} \)

As we see, the anomalous contribution to the divergence of the axial current survives, even in the chiral limit \( m_{0} = 0 \) and in the absence of external fields. This is called an internal anomaly which does not even disappear if vector mesons are integrated out. In fact it has been proven by Wakamatsu (1989), Ruiz-Arriola and Salcedo (1993a), that this anomaly breaks some low energy theorems of QCD.

7.2. Fixing of the Parameters in the Vector Mesonic Sector

As it has been done in previous cases, the parameters have to be fixed by looking at different mesonic properties. As input parameters we use the pion weak decay constant \( f_{\pi} = 93 MeV \) and the meson masses \( m_{\pi} = 139 MeV, m_{\rho} = 770 MeV \) and \( m_{\omega} = 783 MeV \).

The main new point in the meson sector is the occurrence of a mixing between the pion and the axial meson very similar to the one found in the early massive Yang-Mills approach (Lee and Nieh 1967). This causes after redefinition of the physical axial field a finite renormalization of the pion kinetic energy (Kleinert 1978; Ebert and Reinhardt 1986). As a consequence the corresponding cutoff increases with respect to the case without vector mesons. In addition, the
axial mass acquires a substantial contribution from a partial Higgs mechanism. For simplicity we will work on the chiral circle.

**Real Part - Massive Yang-Mills and Hidden Symmetry Approach**

In the second order heat kernel approximation the real part of the effective action reduces to the following expression (Ebert and Reinhardt 1986; Wakamatsu and Weise 1988)

\[ \mathcal{L}_{\text{real}} = \mathcal{L}_{\text{MYM}} = I_2 M^2 \text{tr} \left[ D_\mu U^\dagger D^\mu U + (U^\dagger + U) \right] \]

\[ - \frac{1}{6} I_2 \left\{ g_\rho^2 \text{tr} \left[ (V^R_\mu)^2 + (V^L_\mu)^2 + g_\omega^2 \right] \right\} 
+ \frac{m_\rho^2}{2} \text{tr} [(V^R_\mu)^2 + (V^L_\mu)^2] + \frac{m_\omega^2}{2} \omega_{\mu
u} \]

(7.14)

where the regularization dependent integral \( I_2 (M) \) has been defined in eq. (2.28). The constituent quark mass \( M = g_\pi / f_\pi \) has also been introduced. The covariant derivatives and field strength tensors read

\[ D_\mu U = D^L_\mu U - U D^R_\mu = \partial_\mu U - ig_\rho (V^L_\mu U - UV^R_\mu) \]

\[ V^{RL}_{\mu\nu} = \partial_\mu V^R_{\nu} - \partial_\nu V^R_{\mu} = ig_\rho [V^R_{\mu}, V^R_{\nu}] \]

(7.15)

where

\[ U = \frac{1}{f_\pi} (\sigma + i \mathbb{F} \cdot \overline{\mathbb{F}}) \]

\[ V^R_\mu = \frac{1}{2} (\mathbb{F}_\mu + \mathbb{A}_\mu) \cdot \mathbb{F} \]

\[ V^L_\mu = \frac{1}{2} (\mathbb{F}_\mu - \mathbb{A}_\mu) \cdot \mathbb{F} \]

The former expression for \( \mathcal{L} \) can be identified with the old massive Yang-Mills Lagrangian (Lee and Nieh 1967) if one demands

\[ \frac{1}{g_\rho^2} = \frac{1}{4 g_\omega^2} = \frac{2}{3} I_2 \]

and also that \( m_\rho \) and \( m_\omega \) are the physical vector meson masses. Also it has been shown (Ball 1987; Wakamatsu and Weise 1988) that if one performs a field dependent chiral rotation the hidden symmetry Lagrangian of Bando et al. (1984, 1988) is obtained. As we can see at (7.14) the chiral symmetry of the Lagrangian suggests that the chiral partners \( \rho \) and \( a \) have the same mass. However, due to the spontaneous breaking of chiral symmetry there appears a term of the form \( M \mathbb{A}_\mu \cdot \partial^\mu \overline{\mathbb{F}} \) which would imply an unphysical decay process. This problem may be solved by introducing a new physical axial field \( \mathbb{A}_\mu = \mathbb{A}_\mu + \xi \partial_\mu \overline{\mathbb{F}} \) and fixing the parameter \( \xi \) in a way that the mixing for the new field disappears. This feature is also present in the hidden symmetry approach. In any case the solution to the \( A - \pi \) mixing leads to the following algebraic conditions (Ebert and Reinhardt 1986; Wakamatsu and Weise 1988)

\[ f_\pi^2 = \frac{m_\rho^2}{m_\rho^2 + 6 M^2 I_2} \]

\[ g_\rho^2 = \frac{M^2}{f_\rho^2} \frac{6 m_\rho^2}{m_\rho^2 + 6 M^2} \]

\[ g_\omega^2 = \frac{1}{4} g_\rho^2 \]

(7.18)
and the axial meson mass is given by \( m_A^2 = m_\rho^2 + 6M^2 \). After this the parameters may be fixed in the way described below. For a given constituent quark mass \( M \) the cutoff is adjusted to reproduce the pion decay constant \( f_\pi = 93 \text{ MeV} \) and the vector meson mass \( m_\rho = 770 \text{ MeV} \). Then the rest of the parameters as for instance \( g_\rho \) and \( m_A \) are determined uniquely. It is interesting to note that in the limit \( m_\rho \to \infty \) the above conditions become identical to those of sect. 1.6. For \( m_\rho = \sqrt{5}M \) and if a constituent quark mass of \( M = 315 \text{ MeV} \) is chosen, the KSFR relation \( 2g_\rho^2 f_\pi^2 = m_\rho^2 \) (1967) and the Weinberg sum rule \( m_A^2 = 2m_\rho^2 \) (Weinberg 1967) are fulfilled simultaneously. Let us finally mention that the fact that the massive Yang-Mills Lagrangian and further relations are obtained even for a finite cutoff indeed requires the continuation of vector fields into Euclidean space as it is done in this review.

**Imaginary Part - Gauged Wess-Zumino Term**

As we have already mentioned vector mesons generate an imaginary part for the Euclidean action. This has been computed in the low momentum limit (Dhar et al. 1984, 1985; Ebert and Reinhardt 1986), reproducing the vector gauged Wess-Zumino term (Wess and Zumino 1971; Witten 1983) after rotation to Minkowski space

\[
\mathcal{L}_{im} = \mathcal{L}_{GWZ} = \frac{N_c}{24\pi^2} g_\omega \omega_\mu \epsilon^{\mu\nu\alpha\beta} \text{Tr} \left( U_\nu U_\alpha U_\beta U_\gamma \right) \\
+ 3ig_\rho \partial_\nu \left( \partial_\mu U_\alpha \nu_\beta - U_\mu \partial_\nu U_\beta \nu_\alpha + ig_\rho U_\mu \nu_\beta \nu_\alpha \right)
\]

which is \( SU(2)_V \otimes U(1)_B \) gauge invariant. It must be said that this part of the Lagrangian saturates the anomalous Ward identity (7.13). Hence higher order terms ought to be chirally gauge invariant. Another interesting point is that although such an effective Lagrangian preserves vector gauge invariance it breaks global chiral symmetry and hence breaks some low energy theorems such as the amplitude for the decay \( \gamma \to 3\pi \) (Wakamatsu 1989, Ruiz Arriola and Salcedo 1993a). The main reason can be found in the \( a - \pi \) mixing which produces systematic corrections in any vertex with external pions. This is a clear drawback of the model, also present in the topological soliton model. Nonetheless it reproduces the correct result for the neutral pion decay via intermediate neutral vector mesons \( \pi^0 \to \omega^0, \rho^0 \to \gamma \gamma \) and the experimental value for the strong decay \( \omega \to 3\pi \) (Gomm et al. 1984; Kaymakcalan et al. 1985).

**Meson Propagators**

The calculation of on-shell meson propagators proceeds similarly as sketched in sect. 2.6. for the scalar-pseudoscalar case and has been treated in detail in several works. Here we will follow Jaminon et al. (1992). As in the heat kernel expansion, there appears an \( a - \pi \) mixing term, which can be diagonalized after a proper redefinition of the axial field. In summary, the following conditions are obtained

\[
f_\rho^2 = M^2 + \frac{4N_c F(-m_\rho^2)}{1 + \left( \frac{2M}{m_\rho} \right)^2 \frac{F(-m_\rho^2)}{S(-m_\rho^2)}}, \quad g_\rho^2 = \frac{1}{N_c S(-m_\rho^2)}, \quad g_\omega^2 = \frac{1}{4N_c S(-m_\rho^2)} \tag{7.20}
\]

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with the proper time regularized functions $F(q^2)$ and $S(q^2)$

$$F(q^2) = \frac{1}{16\pi^4} \int_{-1}^{1} \frac{du}{2} \int_{0}^{\infty} \frac{d\tau}{\tau} e^{-\left[M^2 + \frac{1}{2}(1-u^2)|q|^2\right]\tau}$$

$$S(q^2) = \frac{1}{16\pi^4} \int_{-1}^{1} \frac{du}{2} \int_{0}^{\infty} \frac{d\tau}{\tau} \frac{1}{\tau} e^{-\left[M^2 + \frac{1}{2}(1-u^2)|q|^2\right]\tau}$$

(7.21)  

(7.22)

It should be mentioned that the approximation $S(-m^2_{\rho}) \sim S(0) = \frac{2}{3} I_2$ and $F(-m^2_{\rho}) \sim F(0) = I_2$ corresponds to the heat kernel approximation as can be seen by comparison of formulas (7.20) with (7.18). Moreover, in the limit of $\rho - \omega$ degeneracy the relation $g_{\rho} = 2g_{\omega}$ holds. Jaminon et al. (1992) have found that for constituent quark masses lower than $M = 385$ MeV the $\rho$ meson is no longer a bound state. This is a direct consequence of the absence of confinement in the model.

For these masses various remedies have been proposed. Takizawa et al. (1991) suggest to look for the complex solutions of the Bethe-Salpeter equation, where the corresponding imaginary part represents the decay width into free quark-antiquark pairs. Jaminon et al. (1992) propose rather to take advantage of the approximate linear behaviour of the inverse propagator in the time-like region and to use a linear extrapolation into the space-like region. For constituent quark masses around 500 MeV they find the approximate formula

$$m^2_A \sim m^2_{\rho} + 3.1 M^2$$

(7.23)

which differs from the corresponding result in the heat kernel approximation (7.18). Finally, it has been found (Schuuren et al. 1993) that these results do not depend strongly on the particular regularization scheme employed.

**Numerical Results**

Vector mesons do influence the vacuum properties since their masses and the $A - \pi$ mixing enter explicitly the determination of the cutoff. In general the cutoff increases as compared to the case without vector mesons. This change is more dramatic if the parameters are fixed off-shell rather than on-shell (Schuuren et al. 1993). The interesting result is that only in the latter method the results for the vacuum parameters seem to be reasonable and fairly independent of the regularization. For illustration we quote the results for $M = 350$ MeV and in the proper time regularization

$$< \vec{q}_q > = -(271 \text{MeV})^3 \quad m_1 = \frac{1}{2}(m_u + m_d) = 8.2 \text{MeV}$$

The effect of vector mesons on pure pionic properties, i.e. pionic radii and threshold parameters for $\pi \pi$ scattering, has been investigated (Ruiz Arriola 1991a; Schuuren et al. 1993). A very interesting point is that vector mesons generate in a natural way an axial coupling constant for the constituent quarks $g^Q_A$ lower than one as given by the expression (Vogl et al. 1990; Ruiz Arriola 1991a)

$$g^Q_A = 1 - \frac{g_{\rho}^2 f^2}{m^2_{\rho}}$$

(7.24)
due to intermediate axial vector meson contributions to the axial current even in the leading order of the large \( N_c \) expansion. A value smaller than one has been also predicted in somehow different approaches and models (Peris 1991, 1992; Weinberg 1992; Blotz and Goeke 1992) as subleading large \( N_c \) corrections.

The main result found by Schuereen et al. (1993) is that vector mesons do not change noticeably the calculated pionic properties, at least if the \( \rho \) meson mass is fixed to its experimental value. As in the \((\sigma, \pi)\) case a strong dependence of the pion scalar and isovector radii on the constituent quark mass is found. In particular, fitting the pionic radii would require a quark mass of 250 MeV. As we have said, for this value of the mass the \( \rho \) and \( \omega \) mesons lie in the continuum, and as it will be seen below no solitons have been found.

