Uncertainty Relation for a Quantum Open System

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Abstract

We derive the uncertainty relation for a quantum open system comprised of a Brownian particle interacting with a bath of quantum oscillators at finite temperature. We examine how the quantum and thermal fluctuations of the environment contribute to the uncertainty in the canonical variables of the system. We show that upon contact with the bath (assumed ohmic in this paper) the system evolves from a quantum-dominated state to a thermal-dominated state in a time which is the same as the decoherence time in similar models in the discussion of quantum to classical transition. This offers some insight into the physical mechanisms involved in the environment-induced decoherence process. We obtain closed analytic expressions for this generalized uncertainty relation under the conditions of high temperature and weak damping separately. We also consider under these conditions an arbitrarily-squeezed initial state and show how the squeeze parameter enters in the generalized uncertainty relation. Using these results we examine the transition of the system from a quantum pure state to a nonequilibrium quantum statistical state and to an equilibrium quantum statistical state. The three stages are marked by the decoherence time and the relaxation time respectively. With these observations we explicate the physical conditions when the two basic postulates of quantum statistical mechanics become valid. We also comment on the inappropriateness in the usage of the word classicality in many decoherence studies of quantum to classical transition.
Introduction

This paper aims at expounding the relation of quantum and thermal fluctuations and their relative importance in affecting the behavior of a quantum open system [1]. The demarkation of quantum, classical and thermal regimes is not always clearly noted, their usual definitions or usage oftentimes are imprecise and misleading. For example, one hears the vague identification of a high temperature regime as the classical regime. One also hears the loose statement that classical is defined as the regime when the Heisenberg uncertainty principle ceases to hold. Many recent discussions of decoherence via interaction with an environment [2] view the disappearance of the off-diagonal components of a reduced density matrix in some special basis or the selection or emergence of a set of classical histories [3] as signalling a quantum to classical transition. What are the roles played by quantum and thermal fluctuations in all these processes and how are these issues related to each other? [4]

To seek an answer to these questions we focus on the derivation of a generalized uncertainty principle valid at finite temperature. The uncertainty in the canonical variables are caused by both the vacuum and thermal fluctuations of the system and the bath. We study this problem because while simple enough to yield analytic results it also embodies these issues fully. A summary of the main results of this investigation was recorded earlier in [5].

It is a well-known fact in quantum mechanics that a lower bound exists in the product of the variances of pairs of noncommutative observables. Taking the coordinate $x$ and momentum $p$ as examples, the Heisenberg uncertainty principle states that with $(\Delta x)^2 = <x^2>\,-\, <x>^2$, the uncertainty function is

$$U_0^{QM} = (\Delta x)^2(\Delta p)^2 \geq \frac{\hbar^2}{4} \quad (T = 0, \quad quantum \ mechanics) \quad (1.1)$$

The existence of quantum fluctuations is a verified basic physical phenomenon. The origin of the uncertainty relation can be attributed as a mathematical property of Fourier analysis
which describes quantum mechanics as a wave theory. Recent years have seen effort in establishing a stronger relation based on information-theoretical considerations [7]. In particular we refer to the papers of Anderson and Halliwell [8] and Halliwell [9].

In realistic conditions quantum systems are often prepared and studied at finite temperatures where thermal fluctuations permeate. At high temperatures the equipartition theorem of classical statistical mechanics imparts for each degree of freedom an uncertainty of $kT/2$. Thus the uncertainty function for a one-dimensional harmonic oscillator approaches the limit

$$U_T^{MB} \approx \left(\frac{kT}{\Omega}\right)^2 \quad \text{(high } T, \text{ classical statistical mechanics)} \quad (1.2)$$

where $\Omega$ is its natural frequency $\Omega$. This result, obtained by assuming that the system obeys the Maxwell-Boltzmann distribution, is usually regarded as the classical limit. For a system of bosons in equilibrium at temperature $T$, the application of canonical ensemble gives the result in quantum statistical mechanics as

$$U_T^{BE} = \frac{\hbar^2}{4} \left[ \coth \left( \frac{\hbar \Omega}{2kT} \right) \right]^2 \quad \text{(all } T, \text{ quantum statistical mechanics)} \quad (1.3)$$

which interpolates between the two results (1.1) and (1.2) at $T = 0$ and $T \gg \hbar \Omega/k$. This result applies to a system already in equilibrium at temperature $T$.

Under equilibrium and stationary conditions the finite-temperature uncertainty principle is just this simple. It has been studied before via thermofield dynamics [10]. Here we aim to tackle the fully nonequilibrium problem. Using the microdynamics of a quantum system as starting point we view thermal fluctuations as statistical variations of the coarse-grained environmental variables with which the quantum system interacts, the exact microdynamics of the system and the environment obeying only the laws of quantum mechanics.

Our model is that of a collection of coupled harmonic oscillators where one is distinguished as the system of interest and the rest as bath. This model has been studied
extensively before [2][11-17]. We use the influence functional method [12] to incorporate the statistical effect of the bath on the system. At time \( t_0 \) we put the system in contact with a heat bath at temperature \( T \) and follow its time evolution. We consider an initial Gaussian wave packet and calculate the spread \( <\Delta x>^2, <\Delta p>^2 \) and the uncertainty function \( U_T \) due to both quantum and thermal fluctuations. The main result is given formally by Eq. (2.3.20), which, under the conditions of high temperature and weak damping, simplifies to Eq. (3.3.4) and Eq. (3.4.6) respectively. The results for an initial coherent or minimum-uncertainty state was first reported in [5]. Anderson and Halliwell [8] have recently shown that this result is equal to the lower bound of an information-theoretical definition of uncertainty with different initial coherent states. Here we derive the uncertainty relation for the general case of an arbitrarily-squeezed initial state where the squeeze parameter appears explicitly [17].