More recently it has been found that a \( g^Q_\Lambda \neq 1 \) in the Nambu–Jona-Lasinio-model breaks the proper QCD anomalous structure. This can be only obtained for \( g^Q_\Lambda = 1 \) and correspondingly \( g_\rho = 0 \) (Ruiz Arriola and Salcedo 1993). The problem arises whether vector mesons can be described in a NJL type model without breaking the QCD anomaly.

### 7.3. Solitonic Solutions and Nucleon Observables

The solitonic sector of the NJL model has been studied with quark couplings corresponding to \( \rho \)-mesons (Alkofer and Reinhardt 1990), \( \rho \) and \( A \) (Doering et al. 1992; Alkofer et al. 1992), \( \omega \) (Schuereen et al. 1992a; Watabe and Toki 1992; Alkofer et al. 1993) and \( \rho \), \( A \) and \( \omega \) mesons (Doering et al. 1993; Schuereen et al. 1993; Ruiz Arriola et al. 1993; Zueckert et al. 1993). The main difficulty arising in the calculations including the \( \omega \)-meson is the fact that the Euclidean static energy is complex, due to the \( \omega \) meson. In this section we will present a way from the complex valued Euclidean action to the description of selfconsistent solitonic solutions for hedgehog field configurations (Schuereen et al. 1992a; Doering et al. 1993; Ruiz Arriola et al. 1993). At the end of this section we comment on other approaches (Watabe and Toki 1992; Alkofer et al. 1993).

#### Statement of the Problem

To understand the nature of the problem related to the Wick rotation let \( H \) and \( H^\dagger \) denote the single particle Dirac Hamiltonian and its hermitean conjugate given by

\[
H = h + i \left[ -g_\omega \omega_4 + g_\rho \frac{\not{\rho}}{2} (\not{\rho} + \not{A}) \gamma_5 \right]
\]

\[
H^\dagger = h - i \left[ -g_\omega \omega_4 + g_\rho \frac{\not{\rho}}{2} (\not{\rho} + \not{A}) \gamma_5 \right]
\]  

respectively. The hermitean part \( h \) of the hamiltonians \( H \) and \( H^\dagger \) has the form

\[
h = -i \alpha_i \nabla_i + \beta g_\omega (\sigma + \gamma_5 \not{\tau} \cdot \not{\tau}) + \alpha_i \left( -g_\omega \omega_i + g_\rho \frac{\not{\rho}}{2} (\not{\rho} + \not{A} \gamma_5) \right)
\]

(7.26)
Due to the Wick rotation $H$ is in general non-normal, i.e. $[H, H^\dagger] \neq 0$, and therefore $H$ and $H^\dagger$ may not be diagonalized simultaneously. For time independent configurations the real part of the action reads after regularization

$$\text{Re} S_f = -\frac{T}{2} N_c \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_0^{\infty} \frac{d\tau}{\tau} \delta(\tau, \Lambda) \text{tr} e^{i\nu + H}(-i\nu + H^\dagger)$$  \quad (7.27)$$

and the imaginary part

$$\text{Im} S_f = \frac{T}{2} N_c \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \left\{ \text{tr} \log(i\nu + H) + \text{tr} \log(-i\nu + H^\dagger) \right\}$$  \quad (7.28)$$

Now one would expect to proceed similar to perturbation theory, i.e. one should evaluate the trace, compute the $\nu$-integral and rotate back to Minkowski space. Unfortunately, since the one particle Hamiltonian is non-normal, the $\nu$ integration in the real part cannot be done analytically. Thus, the eigenvalues in the exponent should be computed for any value of the variable $\nu$. After that, the analytical continuation to Minkowski space should be done numerically. Needless to say that such an approach is not feasible in practice.

### Total Energy of the B=1 Soliton

The calculation of the functional trace for baryonic systems is conceptually involved and has been discussed in full detail by Schuilen et al. (1992, 1993). The interesting feature is that the analytical behaviour of the spectrum is such that the rotation back to Minkowski space can be performed by solving the eigenvalue problems

$$H^\pm \psi^\pm_\alpha (x) = \epsilon^\pm_\alpha \psi^\pm_\alpha (x)$$  \quad (7.29)$$

where

$$H^\pm = H_0 \pm \left[ g_\omega \omega_0 + g_\mu \frac{2}{\Lambda \gamma_5} \right]$$  \quad (7.30)$$

We just quote the final result. The total energy for a system with baryon number equal to one has been found to be

$$E = E_{\text{val}} + E_R + E_I + E_{\text{mes}}$$  \quad (7.31)$$

where the mesonic contribution

$$E_{\text{mes}} = \frac{\mu^2}{2} \int d^3x (\sigma^2 + \pi^2 - f_{\pi}^2) + \frac{m_\mu^2}{2} \int d^3x (\rho^2 + \rho^2) + \frac{m^2}{2} \int d^4x \omega^2$$  \quad (7.32)$$

the valence contribution

$$E_{\text{val}} = \theta(\epsilon_{\text{val}}) \epsilon^+_\text{val}$$  \quad (7.33)$$

the sea real

$$E_R = -\frac{N_c}{2} \sum_\alpha \left[ \xi_\alpha R(\tau_\alpha, \Lambda) - \epsilon^0_\alpha R(\epsilon^0_\alpha, \Lambda) \right]$$  \quad (7.34)$$

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and the sea imaginary contributions

$$E_I = -\frac{N_c}{4} \sum_\alpha \text{sign}(\tau_\alpha) (\epsilon^+ - \epsilon^-)$$  \hspace{1cm} (7.35)

have been defined. We have also introduced the averaged eigenvalues

$$\bar{\tau}_\alpha = \frac{1}{2} (\epsilon^+ + \epsilon^-)$$  \hspace{1cm} (7.36)

where $\epsilon^+$ and $\epsilon^-$ are the eigenvalues of the Hamiltonians $H^+$ and $H^-$ respectively with the condition that for $g_\omega = 0$ they coincide $\epsilon^+ = \epsilon^-$. It should be mentioned that the expression for the total energy has been checked asymptotically against the heat kernel expansion in the pure $\sigma$, $\pi$ and $\omega$ case for fixed meson profiles (Schuereen et al. 1992b). From the former equations it can be seen that the valence part contributes to the total energy of the system as long as the averaged eigenvalue $\bar{\tau}_{val}$ is non-negative. We will see below that this is in perfect agreement with the way the valence quarks contribute to the baryon number. Since the valence part of the energy is discontinuous for $\bar{\tau}_{val} = 0$, the question arises whether the sum of sea and valence contributions behave continuously. It can be shown analytically and also numerically (Schuereen et al. 1992b) that continuity is fulfilled due to the contribution of the imaginary part. The remarkable effect is the strong evidence for a vanishing imaginary contribution if $\bar{\tau}_{val} > 0$.

Let us remind that in the long wavelength limit the imaginary part of the action reduces to the gauged Wess-Zumino term eq. (7.19). Therefore the valence quarks coupled to all degrees of freedom seem to be complementary to the gauged Wess-Zumino term as long as $\bar{\tau}_{val}$ becomes negative.

Another consistency check is based on the infinite cutoff limit $\Lambda \to \infty$. In this case it is possible to perform the whole calculations also in Minkowski space. The result found coincides exactly with formula in the infinite cutoff limit.

**Currents of the B=1 Soliton**

From the expression for the total action the corresponding mean field quark densities may be obtained as functional derivatives of the quark contribution with respect to the relevant fields. The expectation value of a bilinear in the soliton background is given by

$$< \bar{q}(x) \bar{q}(x) > = \theta(\bar{\tau}_{val}) \bar{\psi}^+ \bar{\psi}^+(\vec{x}) \Gamma \psi^+ \psi^+(\vec{x})$$

$$- \frac{N_c}{2} \sum_\alpha \text{sign}(\bar{\tau}_\alpha) [\bar{\psi}^+ \bar{\psi}^+ \bar{\psi}^+ \bar{\psi}^+]$$

$$- \frac{N_c}{2} \sum_\alpha [\mathcal{R}_\alpha(\bar{\tau}_\alpha, \Lambda) + \bar{\tau}_\alpha \mathcal{R}'(\bar{\tau}_\alpha, \Lambda)] \text{sign}(\bar{\tau}_\alpha)$$

$$[\bar{\psi}^+ \bar{\psi}^+ \bar{\psi}^+ \bar{\psi}^+]$$  \hspace{1cm} (7.37)
with $\Gamma = 1, i\gamma_5\tau, \gamma_\mu$. Here $\mathcal{R}_2(\epsilon, A)$ and $\mathcal{R}_2^f(\epsilon, A)$ represent the regularization function and its derivative as used in sect. 3.3. As we see there are three distinct contributions to any mean field quark density: 1) a valence one provided the averaged eigenvalue $\tau_{val}$ is positive, 2) a real sea contribution which depends on the cutoff and 3) an imaginary part contribution which is not regularized. This also true in particular for the baryon number density. Moreover, if vector fields are switched off the baryon density coincides with the non-regularized baryon density of chapter 3.

**Self-Consistent Solitonic Solutions**

The equations of motion for general static configurations can be obtained by varying the total energy of the soliton with respect to all the meson fields. This yields

\[
\sigma(x) = \frac{g}{\mu^2} < \bar{q}(x) q(x) > \\
\tau(x) = \frac{g}{\mu^2} < \bar{q}(x) i\gamma_5 \rho(x) > \\
\rho^\mu(x) = \frac{g_{\rho}}{2m_\rho} < \bar{q}(x) \gamma^\mu \gamma_5 \rho(x) > \\
\varphi^\mu(x) = \frac{g_{\varphi}}{2m_\varphi} < \bar{q}(x) \gamma^\mu \gamma_5 \varphi(x) > \\
\omega^\mu(x) = \frac{g_\omega}{m_\omega} < \bar{q}(x) \gamma^\mu q(x) >
\]

(7.38)

The first two equations of motion are formally identical to the ones for the $\sigma\pi$ case discussed in chapter 3. For vector mesons the equations of motion reproduce the current field identities, i.e. the $\omega$, $\rho$ and $a$ fields in the nucleon are proportional to the expectation values of the baryon, vector and axial currents in the soliton background respectively (see (7.9)).

For practical calculations we use the hedgehog ansatz

\[
\sigma = f_\sigma \Phi(r) \cos \Theta(r), \quad \tau = x_\sigma f_\varphi \Phi(r) \sin \Theta(r), \quad \rho^i = 2\epsilon_{ijk} x^j \rho(r), \quad \rho^4 = 0, \\
\omega^i = 0, \quad \omega^4 = \omega(r), \quad a^i = 2\epsilon_{ijk} A_S(r) + 2x_\sigma A_T(r) - \frac{1}{3} A_T(r), \quad a^4 = 0
\]

(7.39)

Notice that the chiral circle condition $\sigma^2(x) + \varphi^2(x) = f_\rho^2$ is achieved if the polar field $\Phi(r)$ is taken to be a constant equal to one. It can be checked that this variational ansatz is a true solution of the equations of motion provided the Dirac sea contribution only involves closed shells in grand spin quantum numbers.

**Numerical Results**

The mean field equations of motion have been solved iteratively for the non-linear case ($\Phi(r) = 1$) by Schueren et al. (1993) and in the linear case ($\Phi(r) \neq 1$) by Ruiz Arriola et al. (1993) in a similar way as it was done in chapter 3. To simplify the discussion we will refer
mainly to the full $\sigma_\pi\omega\rho$ a system where the parameters have been fixed by computing the on-shell meson propagators and in the proper-time regularization. We remind that fitting the meson masses and the pion decay leaves the constituent quark mass $M$ as the only free parameter. An interesting result is that vector mesons prevent the soliton to collapse if the chiral circle constraint $\Phi(r)=1$ is relaxed (Ruiz Arriola et al. 1993). This is a consequence of the repulsive nature of the $\omega$ meson. For the simpler $\sigma\pi$ case the collapse has been discussed in sect. 3.5. The results for various mean field observables both in the non-linear and in the linear case are shown in Tab. 7.1, 7.2 for different values of the constituent quark mass.

As it can be seen, the total soliton mass is around 200 MeV higher than without vector mesons (see chap. 3). Unfortunately (see Tab. 7.2 the isoscalar radius and the axial coupling constant are 25% too high and 60% too low corresponding to experimental values respectively being nearly independent of the particular constituent quark mass $M$. Actually in the physical region the computed quantities do not depend strongly on the chiral circle condition, although the deviations from unity at the origin are significant in the linear model. Those become larger if the $\omega$ coupling constant is reduced and lead eventually to a collapse of the soliton for values much lower than the ones needed to fit the $\omega$ meson mass. For instance, for $M=340$ MeV a $g_\omega=2.24$ is required to fit the $\omega$ meson mass, whereas collapse takes place below $g_\omega=0.8$. The experimental isoscalar nucleon mean square radius can only be obtained for $g_\omega=1.60$. For this last value the averaged quark eigenvalue has been found to be positive.