The paper is organized as follows: In Sec. 2, we begin with a short summary of the influence functional formalism applied to the Brownian motion model. (Details can be found in [12-15]. Readers familiar with this can skip to Sec. 2.2.) We then derive the reduced density matrix for a Gaussian wavepacket with nonzero initial position and momentum and arrive at a finite temperature uncertainty function (2.3.20) in terms of the elementary functions \( u_i \) and coefficients \( a_{ij} \) introduced in [15]. In Sec. 3 we assume an ohmic bath and derive the forms of certain coefficients \( f_n, g_n \) related to \( u_i, a_{ij} \), expressing them in terms of the squeeze parameter \( \delta \). We then work out the uncertainty function for a closed system (zero-coupling with the environment) of squeezed states (3.2.5). In Sec 3.3 and 3.4 we obtain the uncertainty function for high temperature and weak damping cases, given respectively by (3.3.4) and (3.4.6). For weak couplings, if the initial state is a minimum-uncertainty state, we recover the result (3.4.7) obtained earlier by us [5] and by Anderson and Halliwell [8]. In Sec. 4 we use our results to discuss the relative importance of quantum and thermal fluctuations and show how this bears on two related issues: One on the relation of uncertainty to decoherence studies [2], and the other on the demarkation
of quantum, classical and thermal regimes. In our results one can identify a time where the thermal fluctuation overtakes the quantum fluctuation. This time is identical to the decoherence time obtained in earlier studies of quantum to classical transition [2]. The other time scale of significance is that of relaxation time. We caution that after the decoherence time, although the system is describable in terms of probabilities, it cannot yet be regarded as classical because of the spin-statistics effect, and it remains in a state of nonequilibrium. After the relaxation time the two basic postulates of statistical mechanics [18] become valid and one can depict the system under the premises of equilibrium (but still quantum) statistical mechanics. The system has to be in a sufficiently high temperature when the spin-statistics can be adequately represented by the Maxwell-Boltzmann distribution (particle becomes distinguishable) for it to be aptly called classical.
2. Quantum and Thermal Fluctuations in an Open System

2.1 Influence Functional for a Brownian Particle in a Harmonic Oscillator Bath

Consider a Brownian particle with mass \( M \) and natural frequency \( \omega_0 \). Its environment is modeled by a set of harmonic oscillators with mass \( m_n \) and natural frequency \( \omega_n \). The Brownian particle is coupled linearly to the \( n \)-th oscillator with strength \( C_n \). The action of the combined system and environment is

\[
S[x, q] = S[x] + S_b[q] + S_{int}[x, q]
\]

\[
= \int_0^t ds \left[ \frac{1}{2} M \left( \dot{x}^2 - \Omega_0^2 x^2 \right) + \sum_n \left\{ \frac{1}{2} m_n \dot{q}_n^2 - \frac{1}{2} m_n \omega_n^2 q_n^2 \right\} + \sum_n \{-C_n x q_n\} \right]
\]

(2.1.1)

where \( x \) and \( q_n \) are the coordinates of the Brownian particle and the \( n \)-th bath oscillators respectively. We are interested in how the environment affects the system in some averaged way. The quantity containing this information is the reduced density matrix of the system \( \rho_r(x, x', t) \) obtained from the full density operator of the system and environment \( \rho(x, q; x', q'; t) \) by tracing out the environmental degrees of freedom \( (q_n, q_n') \)

\[
\rho_r(x, x', t) = \prod_n \int_{-\infty}^{+\infty} dq_n \int_{-\infty}^{+\infty} dq_n' \rho(x, q; x', q', t) \delta(q_n - q_n')
\]

(2.1.2)

The reduced density matrix evolves under the action of the evolutionary operator \( J_r \) in the following way:

\[
\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx_i' J_r(x, x', t | x_i, x_i', 0) \rho_r(x_i, x_i', 0)
\]

(2.1.3)

In general, this is a very complicated expression since \( J_r \) depends on the initial state. If we assume that at a given time \( t = 0 \) the system and the environment are uncorrelated, i.e. that
\[ \dot{\rho}(t = 0) = \dot{\rho}_a(t = 0) \times \dot{\rho}_c(t = 0), \]  
then the evolution operator for the reduced density matrix can be written as

\[ J_r(x_f, x'_f, t | x_i, x'_i, 0) = \int_{x_i}^{x_f} \int_{x'_i}^{x'_f} Dx \int Dx' \exp \left\{ \frac{i}{\hbar} \left\{ S[x] - S[x'] \right\} \right\} F[x, x'] \]  

where \( F[x, x'] \) is the Feynman-Vernon influence functional \([12]\). If the environment is initially in thermal equilibrium at temperature \( T \), for the problem described by (2.1.1), the influence functional can be computed exactly. The result is well known \([12-15]\):

\[ F[x, x'] = \exp \left\{ -\frac{i}{\hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 \left[ x(s_1) - x'(s_1) \right] \eta(s_1 - s_2) \left[ x(s_2) + x'(s_2) \right] \right\} \]

\[ - \frac{1}{\hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 \left[ x(s_1) - x'(s_1) \right] \nu(s_1 - s_2) \left[ x(s_2) - x'(s_2) \right] \]

The non-local kernels \( \eta(s) \) and \( \nu(s) \) are defined as

\[ \nu(s) = \int_0^{\infty} \frac{d\omega}{\pi} I(\omega) \coth \frac{\hbar \omega}{2kT} \cos \omega s \]  

and

\[ \eta(s) = \frac{d}{ds} \gamma(s) \]

where

\[ \gamma(s) = \int_0^{\infty} \frac{d\omega}{\pi} \frac{I(\omega)}{\omega} \cos \omega s \]

Here \( I(\omega) \) is the spectral density function of the environment,

\[ I(\omega) = \sum_n \delta(\omega - \omega_n) \frac{\pi C_n^2}{2m_n \omega_n} \]
The real and imaginary parts of the exponent in the influence functional $F[x, x']$ are understood as responsible for dissipation and noise respectively, thus the names dissipation and noise kernels are given to $\eta(s)$ and $\nu(s)$. The most general environment would thus engender nonlocal dissipation and colored noise. (We refer the reader to Ref [1, 15] for a discussion of the fluctuation-dissipation relation and the time scales of the relevant processes.)

An example of the spectral density is given by [13]

$$I(\omega) = M\gamma_0\omega \left( \frac{\omega}{\tilde{\omega}} \right)^s e^{-\frac{\omega^2}{A^2}}$$  \hspace{1cm} (2.1.9)

where $\tilde{\omega}$ is a frequency scale usually taken to be the cut-off frequency $A$. The environment with this kind of spectral density is classified as

$$\text{ohmic} \quad \text{if} \quad s = 1$$  \hspace{1cm} (2.1.10a)

$$\text{supra-ohmic} \quad \text{if} \quad s > 1$$  \hspace{1cm} (2.1.10b)

$$\text{subohmic} \quad \text{if} \quad s < 1$$  \hspace{1cm} (2.1.10c)

The propagator (2.1.5) has been calculated before [12-15]. (We use the notation of Ref. 15)

$$J_r(x_f, x'_f, t \mid x_i, x'_i, 0) = Z_0(t) \exp \left\{ \frac{i}{\hbar} \left[ \hat{u}_1(0)X_i + \hat{u}_2(0)X_f \right] Y_i ight.$$  

$$- \left[ \hat{u}_1(t)X_i + \hat{u}_2(t)X_f \right] Y_f + ia_{11}(t)Y_i^2$$  \hspace{1cm} (2.1.11)

$$+ i \left[ a_{12}(t) + a_{21}(t) \right] Y_i Y_f + ia_{22}(t)Y_f^2 \right\}$$

Here we have introduced

$$\left\{ \begin{array}{l} X_i = \frac{1}{2}(x_i + x'_i) \\ Y_i = x'_i - x_i \end{array} \right.$$  \hspace{1cm} (2.1.12a)
and

\[
\begin{align*}
X_f &= \frac{1}{2}(x_f + x'_f) \\
Y_f &= x'_f - x_f
\end{align*}
\]  \hspace{1cm} (2.1.12b)

The elementary functions \(u_i(s), i = 1, 2\) in (2.1.11) satisfy the following integro-differential equation

\[
\frac{d^2u_i(s)}{ds^2} + 2\int_0^s ds' \eta((s - s')u_i(s')) + \Omega_0^2 u_i(s) = 0
\]  \hspace{1cm} (2.1.13)

with boundary conditions

\[
\begin{align*}
u_1(0) &= 1 \\
u_1(0) &= 0 \quad \text{and} \quad \begin{align*}u_2(0) &= 0 \\
u_2(0) &= 1\end{align*}
\end{align*}
\]  \hspace{1cm} (2.1.14)

The coefficient functions \(a_{ij}(t), i, j = 1, 2\) are defined by

\[
a_{ij}(t) = \int_0^t ds_1 \int_0^{s_1} ds_2 v_i(s_1)v(s_1 - s_2)v_j(s_2)
\]  \hspace{1cm} (2.1.15)

where

\[
v_1(s) \equiv u_2(t - s)
\]  \hspace{1cm} (2.1.16a)

and

\[
v_2(s) \equiv u_1(t - s)
\]  \hspace{1cm} (2.1.16b)

It is clear that the functions \(u_i(t)\) and \(a_{ij}(t)\) only depend on the kernels \(\eta(s)\) and \(\nu(s)\).
2.2 Reduced Density Matrix For A Gaussian Wavepacket

We now consider a Brownian harmonic oscillator with an initial wave function (Calde-
ria and Leggett 1985, Unruh and Zurek 1987 in [2], Hu, Paz and Zhang 1992 in [15], Paz,
Habib and Zurek 1992 in [16])

\[ \psi(x, 0) = \sqrt{N_0} \exp\left\{-\frac{(x - x_0)^2}{4\sigma^2} + \frac{i}{\hbar} p_0 x\right\} \]  

(2.2.1)

where \( \sigma \) is the initial spread, and \( x_0 \) and \( p_0 \) are the averaged initial position and momentum of
the Gaussian wave packet. The initial reduced density matrix is given by

\[ \rho_r(x_i, x_i', 0) = N_0 \exp\left\{-\frac{(x_i - x_0)^2}{4\sigma^2} - \frac{i}{\hbar} p_0 x_i - \frac{(x_i' - x_0)^2}{4\sigma^2} + \frac{i}{\hbar} p_0 x_i'\right\} \]  

(2.2.2)

\[ = \tilde{N}_0 \exp\left\{-\frac{X_i^2}{2\sigma^2} - \frac{Y_i^2}{8\sigma^2} + \frac{x_0}{\sigma^2} X + \frac{i}{\hbar} p_0 Y\right\} \]

(2.2.3)

Substituting this and the propagator (2.1.11) into (2.1.3), one would get

\[ \rho_r(x_f, x_f', t) = Z_0 \tilde{N}_0 \exp\left\{-\frac{1}{\hbar} \hat{u}_2(t)X_f Y_f + ia_{22}(t)Y_f^2\right\} \times \int_{-\infty}^{+\infty} dX \int_{-\infty}^{+\infty} dY \times \exp\left\{-\frac{1}{2} \begin{pmatrix} X_i \\ Y_i \end{pmatrix}^T \begin{pmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{pmatrix} \begin{pmatrix} X_i \\ Y_i \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}^T \begin{pmatrix} X_i \\ Y_i \end{pmatrix}\right\} \]  

(2.2.4)

where

\[ b_1(t) = -\frac{i}{\hbar} \hat{u}_1(t)Y_f + \frac{x_0}{\sigma^2} \]  

(2.2.5a)

\[ b_2(t) = i \frac{\hbar}{\hat{u}_2(0)X_f} - \frac{1}{\hbar} [a_{12}(t) + a_{21}(t)] Y_f + \frac{i}{\hbar} p_0 \]  

(2.2.5b)
and the matrix elements of $H(t)$ are given by

\begin{align*}
H_{11}(t) &= \frac{1}{\sigma^2} \quad (2.2.6a) \\
H_{12}(t) &= H_{21}(t) = -\frac{i}{\hbar} \hat{u}_1(0) \quad (2.2.6b) \\
H_{22}(t) &= \frac{1}{4\sigma^2} + \frac{2}{\hbar} a_{22}(t) \quad (2.2.6c)
\end{align*}

Performing the Gaussian integrals over $X$ and $Y$ in (2.2.4), one gets

\[
\rho_r(x_f, x'_f, t) = \hat{Z}_0(t) \exp \left\{ -\frac{1}{2} \begin{pmatrix} X_f & Y_f \end{pmatrix}^T \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{pmatrix} \begin{pmatrix} X_f \\ Y_f \end{pmatrix} + \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}^T \begin{pmatrix} X_f \\ Y_f \end{pmatrix} \right\} \tag{2.2.7}
\]

where

\[
B_1(t) = -\frac{i \hat{u}_2(0) H_{12}}{\hbar \sigma^2 \det H(t)} x_0 - \frac{\hat{u}_2(0) H_{11}(t)}{\hbar^2 \det H(t)} p_0, \tag{2.2.8a}
\]

\[
B_2(t) = \frac{1}{\hbar \det H(t)} \left\{ -i H_{22}(t) + H_{12}(t) \left[ a_{12}(t) + a_{21}(t) \right] \right\} x_0 - \frac{1}{\sigma^2 \hbar \det H(t)} \left\{ \hat{u}_1(t) H_{12}(t) + i H_{11}(t) \left[ a_{12}(t) + a_{21}(t) \right] \right\} p_0 \tag{2.2.8b}
\]