For illustration we show in Fig. 7.1 the self-consistent vector meson fields $\omega, \rho, A_S$ and $A_T$ in the non-linear case for the particular values $M=340$ MeV, $g_\rho=4.61$ and $g_\omega=2.24$. In principle, they resemble the corresponding fields found in other soliton models like e.g. the Skyrme model (Meissner UG 1988) and the quark-meson model (Broniowski and Banerjee 1986), and their shapes do not depend strongly on the regularization scheme employed (Schieren et al. 1993).

As we have discussed in chapter 3 one of the virtues of the NJL model is that no assumption is made a priori whether the valence quark picture is correct or not. This is in fact decided upon the results in the calculation of physical observables. This means if a calculation reproduces the relevant experimental data and if then a clearly separated bound single quark state exists in the energy gap between positive and negative continuum, then the valence picture is valid. In SU(2) and SU(3) calculations with non-vector mesonic couplings all calculations by now yield this picture.

When vector mesons are present the validity or invalidity of the valence quark picture depends on the sign of the averaged quark eigenvalue $\bar{\tau}_{val}$. From Tab. 7.2 we see that this is positive both in the linear and in the non-linear model. However in this context all calculations involving vector mesons performed so far do not allow to reach final conclusions because the nucleon observables are not yet reproduced well enough. In particular the sign of $\bar{\tau}_{val}$ depends very much on the particular vector mesons included and the way the parameters are fixed. Schuermen et al. 1993 have shown that in the non-linear model and as long as the nucleon isoscalar radius is used as nucleon
observables to be reproduced this question cannot be decided unambiguously. In other words, it is possible to adjust the radius but with different signs for $\tilde{r}_{\text{vol}}$. In this sense the calculation of other nucleon observables is desirable to reach definite conclusions.

**Alternative Approaches**

For the $\sigma, \pi, \omega, \rho, A_1$-system there are other approaches to the soliton sector of the NJL model with vector mesons. Alkofer et al. (1993) propose to make the whole numerical calculation in Euclidean space without rotating the single particle hamiltonian back to Minkowski space where the problem is originally formulated. Instead they use a particular operational prescription to obtain the mass of the soliton. The parameters are fixed by means of a heat kernel expansion although the masses of the vector mesons are not negligible. These authors find solitons for the whole system. Within the same scheme Zueckert et al. 1993 have also found more recently that vector mesons stabilize the soliton of the $\sigma, \pi, \omega, \rho, A_1$-system in the linear case. Following their calculations it is possible to fit the nucleon radius with a solution having a negative valence quark eigenvalue. Watabe and Toki (1992) made calculations in the $\sigma, \pi, \omega$-system without continuation of the omega field into Euclidean space, which avoids the appearance of an imaginary part of the Euclidean action. Besides an explicit breaking of vector gauge symmetry (which dissappears in the infinite cutoff limit), their prescription corresponds to have a regularized baryon density. In addition they release the chiral circle condition. However neither bound nor stable solitons exist in this scheme.

As we have mentioned at the beginning of this section, the introduction of vector mesons in the soliton sector is by no means a trivial problem from a conceptual point of view. As an unfortunate consequence the various prescriptions proposed by different authors quoted at the beginning of sect. 7.3. lead to different results. This makes the situation unsatisfactory. In this sense it would be highly interesting to study this problem more deeply and to see which or even if any of the prescriptions suggested is the correct one.

In particular the problem of regularization has to be addressed since none of the above authors really starts from a regularized action as basis for all further developments.
Finally we want to compare shortly the NJL soliton consisting of valence quarks and the polarized Dirac sea with other effective chiral models which have been discussed in the literature so far. Both the Skyrme model (sect.8.1.) and the chiral sigma model of Gell-Mann and Levi (sect.8.2.) can be formally related to the NJL model via gradient and heat-kernel expansions. Whereas the Skyrme model contains no explicit valence quark degrees of freedom but relates the baryon number to the topological winding number of the Goldstone fields, the valence quarks are present in the Gell-Mann-Levi model from the very beginning. As we have discussed in chapter 3 both pictures can be incorporated in the NJL model and it is therefore interesting to see how these models are formally and numerically related. The renormalized chiral sigma model containing polarized Dirac sea as well as explicitly the kinetic energy of the mesons shows a translational not invariant ground state and is described in sect.8.3. In sect.8.4 we consider recent extension of the NJL model containing dilaton fields. General aspects concerning the connection between those various effective models are given in sect.8.5.

8.1. The Skyrme model

The Skyrme Model Without Vector Mesons


\[
\mathcal{L}_{SKY} = \frac{1}{4} f^2 \text{Tr} \left[ \left( \partial_\mu U^\dagger \right) \left( \partial^\mu U \right) \right] \\
+ \frac{1}{32 \epsilon^2} \text{Tr} \left\{ \left[ \left( \partial_\mu U^\dagger \right) \partial_\nu U \left( \partial^\mu U^\dagger \partial^\nu U \right) \right] - \left[ \left( \partial_\mu U^\dagger \right) \left( \partial^\mu U \right) \right]^2 \right\}
\]

where \( U(\vec{r}) = e^{i \theta(r) \vec{r}} \) with \( \theta(\infty) = 0 \) denotes the hedgehog chiral field on the chiral circle and \( \epsilon \) is a model parameter, which can be chosen to reproduce certain observables. The topological winding number \( n \) of the chiral field, which is defined by \( \theta(0) = -n \pi \), has to be an integer number in order to obtain a finite energy \( E_{SKY} = \int dt (-\mathcal{L}_{SKY}) \). It can be shown that the 4-divergence of the corresponding current, which is the topological or anomalous current (Goldstone and Wileczek 1981, Witten 1983)

\[
J^{\mu}_{\text{exp}}(x) = \frac{(-1)^n}{24 \pi^2} f^2 \epsilon_{\mu \nu \tau} \text{Tr} \left[ \left( \partial_{\nu_1} U \right) \left( \partial_{\nu_2} U \right) \left( \partial_{\nu_3} U \right) \left( \partial_{\nu_4} U \right) \right]
\]

vanishes and \( n \) is identified with the baryon number of the system. As we have already stated (cf. sect.3.2, app. B), \( J^{\mu}_{\text{exp}}(x) \) is obtained as the 4\(-\)th order gradient expansion of the 1-loop (sea) part of the baryon current \( \langle \hat{b}^\mu(x) \rangle \), whereas the second order vanishes. If we look at he 4th order gradient or heat-kernel expansion of the fermion determinant \( S_{\text{eff}}^F(U) \) of the effective action in the
NJL (eq. (2.16)) itself we find (Aitchison and Frazer 1984, 1985a, 1985b, Dhar et al. 1985, Ebert and Reinhardt 1986):

\[ \mathcal{L}^{(4)} = \mathcal{L}_{SKY}^{(4)} + \mathcal{L}_{D}^{(4)} \]  

(8.3)

with

\[ \mathcal{L}_{SKY}^{(4)} = \frac{N_c}{32\pi^2} \Gamma \left( 2, \left( \frac{M}{X} \right)^2 \right) \frac{1}{6} \text{Tr} \left\{ \left[ (\partial_{\mu} U^\dagger \partial_{\nu} U)(\partial^{\mu} U^\dagger \partial^{\nu} U) \right] - \left[ (\partial_{\mu} U^\dagger)(\partial^{\mu} U) \right]^2 \right\} \]

\[ \mathcal{L}_{D}^{(4)} = \frac{N_c}{32\pi^2} \left\{ \Gamma \left( 1, \left( \frac{M}{X} \right)^2 \right) \frac{1}{3} \text{Tr} \left[ ((\partial U^\dagger) (\partial U)) \right] - \Gamma \left( 2, \left( \frac{M}{X} \right)^2 \right) \frac{1}{6} \text{Tr} \left[ (\partial_{\mu} U^\dagger)(\partial^{\mu} U) \right]^2 \right\} \]  

(8.4)

where we have used the proper time scheme for regularizing the fermion determinant and performed a heat-kernel expansion (cf. sect. 2.5.). \( \Gamma(\alpha, x) \) denotes the incomplete \( \Gamma \)-function:

\[ \Gamma(\alpha, x) = \int_x^\infty dt t^{\alpha-1} e^{-t} \]  

(8.5)

Indeed, \( \mathcal{L}_{SKY}^{(4)} \) has the form of the Skyrme stabilizing term in eq. (8.1) but in addition a 'destabilizing' term \( \mathcal{L}_{D}^{(4)} \) appears (Aitchison et al. 1985). Meissner Th et al. (1990) showed that for small constituent quark masses \( M \leq 700 \text{MeV} \) the total energy in 4th order heat kernel expansion is indeed unstable, because \( E_{D}^{(4)} \) corresponding to \( \mathcal{L}_{D}^{(4)} \) goes to \(-\infty \) for meson profiles with small size \( R \) (cf. eq. (3.7), Fig. 3.2). Due to the behavior of the incomplete \( \Gamma \)-functions \( E_{D}^{(4)} \) changes sign for larger \( M \) and the theory in 4th order heat kernel expansion gets stable. Because for very high constituent quark masses the valence quark is part of the negative spectrum one therefore recovers at least the philosophy of the Skyrme model for \( M \to \infty \). On the other hand we know from sect.3.3 that the physics of the nucleon demands \( M \approx 400 \text{MeV} \), from which we conclude, that the Skyrme model in the Goldstone sector is not a good approximation to the NJL approach.

The Skyrme Model With Vector Mesons

In the mesonic sector the heat kernel approximation of the NJL Lagrangian has been found to resemble the phenomenological chiral Lagrangian of Gomm et al. (1984), where the imaginary part of the effective action of the NJL corresponds to the gauged Wess-Zumino term in the Skyrme like theory, in which Gomm et al. (1984) adjusted the parameters to reproduce the hadronic processes \( \rho \to \pi \pi \) and \( A_1 \to \rho \pi \). Using an on shell mass renormalization (cf. sect. 7.2.) the NJL prediction for these decays is rather good for constituent quark masses around 300 and 350 \text{MeV}.

Reinhardt and Dang (1989) have claimed that the NJL model with vector mesons reduces exactly to the original Skyrme Lagrangian (8.1) if the vector mesons are integrated out, and the nonlinear constraint of the chiral circle \( \sigma^2 + \varphi^2 = f_0^2 \) is assumed. This indeed is true in a second order heat kernel expansion. However, in the 4th order additional terms appear, whose magnitude
are not known. Hence there arises an ambiguity about the proper way to compare with the original
Skyrme Lagrangian. It is important to notice that the way vector mesons are presently described
in the Nambu-Jona-Lasinio-model turns out to be in full agreement with the old massive Yang-
Mills idea (Lee and Nieh 1968) that vector mesons are the gauge bosons of chiral symmetry. Such
an ideology has been widely used within the Skyrme model approach (Meissner UG 1988). Not
surprisingly, the gradient expansion produces a massive Yang-Mills type Lagrangian for the non-
anomalous part and a gauged Wess-Zumino term for the anomalous part, as it is often used in the
Skyrme model (Zahed and Meissner UG 1986, Meissner UG and Zahed 1987).

Finally we want to state that up to now it remains an open question, if the solution of the NJL
including \( \sigma, \bar{\sigma}, \text{and } \vec{A}_1 \) mesons as degrees of freedom supports a bosonized, Skyrme-like picture and
moreover if the explicit inclusion of vector mesons in the NJL is reasonable at all (cf. sect. 7.3.).
In fact it is known that the way they are introduced in the Nambu-Jona-Lasinio-model through
direct vector and axial couplings breaks the proper chiral anomaly (Wakamatsu 1988, Ruiz Arriola
and Sakedo 1993a).