The prefactor

\[
\hat{Z}_0(t) = Z_0(t) \tilde{N}_0 \frac{\pi}{\sqrt{\det H}} \exp \left\{ \frac{H_{22}(t)}{2\sigma^2 \det H(t)} x_0^2 - \frac{i H_{12}(t)}{\hbar \sigma^2 \det H(t)} x_0 p_0 - \frac{H_{11}(t)}{2\hbar^2 \det H(t)} p_0^2 \right\} \tag{2.2.9}
\]

depends only on the parameters $x_0$, $p_0$ and time. The matrix elements of $Q(t)$ are given by

\[
Q_{11}(t) = \frac{[\hat{u}_2(0)]^2}{\hbar^2 \sigma^2 \det H(t)} \quad (2.2.10a)
\]
\[ Q_{12}(t) = Q_{21}(t) = \frac{i}{\hbar} \hat{u}_2(t) + \frac{i}{\hbar^2 \det \mathbf{H}(t)} \left\{ \frac{1}{\sigma^2} \left[ a_{12}(t) + a_{21}(t) \right] - \frac{1}{\hbar} \hat{u}_1(0) \hat{u}_1(t) \right\} \] (2.2.10b)

\[ Q_{22}(t) = \frac{2}{\hbar} a_{22}(t) + \frac{1}{\hbar^2 \det \mathbf{H}(t)} \left\{ \frac{1}{4\sigma^2} + \frac{2}{\hbar} a_{11}(t) \left[ \hat{u}_1(t) \right]^2 \right. \]
\[ + \frac{2}{\hbar} \left[ a_{12}(t) + a_{21}(t) \right] \hat{u}_1(0) \hat{u}_1(t) - \frac{1}{\sigma^2} \left[ a_{12}(t) + a_{21}(t) \right]^2 \} \] (2.2.10c)

2.3 Finite Temperature Quantum Uncertainty Function

To calculate the averages of observables, it is convenient to use the Wigner function associated with the reduced density matrix \( \rho_r \) defined as

\[ W_r(X, p, t) = \int_{-\infty}^{+\infty} dY \ e^{i\pi Y} \ \rho_r(X - \frac{1}{2}Y, X + \frac{1}{2}Y, t), \] (2.3.1a)

with an inverse relation given by

\[ \rho_r(X - \frac{1}{2}Y, X + \frac{1}{2}Y, t) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \ e^{-i\pi Y} \ W_r(X, p, t) \] (2.3.1b)

Applying the above Wigner transform to the reduced density matrix (2.2.7), we get

\[ W_r(X, p, t) = \int_{-\infty}^{+\infty} dY \ e^{i\pi Y} \ \tilde{Z}_0(t) \]
\[ \times \exp \left\{ -\frac{1}{2} \begin{pmatrix} X_f \\ Y_f \end{pmatrix}^T \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{pmatrix} \begin{pmatrix} X_f \\ Y_f \end{pmatrix} + \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}^T \begin{pmatrix} X_f \\ Y_f \end{pmatrix} \right\} \] (2.3.2)

where the superscript T denotes the transpose of the matrix. The quantum statistical averages of an observable of the system, e.g., \( x^n \) or \( p^n \) with respect to the reduced density matrix \( \rho_r(t) \) are given by
\[
< x^n >_T = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} x^n \, W_r(X, p, t)
\]  
(2.3.3)

and

\[
< p^n >_T = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} p^n \, W_r(X, p, t)
\]  
(2.3.4)

where the subscript \( T \) indicates that the environment is at temperature \( T \). Obviously the averages have both quantum and thermal contributions. It is easy to show that

\[
< x >_T = \frac{B_1(t)}{Q_{11}(t)}
\]  
(2.3.5)

and

\[
< x^2 >_T = \frac{1}{Q_{11}(t)} +\left[ < x >_T \right]^2
\]  
(2.3.6)

thus

\[
(\Delta x)^2 = < x^2 >_T - < x >^2_T = \frac{1}{Q_{11}(t)}
\]  
(2.3.7)

Similarly,

\[
< p >_T = i\hbar \left[ \frac{\det Q(t)}{Q_{11}(t)} \right]^{\frac{1}{2}} \left[ B_2(t) - B_1(t) \frac{Q_{12}(t)}{Q_{11}(t)} \right]
\]  
(2.3.8)

and

\[
< p^2 >_T = \hbar^2 \frac{\det Q}{Q_{11}(t)} + \left[ < p >_T \right]^2
\]  
(2.3.9)

thus

\[
(\Delta p)^2_T = < p^2 >_T - < p >^2_T = \hbar^2 \frac{\det Q}{Q_{11}(t)}
\]  
(2.3.10)
The finite temperature quantum uncertainty function defined as

\[ U_T(t) = (\Delta x)_T^2 (\Delta p)_T^2 \]  

(2.3.11)

follows:

\[ U_T(t) = \hbar^2 \frac{\text{det} Q(t)}{|Q_{11}(t)|^2} \]

(2.3.12)

\[ = \hbar^2 \left\{ \frac{Q_{22}(t)}{Q_{11}(t)} - \left[ \frac{Q_{12}(t)}{Q_{11}(t)} \right]^2 \right\} \]

Using (2.2.10a-c), one can rewrite

\[ \frac{Q_{12}(t)}{Q_{11}(t)} = if_0(t) + if_1(t) \]  

(2.3.13)

and

\[ \frac{Q_{22}(t)}{Q_{11}(t)} = g_0(t) + g_1(t) + g_2(t) \]  

(2.3.14)

where the functions \( f_n, g_n \) denote terms of order \( n \) in temperature \( T \). Explicitly they are given by:

\[ f_0(t) = \frac{\sigma^2}{\hbar} \frac{\dot{u}_2(t)}{[\dot{u}_2(0)]^2} \left\{ \frac{\hbar^2}{4\sigma^4} + \frac{\dot{u}_1(0)}{\dot{u}_2(t)} \left[ \ddot{u}_1(0) \dot{u}_2(t) - \ddot{u}_1(t) \dot{u}_2(0) \right] \right\}, \]  

(2.3.15)

\[ f_1(t) = \frac{1}{\dot{u}_2(0)} \left\{ 2 \frac{\dot{u}_2(t)}{\dot{u}_2(0)} a_{11}(t) + a_{12}(t) + a_{21}(t) \right\}, \]  

(2.3.16)

and

\[ g_0(t) = \frac{1}{4} \left[ \dot{u}_1(t) \right]^2, \]  