8.2. The Chiral Quark Meson Model without Vector Mesons

The Chiral Sigma model (CSM) is based on a Lagrangian of Gell-Mann and Levi (1960) and
has been quite successful in the last years for a detailed description of nucleon and \( \Delta \) observables
and form factors. The Lagrangian in \( SU(2) \) and for \( \sigma \) and \( \vec{\pi} \) fields reads in the chiral limit
\( m_{\pi} = 0 \) and on the chiral circle \( \sigma^2 + \vec{\pi}^2 = f_0^2 \):

\[
\mathcal{L} = \bar{q}i\gamma\mu q - g\bar{q}(\sigma + i\vec{\pi}\gamma_5)q + \frac{1}{2}(\partial^\mu\sigma\partial_\mu\sigma + \partial^\mu\vec{\pi}\partial_\mu\vec{\pi})
\]  

(8.6)

In practical calculations (Birse and Banerjee 1984, Birse 1985, Goek et al 1985, Cohen and
Birse 1990, Neuber and Goek 1992) the Dirac sea has been ignored and the \( q \) was taken to be a
spinor of the \( N_c = 3 \) valence quarks. The \( \sigma \) and \( \vec{\pi} \) are first of all classical static fields. In this
approximation the total energy of a soliton reads:

\[
E_{\text{CSM}} = N_c\epsilon_{\text{val}} + \frac{1}{2}\int d^3x \left[ (\nabla^2\sigma)^2 + (\nabla^2\vec{\pi})^2 \right]
\]

(8.7)

where \( \epsilon_{\text{val}} \) is the valence quark energy. The \( \epsilon_{\text{val}} \) and \( \sigma(\vec{x}) \) and \( \vec{\pi}(\vec{x}) \) are obtained easily from the
equations of motion:

\[
\frac{\delta E_{\text{CSM}}}{\delta \bar{q}(\vec{x})} = 0
\]

\[
\frac{\delta E_{\text{CSM}}}{\delta \sigma(\vec{x})} = 0
\]

\[
\frac{\delta E_{\text{CSM}}}{\delta \vec{\pi}(\vec{x})} = 0
\]

(8.8)
in the hedgehog approximation. Solving these equations yields selfconsistent valence quark, \( \sigma \) and
\( \vec{\pi} \) fields of the Lagrangian eq. (8.6). Observables can be calculated in the usual way by coupling of
\( L_{CSM} \) to external currents and inserting the selfconsistent fields into the corresponding expressions, where the quark and meson fields have to be properly quantized, so that the system carries good spin, isospin or momentum quantum numbers, respectively. This quantization has been done in the semiclassical cranking way (Cohen and Broniowski 1986). Another way consists in assuming coherent Fock states for the sigma and pion fields and to use Peierls-Yoccoz projection techniques (Birse 1985, Fiolhais et al. 1987, 1988, Alberto et al. 1988, Neuber and Goeke 1992).

Actually the \( E_{CSM} \) of eq. (8.7) can be directly obtained in the gradient or heat-kernel expansion of the fermion determinant \( S_{eff}^F(\sigma, \bar{\pi}) \) of the effective NJL action eq. (2.16) (Meissner Th et al. 1988, 1990). If one truncates the expansion after the first nonvanishing order one obtains directly the kinetic energy \( \frac{1}{2} \int d^3x \left[ (\vec{\nabla} \sigma)^2 + (\vec{\nabla} \bar{\pi})^2 \right] \) of the meson fields (cf. sect. 2.5.), which, by adding \( N_c v_{vac} \) yields \( E_{CSM} \). This means that the kinetic terms of the mesons in the CSM have been generated from the polarized Dirac sea of the NJL by gradient expansion. A comparison between the two models will be performed now very easily: We once solve the NJL model in the 1 quark loop approximation and evaluate then observables. This is called NJL (exact) in Tab. 8.1. Then we perform a gradient expansion of the energy and several observables up to the first nonvanishing order. We insert the selfconsistent NJL solutions into those gradient expanded expressions yielding NJL (grad.). These numbers are compared with the selfconsistent solutions and the corresponding observables from the CSM. Using in all three cases the semiclassical quantization the results of some relevant observables are compared in Tab. 8.1, where the contributions from the Dirac sea and of the meson cloud are treated on a equal footing. Apparently the total values of the observables agree within 15%, while the separated contributions from valence quarks and mesons (sea quarks) differ. The \( g_A \) is a dear exception, there the gradient or heat kernel expansion seems to converge badly. The \( g_A \) from the Dirac sea of NJL basically vanishes, while the mesonic contribution in the CSM is comparable with the valence quark contribution. Thus a general statement on a comparison between these models is impossible and can only be done separately for each observable.

Finally we want to note that also the full linear version of the CSM, which contains in addition to (8.6) the \textit{mexican hat potential} reading \(- \frac{m_{\pi}^2}{8f_{\pi}^2}(\sigma^2 + \pi^2 - f_{\pi}^2)^2 \) (if \( m_\pi = 0 \)) can be obtained from the NJL by gradient or heat kernel expansion, where the \( \sigma \)-mass comes out to be \( m_{\sigma}^2 = 4(gf_{\pi}^2)^2 \). In contrast to the NJL, where without constraining the meson fields to the chiral circle the soliton collapses (cf. sect. 3.5), the mexican hat potential arising in second order gradient expansion is highly stabilizing and therefore the linear CSM shows perfectly stable solitonic solutions. Moreover it turns out that the nucleon observables calculated from those solutions in the CSM do not differ very much from the ones which are obtained in the nonlinear version.

This fact and the behavior of \( g_A \) as described above show that the \textit{gradient expansion} cannot be regarded as \textit{unreliable in the solitonic sector} as a generally reliable approximation. It has to be used with care.
8.3. THE CHIRAL QUARK MESON MODEL WITHOUT VECTOR MESONS

The chiral quark meson model with vector mesons proposed by Broniowski and Banerjee (1985, 1986) is based on the massive Yang-Mills Lagrangian of Lee and Nieh (1968) supplemented with valence quarks coupled to all mesons \( (\sigma \pi \omega \rho) \). Many practical calculations (Broniowski and Banerjee 1985, 1986; Broniowski and Cohen 1986; Ruiz Arriola et al. 1990; Alberto et al. 1990b; Ruiz Arriola et al. 1993b) neglect the effects of the Dirac sea. Similarly to the case without vector mesons, they may be considered to be included in an approximate way in the kinetic energy of the mesons. It should be also mentioned that this type of models do not include a gauged Wess-Zumino term as it is done in Skyrme type models. From the point of view of the Nambu-Jona-Lasinio model this is not a problem since the gauged Wess-Zumino terms correspond to the valence quarks (Schuhen et al. 1993) (see also the discussions in sect. 7.3.) The dynamical question, whether a chiral quark model with vector mesons is supported by the NJL model, remains an open problem.

8.4. THE RENORMALIZED CHIRAL SIGMA MODEL - VACUUM INSTABILITY

Some authors (Soni 1987, Ripka and Kahana 1987, Li et al. 1989) considered a renormalized CSM, i.e. they solved the Lagrangian (8.6) in the 1-quark loop approximation and subtracted the divergent parts as local counterterms. Such an approach does not contain an UV cutoff by construction. In our language the total energy of the renormalized CSM on the chiral circle reads:

\[
E_{\text{CSM}}^{\text{ren}} = N_c c_{\text{val}} + \frac{1}{2} \int d^3r \left[ (\nabla^2 \sigma)^2 + (\nabla^2 \pi)^2 \right] + \lim_{\Lambda \to \infty} \left\{ E_{\text{sea}}(\Lambda) - (4N_c g^2 l_1^3 \frac{1}{2} \int d^3r \left[ (\nabla^2 \sigma)^2 + (\nabla^2 \pi)^2 \right] \right\}
\]

(8.9)

where \( l_1^3(M) \) is the logarithmically divergent integral from eq. (2.28). Due to the subtraction of the counterterm \( (4N_c g^2 l_1^3 \frac{1}{2} \int d^3r \left[ (\nabla^2 \sigma)^2 + (\nabla^2 \pi)^2 \right] \) from the divergent sea energy \( E_{\text{sea}} \) the whole expression remains finite in the limit \( \Lambda \to \infty \). In the renormalized CSM the kinetic energy of the mesons \( \frac{1}{2} \int d^3r \left[ (\nabla^2 \sigma)^2 + (\nabla^2 \pi)^2 \right] \) is added explicitly rather than generated by the polarized Dirac sea as in the NJL where the cutoff \( \Lambda \) is properly adjusted. Actually one can easily see, that the total energy of the NJL and the renormalized CSM formally only differ by the UV convergent terms. In the NJL those are evaluated with a finite UV cutoff \( \Lambda \) reproducing the pion decay constant, whereas in the renormalized CSM they are treated with infinite \( \Lambda \). Though this small difference has a tremendous effect. The authors quoted above showed that the sea energy \( E_{\text{sea}} \) gets negative for meson profiles with finite size \( R \). This means that the full vacuum is not translationally invariant. Moreover it turns out that \( E_{\text{sea}} \) gets more and more negative, if one increases the topological winding number \( n \) of the pion field and therefore neither a stable vacuum nor a soliton solution with finite baryon number \( B \) exists.

Recently it has been suggested that this problem does not occur if dynamical vector mesons are included in the CSM (Kahana and Ripka 1992) in a massive Yang-Mills scheme and considering the corresponding quantum corrections.
8.5. Scale Invariance and Dilaton Fields

Ripka and Jaminon (1992) extended the NJL to a model which in addition to the spontaneously broken chiral symmetry also exhibits the anomalous breaking of scale invariance in QCD. Following the work of Jain et al. (1987) they introduced to this end a scalar-isoscalar dilaton field $\chi$, which is coupled to the effective action by:

$$I' = \frac{1}{2} \text{Sp} \int_0^{\infty} \frac{ds}{s} \left( e^{-sD^2} - e^{-sD_\mu D^\mu} \right) + \frac{a^2}{2} \int d^4 x \left[ \left( \sigma^2 + \bar{\sigma}^2 \right)^2 - \sigma_\nu^4 \right] + \int d^4 x \mathcal{L}(\chi) \quad (8.10)$$

where $a^2$ is a dimensionless parameter. The first two terms in eq. (8.10) are scale invariant, the scale breaking being related to the $\chi$-field Lagrange $\mathcal{L}(\chi)$. One chooses $\mathcal{L}(\chi)$ in such a way that the divergence of the dilaton current $\partial_\mu s^\mu = \chi^4$ and therefore the vacuum value $\chi_\nu^4$ can be related to the gluon condensate $\langle G^2_{\mu \nu} \rangle$. It has been shown (Jain et al. 1987, Ripka and Jaminon 1992) that, when a vacuum value $\chi_\nu \approx 350 \text{MeV}$ is used, as deduced from the QCD sum rule estimate (Shifman et al. 1979), the $\chi$ field stays almost constant throughout the chiral phase transition, in which the chiral symmetry is restored at high baryon densities or high temperatures.

Motivated by this fact Meissner Th et al. (1993) and Weiss et al. (1993) used a constant dilaton field $\chi = \chi_\nu$ by solving the theory (8.10) in the solitonic sector. The scale invariant form of the mesonic interaction in eq. (8.10), which differs from the mesonic mass term in the original NJL (2.16)

$$B^2 \int d^4 x \left( \sigma^2 + \bar{\sigma}^2 - \sigma_\nu^2 \right)$$

has the effect, that solitonic solutions of eq. (8.10) exist even in the case, when both $\sigma$ and $\bar{\sigma}$ are not restricted to the chiral circle $\sigma^2 + \bar{\sigma}^2 = \sigma_\nu^4$ (non-linear model), but both $\sigma$ and $\bar{\sigma}$ degrees of freedom are fully allowed (linear model). This was not the case in the original version of the NJL, where the constraint of the chiral circle turned out to be necessary, because otherwise the soliton collapses to a configuration with zero size and zero energy but baryon number $B = 1$ (cf. sect. 3.5). One can easily convince oneself, that, with the notation of sect. 3.5, the meson energy in eq. (8.10) gets now proportional to $U^4 R^3 \propto R^{3-4a}$. The soliton energy can then only fall to zero as $R \to 0$ when $a < \frac{3}{2}$. However, as it was shown in sect.3.5, the soliton can only maintain a baryon number $B = 1$ while its total energy tends to zero if $a > 1$. For this reason the $B = 1$ soliton does not collapse when it is calculated with the scale invariant action (8.10).

Furthermore Meissner Th et al. (1993) and Weiss et al. (1993) have shown that for the physical relevant region of the constituent mass $M \approx 400 \text{MeV}$ the deviation of $\sigma$ and $\pi$ obtained as solitonic solutions of (8.10) from the chiral circle is numerically very small. Moreover all calculated observables are nearly identical to the ones which are obtained from the original version of the NJL, where $\sigma$ and $\pi$ are restricted to the chiral circle from the very beginning.
The nonlinear version of the NJL, as it has always been used in the past and on which the present review is based on, can therefore be justified by a model which implements the anomalous breaking of scale invariance in QCD.

8.6. General Aspects

If one looks from a bird’s view on section 8.1, 8.2, 6.4, and 7.3., the following qualitative evaluation can be done: The simple Skyrme model with a pion field only cannot be compared with NJL. In the heat kernel expansion a destabilizing term occurs and it also turns out that the result for semiclassical quantized observables are different. The inclusion of vector mesons in the Skyrme Lagrangian, however, allows a better comparison with the NJL model in the scalar and pseudoscalar sector. The observables are similar and in the collective hamiltonian of $SU(3)$ structurally the same terms appear and the numerical results are close as well. Thus it seems that the role of the valence quarks in the NJL is qualitatively played by the vector mesons in the Skyrme model. Presently one cannot judge the role of vector mesons in NJL. Altogether it seems, and this includes the chiral sigma model as well, that the degrees of freedom relevant for the low energy regime of QCD can effectively be parameterized in rather different ways still reproducing the low energy hadronic phenomena with similar accuracy. One, however, is quite clear: 

*The Goldstone boson fields or Goldstone bosonic quark-quark interactions are playing the dominant role.*
9. Summary

The present review article deals with effective chiral symmetric models of quarks and non-dynamical mesons for the description of mesonic and baryonic ground states. In general effective theories are trying to model low energy QCD phenomena by identifying the appropriate degrees of freedom and the relevant symmetries necessary for a qualitative and if possible, quantitative description of low energy hadronic phenomena. Experience over the last 20 years has cumulated in the common opinion that for baryonic ground states spontaneously broken chiral symmetry and the incorporation of the corresponding Goldstone-boson fields are the important ingredients for a proper effective approach.

The review concentrates on the use of the Nambu-Jona-Lasinio-model and related approaches in the sector with baryon number $B = 1$. In its basic form it is defined by being the simplest pure quark theory with local biquadratic interactions of Goldstone character exhibiting spontaneous breakdown of chiral symmetry. One should note that the Nambu-Jona-Lasinio-model plays a central role because it allows to build conceptual and numerical bridges to other basic approaches, such as fully bosonized approaches of Skyrme type and theories, where valence quarks are coupled to dynamical meson fields.

The Nambu-Jona-Lasinio-model is treated explicitly in the path integral formalism performing a suitable bosonization procedure. Then the theory is described in terms of quarks and composite, non-dynamical meson fields. Actually the model is solved in the zero-boson and one-quark-loop approximation, such that it can be uniquely defined by a so called fermion determinant. In this form and neglecting the scalar fluctuations around the vacuum it reduces to the solutions of the chiral quark loop and agrees with the structure of the Chiral Quark Model. Performing an expansion of the determinant under the assumption of slowly varying pionic fields gives the structure of the Skyrme model, including the whole anomalous sector being described by the Wess-Zumino term.

Actually in the present model and for time-independent chiral fields on the chiral circle the fermion determinant is evaluated exactly and provides a full treatment of the polarization of the Dirac sea. The latter is caused by the presence of a discrete and localized valence level within the single-particle spectrum. The procedure used in the literature consists of three steps: First, a regularization scheme is chosen and the parameters of the model are fixed in the mesonic sector to reproduce PCAC, meson masses and decay constants. Second it is checked that the vacuum condensates and the current quark masses are reasonable. Third, without changing the parameters solitonic solutions are selfconsistently obtained in the sector with baryon number $B = 1$.

In order to describe baryons with the quantum numbers of the spin 1/2 and spin 3/2 multiplets, the hedgehog based quark field ansatz has to be quantized in the collective subspace. Therefore a time dependent rotation in the direction of the symmetry of the model is performed and canonical commutation relations are imposed on the collective coordinates and the canonical momenta. This results in proper commutation relations for the generators of the group, which in the case of SU(2) are the components of the total angular momentum and in SU(3) are complemented by some generators acting on the spin-baryon number space. The hedgehog ansatz for the SU(2) fields,
which manifests the relation between spatial and isospin rotations, results after the quantization in the constraint that the absolute value of spin always equals isospin. However the value of the spin itself can be integer or half integer and therefore the soliton can be quantized as fermion as well as boson. Only the generalization to SU(3) gives a constraint to fermionic solitons as long as the number of colors is odd.

The numerical results for the solitonic sector of the Nambu–Jona-Lasinio-model are basically independent on the regularization scheme used as long as the cut-off is treated as a parameter of the system, which is fixed in the meson sector. The calculations of the literature, all being based on hedgehog structure and physical values for the meson sector, can be summarized as follows (status autumn 1993):

i) SU(2), σ and π field, chiral circle: For a constituent quark mass \( M = 420\,\text{MeV} \) the following observables within 15% are reproduced. Isoscalar and isovector charge squared radius of the nucleon, \( < r^2 >_{T=0} \) and \( < r^2 >_{T=1} \), nucleon sigma term, \( \Sigma \), nucleon axial coupling constant and magnetic moments of proton and neutron (if the recent \( 1/N_c \) corrections are included), \( g_A \), \( \mu_p \) and \( \mu_n \), the nucleon-delta splitting, \( M_{N} - M_{\Delta} \), the \( q \)-dependence of the form factor \( G_E(q^2) \), \( G_M(q^2) \), \( G^n_M(q^2) \), \( g_A(q^2) \). The nucleon energy comes out at about 800MeV if rotational and translational zero modes corrections are included. The neutron squared radius, being a very sensitive quantity, is twice as large as the experimental value, the \( G^n_E(q^2) \) is generally by a factor of two too large at finite \( q^2 \).

ii) SU(3), σ, π, K and η fields, chiral circle for sigma and pion, trivial embedding of SU(2) into SU(3), perturbative treatment of \( m_q \) up to second order or Yabu-Ando approach: Splitting between and within spin 1/2 and spin 3/2 baryons is reproduced within few MeV. The splitting within isospin multiplets is reproduced and a common value for \( m_u - m_d \) is found. If the zero point corrections to translation and the SU(3)-rotations are included the energies of all octet and decuplet baryons are too large by a constant shift of approximately 100 MeV. If recent \( 1/N_c \) corrections are included the \( g_A^0 \), \( g_A^3 \) and \( g_A^8 \) of the nucleon are reproduced slightly outside the experimental errors. The nucleon \( \Sigma \)-term is reproduced with a strangeness content of 15% and the recently measured Gottfried sum comes out as well.

iii) SU(2), σ, π, ρ, \( A_1 \) and \( \omega \) mesons, chiral circle for sigma and pion. None of the present approaches has been developed to an extent that one can judge the influences or the necessity of vector mesonic couplings. The observables calculated by now are the isoscalar charge radius of the nucleon and \( g_A \), both deviating noticeably from experiment. There seems to be also conceptual problems with treating the omega meson. On the other hand the repulsive character of the omega guarantees a stable soliton and one does not require any more the restriction to the chiral circle.

The problem encountered by now can be summarized as following: The semiclassical quantization procedure is not perfectly understood yet, although it is common usage. Its relationship
to well-established quantum-mechanical theories like Peierls-Yoccoz projection approaches is yet unknown. If one looks at $g_A$ and magnetic moments of the nucleon its convergence in the rotational frequency $\Omega$ seems to be slow. Furthermore it has been shown that the PCAC relation is reproduced within the cranking scheme if rotational corrections in linear order of $\Omega$ are included.

The quantum corrections due to zero modes of translation and rotation add up to $30-40\%$ of the resulting baryon mass and hence are by far too large for a correction. The corresponding corrections of non-zero modes are of order $O(N_c)$ as well and they are not known yet. There are indications in the Skyrme model that they are small, but a clear statement in the Nambu-Jona-Lasinio-model is missing.

The treatment of the $\omega$-meson is not quite settled yet. A consistent scheme, which start from the very beginning with an action, the real part of which is regularized in Euclidean space, is still missing. Furthermore one needs some more observables to be evaluated. Hence the question if vector mesons in such a quark theory are indeed necessary to be included as quark-quark couplings is not yet settled.

Altogether one can summarize: Quark models like the Nambu-Jona-Lasinio-model with a polarized Dirac sea and involving Goldstone type couplings (i.e. non-dynamic mesons), have been by now quite successful in describing the ground states of baryons in the octet and decuplet. If one ignores vector mesons, whose relevance in the present model is still under debate, the calculations support a clear picture of the baryons consisting of three localized valence quarks interacting with a moderately polarized Dirac sea. This is in contrast to Skyrme-like models, which are governed by meson fields, exhibiting certain topological properties, and quark-meson models, where valence quarks are coupled to dynamical meson fields.

The problem encountered concern the semiclassical quantization method. It has certain merits, but it deserves improvement. Last but not least, the NJL-model lacks confinement which should be necessary for orbitally excited states of baryons.

Acknowledgements: The work was partially supported by the Alexander von Humboldt Stiftung (Fedor Lynen Program) and the US Department of Energy (under grant DE-FG06-90ER40561) (Th.M.), by the spanish DGICYT under contract PB92-0927 and the Junta de Andalucia (E.R.A.), by the Graduiertensipendium des Landes NRW (A.B.), the Bundesministerium fur Forschung und Technologie(BMFT), the Deutsche Forschungsgemeinschaft(DFG) and the COSY project of the KFA Jülich.

Th.M. and E.R.A. are grateful to the Institut für theorietische Physik II of the Ruhr-Universität Bochum for hospitality.

Finally we would like to thank C.V. Christov, D. Diakonov, V. Petrov, P.V. Pobylitsa, M. Polyakov, M. Praszalowicz, M. Wakamatsu, T. Watabe and R. Wünsch for various discussions along the way and also J. Berger, W. Ricken and C. Schneider for intensive organisatoric help.
Appendix A. Minkowski- and Euclidean Space-Time: Notation and Convention

Tab. A.1 compares the notation for 4-vectors $A_\mu$, the metric tensor $g_{\mu\nu}$, the scalar product between two 4-vectors as well as the Dirac $\gamma$-matrices in Minkowski and Euclidean space.

Generally a Minkowski 4-vector is transformed to an Euclidean 4-vector by performing a Wick rotation:

$$
A^E_4 = (+i)A^M_0 \\
A_{EA} = -A^E_4 = (-i)A^M_0 = (-i)A_{M0} \\
A^E_i = A^M_i \\
A_{Ei} = A_{Mi}
$$

(A.1)

Hereby both $A^E_4$ and $A^M_0$ are assumed to be real numbers, which means that any function $f(A^M_0)$ depending on the real variable $A^M_0$ gets analytically continued to the complex plane. For the 4-space vector and the 4-momentum vector we have especially:

$$
x^M_\mu = (t, \vec{r}) \\
x^E_\mu = (\tau, \vec{r}) \\
\tau = (+i)t \\
\int d^4x_E = (+i)\int d^4x_M \quad \frac{\partial}{\partial \tau} = (-i)\frac{\partial}{\partial t}
$$

(A.2)

and

$$
p^M_\mu = (\nu, \vec{p}) \\
p^E_\mu = (u, \vec{p}) \\
u = (+i)\nu \\
\int d^4p_E = (+i)\int d^4p_M \quad \frac{\partial}{\partial u} = (-i)\frac{\partial}{\partial \nu}
$$

(A.3)

with the notations:

$t$: Minkowski time  \quad \tau$: Euclidean time

$\nu$: Minkowski frequency  \quad u$: Euclidean frequency

Further examples handled in this way are the cranking frequency $\Omega$ (cf. chap. 3), which couples like the time component of an isovector vector meson ($\vec{p}$ meson), or the $\omega$ meson, which is time component of an isoscalar vector meson (cf. chap. 7).

The transformation of a scalar product reads:

$$
(A_M \cdot B_M) = \\
A^\mu M B^\mu_M = A^M_0 A_{M0} + A^i M A_{Mi} = A^M_0 A^M_0 - A^i M A^i M = \\
- A^E_4 A^E_4 - A^i E A^i E = A^E_4 A_{E4} + A^i E A_{Ei} = A^E\mu B^E_{\mu} = \\
(A_E \cdot B_E)
$$

i.e.: 

$$
(A_M \cdot B_M) = (A_E \cdot B_E) \quad (A.4)
$$
Furthermore we notate:

\[(A_M)^2 = (A_M \cdot A_M)\]
\[(A_E)^2 = A_E^\mu A_E^\mu = -(A_E \cdot A_E)\]  \hspace{1cm} (A.5)

giving:

\[(A_M)^2 = -(A_E)^2\]  \hspace{1cm} (A.6)

Fig. A.1 shows the contours in the complex plane arising for the calculation of static observables of the soliton in Minkowski space:

\[
\int \frac{du}{2\pi} \sum_\lambda \frac{1}{u - \epsilon_\lambda} \ldots
\]  \hspace{1cm} (A.7)

and in Euclidean space:

\[
\int \frac{du}{2\pi} \sum_\lambda \frac{1}{iu - \epsilon_\lambda} \ldots
\]  \hspace{1cm} (A.8)

if the one neglects regularization (cf. eq. (3.16)).