(2.3.17)

\[ g_1(t) = \frac{2\sigma^2}{\hbar} \frac{\dot{u}_2(t)}{[\dot{u}_2(0)]^2} \left\{ \left[ \dot{u}_1(t) \right]^2 a_{11}(t) + \dot{u}_1(0) \ddot{u}_1(t) \right\} \]

\[ \times a_{12}(t) + a_{21}(t) + \left[ \frac{\hbar^2}{4\sigma^4} + \left[ \dot{u}_1(0) \right]^2 \right] a_{22}(t) \}, \]  

(2.3.18)
\[
g_2(t) = \frac{1}{[\dot{u}_2(0)]^2} \left\{ 4a_{11}(t)a_{22}(t) - [a_{12}(t) + a_{21}(t)]^2 \right\}. \tag{2.3.19}
\]

The finite temperature uncertainty function is then formally given by

\[
U_T(t) = \left[ g_0(t) + g_1(t) + g_2(t) \right] + \left[ f_0(t) + f_1(t) \right]^2 \tag{2.3.20}
\]
3. Uncertainty Principle and Squeezed States

3.1 Finite Temperature Ohmic Environment

We now proceed to analyze the finite temperature uncertainty function for an ohmic environment. In the cases of high temperature and weak couplings, one can obtain relatively simple analytic expressions for the uncertainty relation for a general squeezed states. Results for the (unsqueezed) coherent state or the minimal uncertainty state were obtained by Hu and Zhang [5] earlier. For simplicity, we set the mass of the Brownian harmonic oscillator $M = 1$. The spectral density for ohmic dissipation is

$$I(\omega) = \gamma_0 \omega \ e^{-\frac{\omega^2}{\Lambda}} \quad (3.1.1)$$

With this, the dissipation kernel becomes

$$\gamma(s) = \gamma_0 \int_0^{+\infty} \frac{d\omega}{\pi} \ \omega \ \cos \omega s \ \ e^{-\frac{\omega^2}{\Lambda}}$$

$$= \gamma_0 \frac{\Lambda}{2\pi^{1/2}} e^{-\frac{1}{2}\Lambda^2 s^2} \quad (3.1.2)$$

If we assume that the cutoff frequency $\Lambda$ is very large, then

$$\gamma(s) \simeq \gamma_0 \delta(s) \quad (3.1.3)$$

so the damping becomes local. However, the noise kernel

$$\nu(s) = \gamma_0 \int_0^{+\infty} \frac{d\omega}{\pi} \ \omega \ \coth \frac{\omega}{2kT} \ e^{-\frac{\omega^2}{2\Lambda^2}} \ \cos \omega s$$

$$= \gamma_0 \frac{\hbar \Lambda}{2kT} e^{-\frac{\hbar \Lambda}{kT}} \ \cos \omega s \quad (3.1.4)$$

is generally nonlocal except at very high temperatures,

$$\frac{\hbar \Lambda}{kT} << 1 \quad (3.1.5)$$
whence

\[ \nu(s) \simeq \frac{2kT}{\hbar} \gamma_0 \delta(s) \]  \hspace{0.5cm} (3.1.6)

corresponding to white noise.

For local dissipation, equation (2.1.13) becomes

\[ \frac{d^2 u_i(s)}{ds^2} + \gamma_0 \frac{du_i(s)}{ds} + \left[ \Omega_0^2 - 2\gamma_0 \delta(0) \right] u_i(s) = -2\delta(s)u_i(0) \]  \hspace{0.5cm} (3.1.7)

The boundary term on the right side of equation (3.1.7) can be neglected since it gives no contribution to the solutions with the boundary conditions (2.1.14). We also can redefine the natural frequency by

\[ \left\{ \Omega_0^2 - 2\gamma_0 \delta(0) \right\} \to \Omega_0^2, \]  \hspace{0.5cm} (3.1.8)

then equation (3.1.7) becomes the familiar equation of motion for a damped harmonic oscillator. Let us define a damping parameter

\[ \alpha \equiv \frac{\gamma_0}{2\Omega} \]  \hspace{0.5cm} (3.1.9)

and the effective frequency [15]

\[ \Omega = (\Omega_0^2 - \frac{1}{4}\gamma_0^2)^{1/2} \]  \hspace{0.5cm} (3.1.10)

Then in the case of underdamping, \( \alpha < 1 \) the elementary functions simplify to

\[ u_1(s) = \frac{\sin \Omega(t - s)}{\sin \Omega t} e^{-\frac{1}{2}\gamma_0 s} \]  \hspace{0.5cm} (3.1.11a)

and

\[ u_2(s) = \frac{\sin \Omega s}{\sin \Omega t} e^{\frac{1}{2}\gamma_0 (t-s)} \]  \hspace{0.5cm} (3.1.11b)
The derivatives of these elementary functions are

\[
\begin{align*}
\dot{u}_1(0) &= -\frac{\Omega}{\sin\Omega t} - \frac{1}{2}\gamma_0 \\
\dot{u}_1(t) &= -\frac{\Omega}{\sin\Omega t} e^{-\frac{1}{2}\gamma_0 t}
\end{align*}
\tag{3.1.12a}
\]

and

\[
\begin{align*}
\dot{u}_2(0) &= \frac{\Omega}{\sin\Omega t} e^{\frac{1}{2}\gamma_0 t} \\
\dot{u}_2(t) &= \frac{\Omega}{\sin\Omega t} e^{-\frac{1}{2}\gamma_0 t} - \frac{1}{2}\gamma_0 
\end{align*}
\tag{3.1.12b}
\]

By using Eq. (3.1.12a-d), it can be shown that

\[
f_0(t) = \frac{\Omega_0}{\Omega} \left\{ 1 - \frac{\delta^2}{4\delta} \sin 2\Omega t - \alpha \frac{1 + \delta^2}{2\delta} \sin^2 \Omega t \right\} e^{-\gamma_0 t}
\tag{3.1.13}
\]

and

\[
g_0(t) = \frac{1}{4} e^{-2\gamma_0 t}
\tag{3.1.14}
\]

where we have introduced a squeeze parameter

\[
\delta = \frac{2\Omega_0\sigma^2}{\hbar}
\tag{3.1.15}
\]

which measures the spread in the initial Gaussian wavepacket. (\(\delta = 1\) corresponds to a coherent state or minimal-uncertainty state.)