In order to obtain for a certain expression the same results in both cases one has to make the central assumption, that during rotating the contour-line from Minkowski into Euclidean space no singularity in the complex plane is hit. This is clearly true for the non-interacting case, but has to be postulated in general (Euclidean Postulate).

A system with baryon number \(B = 1\) can be constructed either by extending the corresponding contours as shown in Fig. A.1 (dashed lines) or by introducing a thermodynamic potential \(\mu\), which causes a shift:

\[\epsilon_\lambda \rightarrow \epsilon_\lambda - \mu\]  \hspace{1cm} (A.9)

in the single particle orbitals and therefore increases the number of singularities in the closed contour line by 1, namely by the valence orbit (cf. sect. 3.2). It is clear that this prescription only works, if \(\mu\) is both in the Minkowski space and the Euclidean space is a real number and therefore no analytic continuation (Wick rotation), as it was posted in eq. (4.1), may be performed. The treatment of the thermodynamic potential \(\mu\) in Euclidean space is therefore different from that of the time component of a physical 4-vector, as it is the \(\omega\) meson.

Also the Dirac \(\gamma\)-matrices can be formally transformed into Euclidean space:

\[
\gamma_E^i = \gamma_M^i = \gamma^i
\]  \hspace{1cm} (A.10)

Though the \(\gamma\)-matrices are no physical objects, which transform under Lorentztransformations as a Lorentz 4-vector. The eq. (A.10) is therefore only a simple redefinition, which means, that \(\gamma_E^4\)
is antihermmitian as it is $\gamma^i$. The reason for doing this is just because the rule for transforming a scalar product (A.4) can now also be applied to the expression $\mathcal{A}$:

$$\mathcal{A}_M = \mathcal{A}_E$$

(A.11)

For the matrices $\beta$ and $\gamma_5$ we will not define Euclidean values (as it is sometimes done especially for $\gamma_5$), but always use the same matrices, namely:

$$\beta = \gamma_M^0 = (-i)\gamma_E^4$$

(A.12)

and

$$\gamma_5 = \gamma^5 = (+i)\gamma_M^0\gamma_M^1\gamma_M^2\gamma_M^3 = \gamma_E^1\gamma_E^2\gamma_E^3\gamma_E^4$$

(A.13)

Generally we will use the Bjorken-Drell notation (1965):

$$\gamma_M^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \gamma_M^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

(A.14)

with the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(A.15)

and especially:

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \gamma_5 = \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

(A.16)

as well as:

$$\gamma^i = \beta \sigma^i$$

(A.17)
Appendix B. Gradient Expansion of the Baryon Current

Because of the importance for our model we show explicitly the gradient expansion of the baryon current \( b_\mu(x) \) and especially the baryon number \( B = \int d^3 x \delta_0(x) \), as it has been performed by Goldstone and Wilczek (1981) and discussed in detail by Witten (1983) in context of the topological quantization of \( B \).

From the general treatment given in sect.3.2 we have for the expectation value of \( b_\mu \):

\[
\langle b_\mu(x) \rangle = \frac{1}{N_c} \langle x \left| \text{Tr}_{\gamma \lambda c} \left[ (iD)^{-1} \gamma^\mu \right] \right| x \rangle - \text{vac. contr.} \tag{B.1}
\]

Hereby the \( U \) denotes a \( SU(n) \) chiral field which is parameterized in terms of the Goldstone field \( \xi(x) \)

\[
U(x) = e^{i \xi(x)} \tag{B.2}
\]

\( \xi(x) \) is hermitian and traceless, i.e.

\[
\xi(x) = \sum_{a=1}^{n^2-1} \xi^a(x) \frac{\lambda^a}{2} \tag{B.3}
\]

where the \( \lambda^a \) being the generators of \( SU(n) \). \( iD \) is the quark Dirac operator with the chiral field \( U \):

\[
iD = i \bar{\psi} - MU_5(x) \]

\[
(iD)^{-1} = [i \bar{\psi} - MU_5]^{-1} = (-) \left[ (\partial_\nu \partial^\nu + M^2) + iMU_5 \right]^{-1} (i \bar{\psi} + MU_5) \tag{B.4}
\]

with:

\[
U_5 = \frac{1}{2} (U + U^\dagger) + \frac{1}{2} \gamma_5 (U - U^\dagger) \tag{B.5}
\]

We therefore find:

\[
\langle b_\mu(x) \rangle = \langle b_\mu(x) \rangle \big|_V = (-) \frac{1}{N_c} \left\{ \langle x \left| \text{Tr}_{\gamma \lambda c} \left[ \gamma^\mu (\partial_\nu \partial^\nu + M^2) + iMU_5 \right]^{-1} (i \bar{\psi} + MU_5) \right| x \rangle - \langle x \left| \text{Tr}_{\gamma \lambda c} \left[ \gamma^\mu (\partial_\nu \partial^\nu + M^2)^{-1} (i \bar{\psi} + M) \right] \right| x \rangle \right\} \tag{B.6}
\]

In this expression we can now perform the power expansion:

\[
(A + B)^{-1} = A^{-1} - A^{-1}BA^{-1} + \ldots = A^{-1} \sum_{m=0}^{\infty} (-)^m (BA^{-1})^m \tag{B.7}
\]

with:

\[
A := (\partial_\nu \partial^\nu + M^2) =: G^{-1} \]

\[
B := i \bar{\psi} U_5 \tag{B.8}
\]
Because of $\text{Tr}_\lambda \xi = 0$ we find that all terms in eq. (B.8) up to order $m = 2$ vanish. For $m = 3$ we get:

$$\langle \langle b^\mu (x) \rangle - \langle b^\mu (x) \rangle_0 \rangle = i^3 M^4 \text{Tr}_\gamma \lambda \left[ \gamma^\mu G(\bar{\theta} U_5) G(\bar{\theta} U_5) G(\bar{\theta} U_5) G U_5 \right] | x \rangle + \ldots \quad (B.9)$$

The $G$ is diagonal in the momentum space basis $| p E \rangle$. In the lowest order gradient expansion the $\bar{\theta} U_5$ does not vary with the coordinate $x$ (cf. sect. 2.5). After inserting complete sets of $| x E \rangle$ and $| p E \rangle$, respectively, one obtains:

$$\langle \langle b^\mu (x) \rangle - \langle b^\mu (x) \rangle_0 \rangle = i^3 M^4 \int \frac{d^4 p_E}{(2\pi)^4} \left( \frac{1}{p_E^2 + M^2} \right)^4 \text{Tr}_\gamma \lambda \left[ \gamma^\mu (\bar{\theta} U_5)(\bar{\theta} U_5)(\bar{\theta} U_5) U_5^\dagger \right] + \ldots \quad (B.10)$$

and finally after performing the $\gamma$ trace:

$$\langle \langle b^\mu (x) \rangle - \langle b^\mu (x) \rangle_0 \rangle = \frac{(-1)}{24\pi^2} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \text{Tr}_\lambda \left[ (\partial_\nu U)(\partial_\sigma U)(\partial_\rho U) U^\dagger \right] \quad (B.11)$$

which is the anomalous baryon current of Goldstone and Wilczek (1980).

The same result can be obtained, if one starts from the imaginary part of the Euclidean effective action and performs a gradient expansion (cf. sect. 2.4), which leads to the famous Wess-Zumino-Witten term (2.37). The algebraic manipulation for doing this is very similar to the one, which was used above for gradient expanding the baryon current itself. 'U(1)-gauging' the Wess-Zumino-Witten term (2.37), as it was shown by Witten (1983), gives also the expression in eq. (B.11) as conserved current.

In the special case of the $SU(2)$-hedgehog form for $U$:

$$U(\vec{r}) = \cos \theta(\vec{r}) + i(\vec{n} \times \vec{r}) \sin \theta(\vec{r}) \quad (B.12)$$

one finds for the time component of eq. (B.11):

$$\langle \langle b^\phi (x) \rangle - \langle b^\phi (x) \rangle_0 \rangle = \frac{1}{2\pi^2} \frac{d\theta}{dr} \sin^2 \theta + \ldots \quad (B.13)$$

and after integration $\int d^3 r$ and using the fact, that $\theta(\infty) = 0$:

$$\langle B \rangle = \frac{\sin(2\theta(0))}{2\pi} - \frac{\theta(0)}{\pi} \quad (B.14)$$

For $\theta(0) = -n \pi$ one recognizes that in case of integer $n$ the gradient expanded baryon number $\langle B \rangle$ coincides with $n$ and therefore the gradient expansion is valid exactly. On the other hand for non integer $n$ the gradient expanded expression (B.14) is also non integer in contrast to the exactly calculated $\langle B \rangle$ and therefore the gradient expansion does not converge in this case.
A necessary condition for the validity of the expansion (B.7) as well as the negligibility of the space dependence of \( \partial U \) in (B.9) is:

\[
|\partial U| \ll M \tag{B.15}
\]

which has to hold at any point \( r \) in the radial integral \( \langle B \rangle = \int \delta^3 r (k_0(\vec{r})) \). If the meson profile \( U(\vec{r}) \) is characterized by some size parameter \( R \) (as e.g. in Fig. 3.1, 3.2) one finds therefore:

\[
|h| \pi \ll M \cdot R \tag{B.16}
\]

which means that the gradient expansion is good either for large profile sizes or also in case of selfconsistent solutions for high constituent masses \( M \).

Appendix C. Lorentz Transformation of the Mean Field Soliton Energy

In this appendix we want to show explicitly that the soliton mean field energy behaves under Lorentz boost transformations like the time component of a Lorentz 4-vector with vanishing space components, and therefore prove formula eq. (4.43), which is the main ingredient in the pushing approach.

Let us to this end consider a Lorentz boost in \( x_1 \)-direction with velocity \( v \):

\[
x^\mu \rightarrow x'^\mu = \Lambda^\mu_{\nu}(\vec{\omega}) x^\nu \tag{C.1}
\]

where:

\[
\vec{\omega} = \arctanh(v) \cdot v
\]

\[
\bar{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

and the contra- and covariant Lorentz transformation read:

\[
\Lambda^\mu_{\nu}(\vec{\omega}) = \begin{pmatrix}
\cosh \omega & -\sinh \omega & 0 & 0 \\
-\sinh \omega & \cosh \omega & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \]

\[
\Lambda_{\mu\nu}(\vec{\omega}) = \begin{pmatrix}
\cosh \omega & \sinh \omega & 0 & 0 \\
\sinh \omega & \cosh \omega & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \tag{C.3}
\]

respectively. The generator of this transformation is the unitary matrix:

\[
B(\vec{\omega}) = e^{-\vec{\omega}R} \tag{CA}
\]
where the infinitesimal boost generators $K^i$ are defined by:

$$K^i = \frac{t^i}{i} \frac{\partial}{\partial \tau^i} - x^i \frac{\partial}{\partial \tau^i} + i \frac{\alpha^i}{2}$$

(C.5)

One can easily convince oneself that:

$$B^{-1}(\xi) x^\mu B(\xi) = x'^\mu \equiv \Lambda^\mu_{\mu} (\xi) x^\mu$$

$$B^{-1}(\xi) \partial_{\mu} B(\xi) = \partial'_{\mu} \equiv \Lambda^\mu_{\mu} (\xi) \partial_{\mu}$$

$$B^{-1}(\xi) \gamma_{\mu} B(\xi) = \Lambda^\mu_{\mu} (\xi) \gamma^\mu$$

and therefore derive the action of $B(\xi)$ on the Dirac operator $iD$:

$$B^{-1}(\xi) \left[ i \gamma^\mu \partial_{\mu} - g \left( \sigma(x) + i \gamma_5 \bar{\tau}(x) \gamma_5 \right) \right] B(\xi) \equiv \left[ i \gamma^\mu \partial_{\mu} - g \left( \sigma(x') + i \gamma_5 \bar{\tau}(x') \gamma_5 \right) \right]$$

(C.7)

In the 1-quark loop approximation the fermionic part of the energy of a system with meson field configuration $\{\sigma(x), \bar{\tau}(x)\}$ is defined as the expectation value of the 1-particle hamiltonian $h = \frac{\Delta_{\mu}^2}{1} + g[\sigma(x) + i \gamma_5 \bar{\tau}(x) \gamma_5]$ due to eq. (3.14):

$$\int dt \langle h(x) \rangle = \frac{DqD\bar{q}}{\beta} \int d^4x \bigl[ \bar{q}(x) \beta h q(x) \bigr] e^i \int d^4x \bar{q}(x) \bigl[ iD[\sigma(x) \bar{\tau}(x)]\sigma(x) \bigr]$$

$$= i \frac{\partial}{\partial T} \text{Sln} [i \partial_{\tau} - h(x) - T h(x)] \bigg|_{T=0}$$

(C.8)

In case of time independent fields $\sigma(x)$ and $\bar{\tau}(x)$ this reduces to:

$$\int dt \langle h \rangle_{\text{stat}} = (-iT) \int \frac{d\nu}{2\pi} \sum_{\lambda} \frac{\epsilon_\lambda}{\bar{u} - \epsilon_\lambda} = T \sum_{\lambda \in \mathbb{C}_B} \epsilon_\lambda = T E_B$$

(C.9)

which is of course the same as the unregularized expression obtained in Euclidean space (eq. (3.11b)), if the integration contour $\mathbb{C}_B$ is chosen in such a way that the appropriate number of orbitals necessary for obtaining a system with baryon number $B$ gets occupied. In Fig. A.1 this has been indicated for the cases $B = 0$ and $B = 1$.