It can be shown that (see Appendix)

\[
f_1(t) = \frac{2}{\Omega} \left\{ -a_0 [ss] + [sc] \right\}
\tag{3.1.16}
\]

\[
g_1(t) = \frac{\delta}{2\Omega^2} \left\{ \left[ 1 + \frac{1}{\delta^2} \right] - \frac{\gamma_0 \Omega}{\Omega^2_0} \sin 2\Omega t \right\} [ss] \\
+ \frac{\delta}{4\Omega^2_0} \left\{ \left[ 1 - \frac{1}{\delta^2} - 2 \frac{\gamma_0^2}{\Omega^2_0} \sin 2\Omega t \right] \sin^2 \Omega t \right\} [sc] \\
+ \frac{\delta}{4\Omega^2_0} \left\{ \left[ 1 + \frac{1}{\delta^2} + \frac{\gamma_0 \Omega}{\Omega^2_0} \sin 2\Omega t \right] + \left[ 1 - \frac{1}{\delta^2} - \frac{\gamma_0^2}{\Omega^2_0} \cos 2\Omega t \right] \right\} [cc]
\tag{3.1.17}
\]
and
\[
g_2(t) = \frac{4}{\Omega^2} \left\{ [ss][cc] - [sc]^2 \right\}
\]  \hspace{1cm} (3.1.18)

where the functions $[ss]$, $[sc]$ and $[cc]$ are defined in (A.1)

### 3.2 Zero Coupling Limit

Let us first examine the simplest case of zero-coupling. It corresponds to an isolated harmonic oscillator taken as a closed quantum system. We expect to recover the familiar results in quantum mechanics.

Assuming $\gamma_0 = 0$, then $\gamma(s) = 0$ and $\nu(s) = 0$. Therefore,

\[
f_1(s) = g_1(t) = g_2(t) = 0
\]  \hspace{1cm} (3.2.1)

and

\[
\Omega = \Omega_0
\]  \hspace{1cm} (3.2.2)

From Eq. (3.1.14) and (3.1.15) one gets

\[
f_0(t) = \frac{1 - \delta^2}{4\delta} \sin 2\Omega_0 t
\]  \hspace{1cm} (3.2.3)

and

\[
g_0(t) = \frac{1}{4}
\]  \hspace{1cm} (3.2.4)

We find the quantum uncertainty function for an initial squeezed states to be

\[
U_T(t) = \frac{\hbar^2}{4} \left\{ 1 + \frac{(1 - \delta^2)^2}{4\delta^2} \sin^2 2\Omega_0 t \right\} \geq \frac{\hbar^2}{4}
\]  \hspace{1cm} (3.2.5)
The time-dependent term is the result of quantum dispersion. For the (unsqueezed) coherent state, 
\[ \delta = 1 \]  
we recover the Heisenberg uncertainty relation 
\[ U_T(t) = \frac{\hbar^2}{4} \]  
With this we can also understand why \( \delta = 1 \) is called a minimum-uncertainty state.

### 3.3 High Temperature Limit

As one can see from (3.1.3) and (3.1.6), at high temperatures both the noise kernel and the dissipation kernel for ohmic dissipation become local. In this limit the functions \( f \) and \( g \) simplify to (See Appendix)
\[ f_1(t) = \alpha \tau_0 \sin^2 \Omega t e^{-\gamma_0 t} \]  
\[ g_1(t) = \tau_0 \left\{ \frac{1 + \delta^2}{4\delta} \left[ 1 - e^{-\gamma_0 t} \right] + \frac{1 + \delta^2}{2\delta} \alpha^2 \sin^2 \Omega t - \frac{1 - \delta^2}{4\delta} \alpha \sin 2\Omega t \right\} e^{-\gamma_0 t} \]  
and
\[ g_2(t) = \tau_0^2 \left\{ \frac{1}{4} - \left[ \frac{1}{2} + \alpha^2 \sin^2 \Omega t \right] e^{-\gamma_0 t} + \frac{1}{4} e^{2\gamma_0 t} \right\} \]  
where we have introduced the dimensionless parameter
\[ \tau_0 \equiv \frac{2k_B T}{\hbar \Omega_0} \equiv \frac{1}{\epsilon_0} \]  
We will also use \( \epsilon(\omega) \) and \( \epsilon \) to denote the same quantities as \( \epsilon_0 \), but with \( \omega \) and \( \Omega \) replacing \( \Omega_0 \) in (3.3.3) respectively. The uncertainty function for ohmic dissipation at the high temperature limit is then given by
\[
\frac{1}{\hbar^2} U_T(t) = \frac{1}{4} \left\{ e^{-\gamma_0 t} + \tau_0 \left[ 1 - e^{-\gamma_0 t} \right] \right\}^2 \\
+ \tau_0 \frac{(1 - \delta)^2}{4\delta} \left[ 1 - e^{-\gamma_0 t} \right] e^{-\gamma_0 t} \\
- \tau_0 \alpha \left\{ \frac{1 - \delta^2}{4\delta} \sin 2\Omega t + \alpha \left[ \tau_0 - \frac{1 + \delta^2}{2\delta} \right] \sin^2 \Omega t \right\} e^{-\gamma_0 t} \\
+ \frac{\Omega_0^2}{\Omega^2} \left\{ \frac{1 - \delta^2}{4\delta} \sin 2\Omega t + \alpha \left[ \tau_0 - \frac{1 + \delta^2}{2\delta} \right] \sin^2 \Omega t \right\}^2 e^{-2\gamma_0 t}
\]

(3.3.4)

This is the first main result of this paper.

Alternatively, since the noise kernel is also a Dirac delta function, from (2.1.15), we have

\[
a_{ij}(t) \simeq \frac{kT}{\hbar} \gamma_0 \int_0^t ds \, v_i(s)v_j(s)
\]

(3.3.5)

They are easily computed to be

\[
a_{11}(t) \simeq \frac{kT}{\hbar} \frac{\Omega^2}{2\Omega_0^2 \sin^2 \Omega t} e^{\frac{1}{2} \gamma_0 t} \left\{ 1 - \left[ \frac{\Omega_0^2}{\Omega^2} - \alpha^2 \cos 2\Omega t + \alpha \sin 2\Omega t \right] e^{-\gamma_0 t} \right\}
\]

(3.3.6a)

\[
a_{12}(t) \simeq \frac{kT}{\hbar} \frac{\Omega^2}{2\Omega_0^2 \sin^2 \Omega t} e^{\frac{1}{2} \gamma_0 t} \left\{ - \left[ \cos \Omega t - \alpha \sin \Omega t \right] + \left[ \cos \Omega t + \alpha \sin \Omega t \right] e^{-\gamma_0 t} \right\}
\]

(3.3.6b)

\[
a_{22}(t) \simeq \frac{kT}{\hbar} \frac{\Omega^2}{2\Omega_0^2 \sin^2 \Omega t} \left\{ \left[ \frac{\Omega_0^2}{\Omega^2} - \alpha^2 \cos 2\Omega t - \alpha \sin 2\Omega t \right] e^{-2\gamma_0 t} \right\}
\]

(3.3.6c)

Substituting these and (3.1.12a-d) into (2.3.16) and (2.3.18-20), we can also arrive at Eq.(3.3.2-4).

We see that there are two factors at play here: time and temperature. Time is measured in units of the relaxation time proportional to \( t_{rel} = \gamma_0^{-1} \), and temperature is measured with reference to the ground state energy \( \hbar \Omega_0/2 \) of the system. There are also
two parameters involved, \(\alpha\) and \(\delta\). Let us now take a closer look at this high temperature uncertainty function under different conditions:

a) At \(t = 0\), we find

\[
U_T(0) = \frac{\hbar^2}{4}
\]  

(3.3.7)

At \(t \gg \gamma_0^{-1}\), we find

\[
U_T(t) \approx \left(\frac{kT}{\Omega_0}\right)^2
\]  

(3.3.8)

These comply with the expected results from quantum mechanics and classical statistical mechanics as stated in the Introduction.

b) At short times \(t \ll \gamma_0^{-1}\), we can expand the uncertainty function (3.3.4) as

\[
U_T(t) \approx \frac{\hbar^2}{4} \left[1 + 2(\tau_0 \delta - 1)\gamma_0 t + O(t^2)\right]
\]  

(3.3.9)

From this we can identify the time when thermal fluctuations overtake quantum fluctuations, i.e.,

\[
t_1 \approx \frac{\hbar \Omega_0}{4kT\gamma_0\delta} \quad (high\ temperature)
\]  

(3.3.10)

which is seen to depend on \(\delta\), the squeeze parameter of the initial state. This time is identical to the decoherence time derived in [2] [15, 16]. That the transition time from quantum to thermal dominance in the uncertainty relation is related to the decoherence time scale was first noted by the present authors in [5].

c) For a minimum uncertainty initial state (\(\delta = 1\),

23
\[ \frac{1}{\hbar^2} U_T(t) = \frac{1}{4} \left\{ e^{-\gamma_0 t} + \tau_0 \left[ 1 - e^{-\gamma_0 t} \right] \right\}^2 \\
- \tau_0 \alpha^2 (\tau_0 - 1) \sin^2 \Omega t e^{-\gamma_0 t} \\
+ \frac{\Omega_0^2}{\Omega^2} \alpha^2 (\tau_0 - 1)^2 \sin^4 \Omega t e^{-2\gamma_0 t} \] (3.3.11)

For weak damping, one can neglect all \( \gamma_0 / \Omega_0 \) terms and find

\[ \frac{1}{\hbar^2} U_T(t) = \frac{1}{4} \left\{ e^{-\gamma_0 t} + \tau_0 \left[ 1 - e^{-\gamma_0 t} \right] \right\}^2 \\
+ \tau_0 (1 - \delta)^2 \left[ 1 - e^{-\gamma_0 t} \right] e^{-\gamma_0 t} \\
+ \frac{(1 - \delta^2)^2}{16 \delta^2} \sin^4 \Omega t e^{-2\gamma_0 t} \] (3.3.12)

e) For weak damping, and for an initial minimum-uncertainty state,

\[ U_T(t) = \frac{\hbar^2}{4} \left\{ e^{-\gamma_0 t} + \tau_0 \left[ 1 - e^{-\gamma_0 t} \right] \right\}^2 \] (3.3.13)

It is seen that the first term is of purely quantum nature whereas the second term is of thermal nature, their contributions to the uncertainty of the system arise from quantum and thermal fluctuations respectively.

### 3.4 Weak Damping Limit

At arbitrary but finite temperatures, the noise kernel (3.1.4) is not a Dirac delta function, and in general there is no closed form for the uncertainty function. However, if the damping is weak, we can find an approximate expression for the uncertainty function.

Assuming that

\[ \alpha \equiv \frac{\gamma_0}{2\Omega} \ll 1 \] (3.4.1)

then

\[ \frac{\frac{1}{2} \gamma_0}{(\frac{1}{2} \gamma_0)^2 + (\Omega - \omega)^2} \sim \delta(\Omega - \omega), \] (3.4.2)
which suggests that the major contributions to the integrals in (A.4a-c) come from a small region near \( \omega = \Omega \). Using Eqs. (A.14) for (3.1.16-18), we obtain

\[
f_1(t) \sim \frac{\gamma_0}{2\Omega_0} \coth \epsilon_r \sin^2 \Omega t \ e^{-\gamma_0 t} \tag{3.4.3}
\]

\[
g_1(t) \sim \frac{1}{2} \coth \epsilon_r \left\{ \frac{1 + \delta^2}{2\delta} \left[ 1 - e^{-\gamma_0 t} \right] - \frac{1 - \delta^2}{2\delta} \alpha \sin 2\Omega t \right\} e^{-\gamma_0 t} \tag{3.4.4}
\]

and

\[
g_2(t) \sim \left[ \frac{1}{2} \coth \epsilon_r (1 - e^{-2\gamma_0 t}) \right]^2 \tag{3.4.5}
\]

The uncertainty function (2.3.20) is then given by

\[
\frac{1}{\hbar^2} U_T(t) \sim \frac{1}{4} \left\{ e^{-\gamma_0 t} + \coth \epsilon_r \left[ 1 - e^{-\gamma_0 t} \right] \right\}^2 \\
+ \coth \epsilon \left\{ \frac{(1 - \delta)^2}{4\delta} \left[ 1 - e^{-\gamma_0 t} \right] - \frac{1 - \delta^2}{4\delta} \alpha \sin 2\Omega t \right\} e^{-\gamma_0 t} \\
+ \left\{ \frac{1 - \delta^2}{4\delta} \sin 2\Omega t + \alpha \left[ \coth \epsilon_r - \frac{1 + \delta^2}{2\delta} \right] \sin^2 \Omega t \right\}^2 e^{-2\gamma_0 t} \tag{3.4.6}
\]

This is the second main result of this paper. One can deduce various limits from this expression. For example, it is obvious that again at \( t = 0 \), when the initial uncorrelated conditions is assumed valid, \( U_T(0) = \hbar^2 / 4 \), which is the Heisenberg relation (1.1). At very long time (\( t >> \gamma_0^{-1} \)), \( U_T(t) \) is insensitive to \( \delta \) and approaches \( U_{T}^{RE} \) as in (1.3) at finite temperature (as \( \Omega \sim \Omega_0 \)). That means the Brownian particle approaches an equilibrium quantum statistical system. (For supraohmic bath this may not always be true). We can also see that for a \( T = 0 \) bath (\( \coth \epsilon = 1 \)), \( U_T(t) \) has a leading term given by \( \hbar^2 / 4 \) (the Heisenberg relation) followed, for squeezed states \( \delta \neq 1 \), by terms of order \( \alpha^0 \) and \( \alpha \) depicting both decay and oscillatory behavior. This is, of course, due to the action of quantum fluctuations alone.