On the other hand in the boosted system (') we have the expectation value of the boosted hamiltonian $h_{\omega}(x) = \frac{\Delta_{\mu}^2}{1} + g[\sigma(x') + i \gamma_5 \bar{\tau}(x') \gamma_5]$

$$\int dt \langle h_{\omega}(x') \rangle =$$

$$= i \frac{\partial}{\partial T} \text{Sln} \left[ i \gamma^\mu \partial_{\mu} - g(\sigma(x') + i \gamma_5 \bar{\tau}(x') \gamma_5) - T \beta h_{\omega}(x') \right] \bigg|_{T=0}$$

$$= i \frac{\partial}{\partial T} \text{Sln} \left\{ B^{-1}(-\xi) \left[ i \gamma^\mu \partial_{\mu} - g(\sigma(x') + i \gamma_5 \bar{\tau}(x') \gamma_5) - T \beta h_{\omega}(x') \right] B(-\xi) \right\} \bigg|_{T=0}$$

(C.10)
Applying eq. (C.6) one finds (Betz and Goldflam 1983):

\[
\int dt \langle h_{\omega}(x'(x)) \rangle = T(\omega) E(\omega) =
\]
\[
\int dt \langle -i \partial_t + h(x) \rangle \quad \text{(C.11)}
\]

\[
+ (T \cosh(\omega)) \cdot \left[ \cosh(\omega) \langle h(x) \rangle - \sinh(\omega)(\bar{\alpha} \omega) h(x) \right] \\
+ \sinh(\omega)(\bar{\alpha} \omega) h(x) - \tanh(\omega) \sinh(\omega)(\bar{\alpha} \omega) \langle h(x) \rangle \right]
\]

For time independent meson field it holds:

\[
\langle -i \partial_t + h(x) \rangle = 0
\]
\[
\langle \vec{p} \rangle = \frac{i}{2} \langle [\vec{h}^2, \vec{\sigma}] \rangle = 0
\]
\[
\langle \bar{\sigma} h \rangle = \langle \vec{p} \rangle = 0 \quad \text{(C.12)}
\]

Furthermore for fixed index \( i \) (e.g. \( i = 1 \) in case of eq. (C.2)) one has:

\[
\langle \alpha_i \beta_i \rangle = \langle x_i \cdot [g_i \beta_i, \sigma(x) + i \vec{\tau} \vec{\pi}(x) \gamma_5] \rangle 
\]
\[
\text{(C.13)}
\]

which can easily be shown to vanish if the meson fields fulfill the classical mean field equations of motion (3.51) reflecting a virial theorem (Rafelski 1977).

Therefore we end up finally with:

\[
\int dt \langle h_{\omega}(x'(x)) \rangle = T(\omega) E(\omega) = (T \cosh(\omega))(E_{MF} \cosh(\omega)) \quad \text{(C.14)}
\]

Because the time interval \( T \) as itself is boosted to \( T_\omega = \cosh(\omega)T \), we obtain the desired relation (4.43) between the energy in the boosted (moving) system \( E_\omega \) and the static mean field energy \( E_{MF} \):

\[
E(\omega) = \cosh(\omega) E_{MF} 
\]
\[
\text{(C.15)}
\]

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Appendix D. Quantization condition for $N_c$ and why solitons are fermions

We know from the action of right generators $R_A$ of Sect. 6 on the baryonic wave-functions $\Psi(A)$, that (cf. Balachandran et al. 1984)

$$\exp i\Theta_a R_a \Psi(A) = \Psi(A \exp (i\Theta_a \frac{1}{2} \lambda_a)) \quad (D.1)$$

For the special case of a rotation around the 8th axis, we have to take into account that $R_S = N_c B / (2\sqrt{3})$ is constraint. Using the fact $\exp (i2\pi \sqrt{3}\lambda_8) = 1$, which corresponds to $\Theta_S = 4\pi\sqrt{3}$ in eq. (D.1), one obtains

$$\exp i4\pi\frac{1}{2} BN_c \Psi(A) = \Psi(A) \quad (D.2)$$

so that $B \times N_c$ is integer. The number of colors therefore is an integer quantity. This corresponds to the quantization condition for electric and magnetic charges, when we interpret the Lagrangean $L_{rot}^M$ as describing the moment of a charged particle in a monopole gauge field (Jackiw 1983). But we can deduce another quantization result. Under a rotation of $2\pi$ around the 3th axis, we observe that

$$\exp (i\Theta_3 \frac{1}{2} \lambda_3) = \exp (i\Theta_3 \frac{1}{2} \lambda_3) \quad (D.3)$$

for $\Theta_S = 2\pi\sqrt{3}$ and $\Theta_3 = 2\pi$. Using eq. (D.1) again one gets on the one hand

$$\exp (i\Theta_S R_S) \Psi(A) = \exp (i\pi) \Psi(A) = -\Psi(A) \quad (D.4)$$

and on the other hand using eq. (D.3) for $N_c = 3$

$$\exp (i\Theta_S R_S) \Psi(A) = \Psi(A \exp (i\Theta_3 \frac{1}{2} \lambda_3)) \quad (D.5)$$

Comparison of eqs. (D.4, D.5) gives that the wave functions $\Psi(A)$ changes sign under a space rotation of $2\pi$. Therefore in the actual case of $N_c = 3$, and in fact for an arbitrary odd value of the number of colors, we have quantized the soliton as a fermion (and as a boson in the case of an even number of colors). It was noted already by Finkelstein and Rubinstein (1968) and Witten (1983b), that in the case of SU(2), where the corresponding Wess-Zumino action - which leads to the constraint for the eighth generator - is vanishing, the soliton can be quantized as fermion as well as boson.
REFERENCES

Adler S L 1969 Phys.Rev. 177 2426
Adler S L and Davies A C 1984 Nucl.Phys. B 244 469
Alkofer R and Reinhardt H 1989 Z.Phys. C45 275
Alkofer R and Weigel H 1993, '1/N_c Corrections to g_A in the Light of PCAC', TU report No. UNI TU-TH E P-9/1993
Ball R 1987 Workshop on "Skyrmions and Anomalies", Mogilany, Poland, eds. M. Jezabek and M. Praszalowicz (World Scientific, Singapore)
Ball R 1989 Phys. Rep. 182 1
Bando M Kugo T Uehara S and Yanagida T 1984 Phys.Rev.Lett. 54 1215
Bardeen W 1969 Phys.Rev. 184 1848
Bell J S and Jackiw R 1969 Nuov. Cim. 60A 47
Bernard V 1986 Phys.Rev. D 34 1601

110
Bernstein J 1968 Elementary Particles and their Currents (San Francisco: Freeman)
Birse M C 1985 Phys. Rev. D 31 118
Birse M C and Banerjee M K 1985 Phys. Rev. D 34 118
Bhaduri R K 1988 Models of the Nucleon: From Quarks to Soliton (Reading Addison-Wesley)
Blotz A Doering F Meissner Th and Goekoe K 1990 Phys. Lett. B 251 235
Broniowski W and Banerjee M 1985, Phys. Lett. B 158 335
Broniowski W and Banerjee M 1986, Phys. Rev. D 34 849
Cahill R T and Roberts C 1985 Phys. Rev. D 32 2419
Cahill R T 1992 Nucl. Phys. A 543 63c
Chan L H 1985 Phys. Rev. Lett. 57 1199
Chemtob M 1985 Nucl. Phys. B 256 600
Chon K-C 1961 Sov. Phys. JETP 12 492
Cohen T D 1986 Phys. Rev. D 34 2187
Cohen T D and Bronkowski W 1986 Phys. Rev. D 34 3472
Dashen R 1969 Phys. Rev. 183 1245
Dashen R and Weinstei M 1969 Phys. Rev. 183 1261
Deister S Gari M F Krueempelmann W and Mahlke M 1991 Few-Body Syst. 10 1-36
De Swert J J 1963 Rev. Mod. Phys. 35 916
Dethier J L et. al. 1983 Phys. Rev. D 27 2191
Ebert D and Reinhardt H 1986 Nucl. Phys. B 271 188

112
Ellis J 1992 Nucl. Phys. A 546 447c
Fonda L and Ghirardi G 1972 Symmetry principles in Quantum Physics ed. A. Barut (New York: Dakkar)
Gasiorowicz S and Geffen D A 1969 Rev. Mod. Phys. 41 531
Gasser J and Leutwyler H 1984 Ann. of Phys. 158 142
Gell-Mann M 1962 Phys. Rev. 125 1067
Gell-Mann M Oakes RJ and Renner B 1968 Phys. Rev. 175 2195
Gell-Mann M and Ne’eman Y 1964 The Eightfold Way (New York: Benjamin)
Goldberger M L and Treiman S B 1958 Phys. Rev. 110 1178
Goldstone J 1961 Nuov. Cim. 19 154

113
Hatsuda T and Kunihiro T 1985 Prog. Theor. Phys. 74 765
Heisenberg W 1932 Z. Phys. 77 1
Holinde K. 1992 Nucl. Phys. A 543 143c
Holland D F 1968 J. Math. Phys. 10 531
Huarg K 1987 Statistical Mechanics (New York Wiley)
Jackiw R 1983 Gauge Theories of the Eighties eds. R. Raitio and J. Lindfors (Berlin: Springer)
Jain S and Wadia S R 1985 Nucl. Phys. B 258 713
Jain P Johnson R and Schechter J 1988b, Phys. Rev. D 38 1571
Jaminon M and Ripka G 1993, Saclay preprint T93/014
Kaymakcalan O and Schechter J 1985 Phys. Rev. D 31 1109
Kikkawa K 1976 Prog. Theor. Phys. 56 947
Klebanov I Strangeness in the Skyrme Model, Princeton report PUPT-1158 (unpublished)
Nambu Y and Jona-Lasinio G 1961a Phys. Rev. 122 345
Negele J and Orland H 1987 Quantum Many Particle Systems (Reading: Addison Wesley)
Okubo S 1962 Progr. Theor. Phys. 27 949
Polyakov M V 1990 Sov. J. Nucl. Phys. 51 711
Praszalowicz M 1990 Phys. Rev. D 42 216
Rajaraman R 1982 Solitons and Instantons (North Holland, Amsterdam)
Reinhardt H 1989 Nucl. Phys. A 503 825
Riazuddin and Fayyazuddin 1967 Phys. Rev. 18 507
Ring P and Schuck P 1980 The Nuclear Many Body Problem Springer Verlag
Ruiz Arriola E Doering F Schueren C and Goeke K 1993c, Bonn preprint
Sakurai J J 1969 Currents and Mesons (University of Chicago Press, Chicago)
Salam A and Strathdee J 1982 Ann. Phys. 141 316
Schneider C 1994, Diploma thesis RUB (unpublished)
Schueren C Doering F Ruiz Arriola E and Goeke K, to be published (1993)
Skyrme T H R 1962 Nucl. Phys. 31 556
Toyot% N 1987 Prog. Theor. Phys. 77 688
Veneziano G 1979 *Nucl. Phys.* **B159** 213


Wakamatsu M 1990a *Phys. Lett.* B **234** 223

Wakamatsu M 1990b *Phys. Rev.* D **42** 2427

Wakamatsu M 1992a *Phys. Lett.* B **280** 97

Wakamatsu M and Yoshiiki H 1991 *Nucl. Phys.* A **524** 561

Wakamatsu M 1992b *Phys. Rev.* D **46** 3762

Wakamatsu M 1993 *Phys. Lett.* B **300** 152


Wakamatsu M and Weise W 1988 *Z. Phys.* A **331** 173


Weigel H, Schwesinger B, and Hayashi A 1987 *Phys. Lett.* B **197** 11

Weigel H 1988 *Phys. Lett.* B **215** 24

Weigel H, Schuderer J, Park NW, and Meissner U-G 1990 *Phys. Rev.* D **42** 3177


Weinberg S 1967 *Phys. Rev. Lett.* **18** 188

Weinberg S 1968 *Phys. Rev.* **166** 1568

Weinberg S 1979 *Physica* **96A** 327


Wess J and Zumino B 1971 *Phys. Lett.* B **37** 95

Wilson K 1974 *Phys. Rev.* D **10** 2445

Witten E 1979a *Nucl. Phys.* B **156** 269

Witten E 1979b *Nucl. Phys.* B **160** 57

Witten E 1983a *Nucl. Phys.* B **223** 422

Witten E 1983b *Nucl. Phys.* B **223** 433

Yabu H and Ando K 1988 *Nucl. Phys.* B **301** 601

Yabu H 1989 *Phys. Lett.* B **218** 124

Zahed I and Brown GE 1986 *Phys. Rep.* **142** 1


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FIGURE CAPTIONS

1. Free Dirac spectrum for current quarks (chiral symmetry spontaneously unbroken)

2. Free Dirac spectrum for constituent quarks with mass $M$ (chiral symmetry spontaneously broken)

3. Dirac spectrum for the soliton. The bound valence quarks $\epsilon_{\text{val}} > 0$ polarize the Dirac sea weakly (valence quark picture).

4. Dirac spectrum for the soliton. The bound valence quarks $\epsilon_{\text{val}} < 0$ polarize the Dirac sea strongly (fully bosonized picture).

2.1. The cutoff $\Lambda$ for the the non-covariant (O(3)), covariant (O(4)) and single (PT) step proper time regularization as a function of the constituent quark mass.

3. The bound state spectrum of the 1-particle hamiltonian $h$ for an exponential profile form $\theta(r) = -n\pi e^{\frac{r}{R}}$ with $n = 1$. $\epsilon_{\lambda}$ and $R$ are given in scaled units ($\frac{\Lambda}{M}$ and $R \cdot M$). The valence orbit (0+) comes down from the positive into the negative spectrum (Kahana and Ripka 1984). Furthermore a value for the thermochemical potential $\mu$ is shown.

3.1. Sea energy $E_{\text{sea}}$, valence energy $\epsilon_{\text{val}}$ and total energy $E_{\text{tot}}$ as well as the corresponding baryon numbers $B_{\text{sea}}$, $B_{\text{val}}$ and $B_{\text{tot}} = B_{\text{sea}} + B_{\text{val}}$, respectively, for a linear profile form

$$\theta(r) = \begin{cases} -n\pi(1 - \frac{r}{R}) & \text{if } r < R \\ 0 & \text{if } r \geq R \end{cases}$$

with $n = 1$ at a constituent mass of $M = 555$ MeV. The total energy $E_{\text{tot}}(R)$ shows a local minimum at $R = 0.8$ fm. $E_{\text{val}}$ and $E_{\text{sea}}$ both jump if $\epsilon_{\text{val}}$ gets negative ($R \approx 1.3$ fm), whereas $B_{\text{tot}} = 1$ in any case. (Diakonov et al. 1988, Meissner Th et al. 1989)

3.2. The mean field energy $E_{MF}$ of the selfconsistent solution in the non-linear model ($m_\pi = 0$) (full line) splitted in valence (dashed-dotted) and sea (dotted) part in dependence of the constituent quark mass $M$. (Meissner Th et al. 1989)

3.3. The quadratic isoscalar electric radius $< R^2 > = < R^2 >_p + < R^2 >_n$ of the selfconsistent solution in the non-linear model ($m_\pi = 0$) (full line) splitted in valence (dashed-dotted) and sea (dotted) part in dependence of the constituent quark mass $M$. (Meissner Th et al. 1989)
3.5. The selfconsistent $\sigma$ field (normalized to $f_\pi$) in the non-linear model with $m_\pi = 0$ for 3 different constituent quark masses $M$ (Meissner Th and Goeke 1991).

3.6. The selfconsistent $\pi$ field (normalized to $f_\pi$) in the non-linear model with $m_\pi = 0$ for 3 different constituent quark masses $M$ (Meissner Th and Goeke 1991).

3.7. The baryon densities $b_0(r)$ of the selfconsistent solutions in the non-linear model with $m_\pi = 0$ for 2 different constituent quark masses $M$ (full lines). The valence contributions (dashed-dotted) are explicitly shown as well (Meissner Th and Goeke 1991). The upper two curves correspond to $M = 725\text{MeV}$ and the lower two to $M = 363\text{MeV}$.

4.1. The rotational moment of inertia (right y-axis), its valence and sea-quark contribution, and the nucleon-delta-splitting (left y-axis) is given in dependence of the constituent quark mass $M$. The pion mass is chosen to be $139\text{MeV}$.

4.2. The mean field energy $E_{MF}$ (HEDGEHOG) as well as the energies of nucleon $E_N$ (NUCLEON) and delta $E_\Delta$ (DELTA) and the translational and rotational corrections of the selfconsistent solution in the non-linear model ($m_\pi = 139\text{MeV}$) in dependence of the constituent quark mass $M$ (Pobylitsa et al. 1992).

5.1. The quadratic radii of the proton and the neutron are given in dependence of the constituent quark mass $M$. The experimental values are indicated.

5.2. The proton electric form factor $G_E^p(Q^2)$ for 4 different values of the constituent mass $M$ (Gorski et al.1992).

5.3. The neutron electric form factor $G_E^n(Q^2)$ for 4 different values of the constituent mass $M$ (Gorski et al.1992).

5.4. The electric charge distribution of the proton for a constituent mass of $M = 465\text{MeV}$ (full line) splitted in valence (short dashed) and sea (long-short dashed) (Wakamatsu 1991, Gorski et al.1992).

5.5. The electric charge distribution of the neutron for a constituent mass of $M = 465\text{MeV}$ (full line) splitted in valence (short dashed) and sea (long-short dashed) (Wakamatsu 1991, Gorski et al.1992).

5.6. The proton magnetic form factor $G_M^p(Q^2)$ for 4 different values of the constituent mass...
5.7. The neutron magnetic form factor $G_{nM}^{2n}(Q^2)$ for $4$ different values of the constituent mass $M$ (Christov et al. 1993a).

5.8. The magnetic moments for neutron and proton including contributions up to first order in the rotational frequency $\Omega$ are presented in dependence of the constituent quark mass $M$. The calculations are performed with $m_\pi = 139$ MeV.

5.9. The axial coupling constant $g_A$ including contributions up to first order in the rotational frequency $\Omega$ are presented in dependence of the constituent quark mass $M$. The calculations are performed with $m_\pi = 139$ MeV.

6.1. The deviation of the theoretical mass from the experimental one is shown for the hyperon spectrum and for the Yabu-Ando treatment and a constituent quark mass of $M = 419$ MeV (Blotz et al. 1993b).

6.2. The deviation of the theoretical mass from the experimental one is shown for the $\Sigma$ and the $\Lambda$, comparing the perturbative and the Yabu-Ando method for $M = 419$ MeV (Blotz et al. 1993b).

6.3. The hadronic part of the isospin splitting for the octet baryons from our theory (Praszalowics et al. 1993) compared with the experimental ranges for these splittings according to Gasser and Leutwyler (1982).

7.1. The selfconsistent vector and axial vector fields $\omega, \rho, A_\omega$ and $A_T$ for a constituent quark mass $M = 340$ MeV. The parameters are fixed according to the on-shell definition for the two-point functions, i.e. $\Lambda = 877$ MeV, $g_\rho = 4.61$ and $g_\omega = 2.24$.

7.2. The polar field $A(\tau) = \frac{1}{f^*} \sqrt{\sigma^2(\tau) + \sigma^2(\tau)}$ (which equals one in the non linear model) of the linear model is shown for different constituent quark masses.

A.1. Contour integration for evaluating static observables of a system with baryon number $B = 0$ (closed lines) and $B = 1$ (dashed lines) in Minkowski and Euclidean space.
TABLE CAPTIONS

1.1. The current quark masses and corresponding charges from the QCD vacuum (Gasser and Leutwyler 1982)

2.1. The cutoff Λ, the μ^2 parameter, the original NJL coupling G, the quark condensate θσ, the current quark mass m_0 and the vacuum energy density for the non-covariant (O(3)), covariant (O(4)), single (PT1) and double (PT2) step proper time and the Pauli-Villars (PV) regularization, defined in Sect. 2 and Sect. 6.

3.1. The critical values of the constituent quark mass M_{cr} and the corresponding values for the coupling constant g_{cr} = \frac{M_{cr}}{\bar{q}_{\pi}} as well as the ratio \lambda_{cr} = \frac{\Lambda(M_{cr})}{\Lambda(M_{cr})} for m_0 = 0 (chiral limit) and m_0 = 139 MeV. Solitonic solutions of the non-linear model exist if M > M_{cr} (g > g_{cr}, \lambda < \lambda_{cr}) (Meissner Th and Goek 1991).

5.1. The isoscalar μ^T=0 and the isovector μ^T=1 magnetic moment for 4 values of the constituent quark mass M. The value in brackets denotes the contribution of the sea quarks. (Wakamatsu and Yoshiki 1990). The calculations are performed in the lowest non-vanishing order in the cranking frequency, i.e., up to O(Ω^{11}) for the isoscalar quantity and up to O(Ω^{10}) for the isovector magnetic moment.

5.2. The pion nucleon coupling constant g_{\pi NN} on shell q^2 = m_{\pi}^2 and at q^2 = 0 in the lowest order of the cranking frequency (Ω^{0}) for 3 values of the constituent quark mass M. For the calculation the nucleon masses M_N from Fig.4.2 have been used. The experimental value of g_{\pi NN}(q^2 = 0) is extracted from the Goldberger-Treiman relation g_{\pi NN}(0) = \frac{M_N}{\bar{q}_{\pi}}g_A. (Meissner Th and Goek 1991).

6.1. The contribution of the valence and the sea part of moments of inertia for M = 391.5 MeV and M = 418.5 MeV.


6.3. The deviation of the theoretical mass from the experimental value is shown for the perturbative treatment with a constituent quark mass M = 390.6 MeV and the Yabu-Ando method with a constituent quark mass M = 418.5 MeV. For two values of the strange current quark mass, these deviations are shown together with the absolute experimental mass. The theoretical value of the Σ^*-mass is adjusted to the experimental one.
6.4. The axial vector coupling constants $g_A^0$, $g_A^3$ and $g_A^8$ for $M = 423.5\,\text{MeV}$, given in the lowest order contribution ($\Omega^0$) and with linear rotational ($\Omega^1$) and additional strange quark mass ($m_s^1$) corrections from the effective action, evaluated for wave functions, which contain linear $m_s$ corrections. These are compared with ’experimental’ numbers from recent EMC and SMC measurements (EMC, Ashman et al. 1988, 1989; SMC, Adeve et al. 1993).

7.1. A comparison of the mean field energies (in MeV), the axial nucleon coupling constant $g_A$, the isoscalar mean squared radius $<r^2>$ (in fm$^2$) and the value of the polar field at the origin $\Phi(0) = \frac{1}{\pi} \sqrt{\sigma^2(0) + \pi(0)^2}$. For a constituent quark mass $M = 340\,\text{MeV}$ the $\omega$ coupling constant $g_\omega$ is increased up to the correct value $g_\omega = 2.24$, which is in accordance with the mesonic sector. The linear model is used for the calculation.

7.2. The same as Tab. 7.1 in the linear and non linear version of the Nambu–Jona–Lasinio–model with $\sigma\tau\omega$ and $A_1$ mesons for different constituent quark masses $M$.

8.1. Various observables (energy mean field energy $E_{MF}$, isoscalar electric mean square radius $<R^2>_{E=0}$, moment of inertia $\Theta$, axial vector coupling $g_A$) for the selfconsistently solved NJL (valence part and sea part) (1.column), the selfconsistently solved NJL, where the sea part of the observables has been approximated by the 2nd order heat kernel or gradient expansion (valence part and mesonic approximation of the sea part) (2.column) and the selfconsistently solved CSM (valence part and mesonic part) (3.column).

A.1. Notation for 4-vectors $A_\mu$, the metric tensor $g_{\mu\nu}$, the scalar product of two 4-vectors and the Dirac $\gamma$-matrices in Minkowski and Euclidean space. Hereby Greek indices run from $0 - 3$ in Minkowski space and from $1 - 4$ in Euclidean space, whereas Latin indices generally run from $1 - 3$. 

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