For a minimum-uncertainty initial state (\( \delta = 1 \)), we get for all finite temperatures

\[
U_T(t) = \frac{\hbar^2}{4} \left\{ e^{-\gamma_0 t} + \coth \epsilon \left[ 1 - e^{-\gamma_0 t} \right] \right\}^2 + O(\alpha^2) \quad \text{(weak coupling)} \tag{3.4.7}
\]
This result was obtained in [5] earlier. Obviously, taking the high temperature limit of (3.4.7) reduces to (3.3.13) above.

Notice that in (3.4.7) there is no linear order damping term. In this calculation we introduced a cutoff frequency $\Lambda$ in the spectral density $I(\omega) \sim \omega \exp(-\omega^2/\Lambda^2)$ which leads to a divergent term dependent on $\Lambda$. It is removed by a frequency renormalization procedure standard in field theory. (See Eq. (A.12c) and discussions in the 1992 paper of [15] and Appendix B of the 1983 paper of [13]).

At short times ($t << \gamma_0^{-1}$),

$$U_T(t) \simeq \frac{\hbar^2}{4} \left[ 1 + 2(\delta \coth \frac{\hbar \Omega}{2kT} - 1)\gamma_0t + O(t^2) \right]$$

This simple expression is revealing in several aspects: The first term is the ubiquitous quantum fluctuation, the second term is the thermal contribution, which depends on the initial spread and increases with increasing dissipation and temperature. The time when thermal fluctuations overtake quantum fluctuations is (assuming the temperature is higher than the ground state energy):

$$t_1 = \frac{1}{2\gamma_0(\delta \coth \frac{\hbar \Omega}{2kT} - 1)}$$

Even though this result has not appeared in decoherence studies of quantum to classical transitions [15], we expect this to be equal to the decoherence time scale $t_{\text{dec}}$ calculated for weak coupling for all temperatures.
4. Discussion

Combining the generalized uncertainty relations obtained here with the recent findings of the environment-induced decoherence studies based on the same open-system framework, where two characteristic times—the decoherence time $t_{\text{dec}}$ and the relaxation time $t_{\text{rel}}$—are defined, one can address two basic issues in quantum mechanics and in (nonequilibrium) quantum statistical mechanics i.e., 1) expound the relation between quantum, thermal and classical: more specifically, depict the role of quantum and thermal fluctuations in the quantum to classical transition; and 2) explicate the physical conditions for the realization of the two basic tenets of (equilibrium) quantum statistical mechanics from quantum dynamics.

Quantum statistical mechanics of a macroscopic system is derived from the quantum dynamics of its microscopic constituents under two basic postulates \cite{18}: i) random phase, and ii) equal \textit{a priori} probability. The first condition enables one to assign probability distributions to a system occupying certain quantum states. It requires the suppression of interference terms in the wave function or that the reduced density matrix of the system be approximately diagonal. The second condition is the basis for the microcanonical ensemble. When applied to the stipulations of a canonical ensemble, it ensures that under general conditions the system when put in contact with a large bath would eventually equilibrate with it. We want to examine the processes by which these two conditions are attained from a more basic level, starting with the microdynamics of a system of quantum particles. Specifically, we want to see if there is a characteristic time when the phase information is lost (Postulate i) and another time when the system attains equilibrium with its surrounding so that all accessible states (in the combined system and environment) are equally probable (Postulate ii).

Now for the first issue which embodies the question we raised in the beginning and which underlies the premises of the second issue: On the one hand, one often regards the regime when thermal fluctuations begin to surpass quantum fluctuations as the transition
point from quantum to classical. On the other hand, from the wave picture of quantum mechanics we know that a necessary condition for a system to behave classically is that the interference terms in its wave function have to diminish below a certain level, so that probability can be assigned to classical events [2] or that classical decoherent histories can be well-defined [3]. This is known as the decoherence process. One should ask if there exists any relation between these two criteria of classicality – one based on the uncertainty relation and the other on decoherence. In particular we want to see how the uncertainty function $U_T(t)$ changes from the initial quantum fluctuation-dominated condition to a later thermal fluctuation-dominated condition. In Eq. (3.3.13) we show that under the stipulated conditions they are indeed equivalent: The time the quantum system ‘decoheres’ is also the time when thermal fluctuation overtakes quantum fluctuations.

However, we issue a warning here that the regime after thermal fluctuations dominate should not be called classical, as is customary in many quantum to classical transition studies. In fact, after the decoherence time only the first postulate of quantum statistical mechanics (QSM) is satisfied, the system can be described by *non-equilibrium* QSM. Only after the relaxation time, when the second postulate is satisfied, can the system be correctly described by *equilibrium* QSM. The classical regime starts at a much later time and it would have to be at a sufficiently high temperature. It is well-known that quantum statistical effects can be important at very high temperatures (e.g., Fermi temperature for metals). This is due to exchange interactions of identical particles, a distinctly quantum effect. Only when the statistical properties of fermions and bosons can be approximated by distinguishable particles, usually at high temperatures when the Fermi-Dirac or Bose-Einstein statistics approaches the Maxwell-Boltzmann statistics, can the system be rightfully called classical. In this regard quantum carries two meanings, one refers to the interference effect and the other refers to spin-statistics effect.

The analysis of nonohmic environments will be reported later [19]. There one expects to see some qualitatively new behavior in the case of supraohmic spectral density function
and at low temperatures, when quantum fluctuations can play a more determinant role which can make the decoherence time longer and the relaxation of the system to equilibrium more difficult to achieve.

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Appendix

Let us define

\[
[s,s] = \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 \sin \Omega s_1 \sin \Omega s_2 \ e^{-\frac{1}{2} \gamma_0 (s_1 + s_2)} \\
\quad \times \gamma_0 \int_0^{+\infty} \frac{d\omega}{\pi} \coth \epsilon(\omega) \cos \omega (s_1 - s_2)
\]  
(A.1a)

\[
[sc] = \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 \sin \Omega s_1 \cos \Omega s_2 \ e^{-\frac{1}{2} \gamma_0 (s_1 + s_2)} \\
\quad \times \gamma_0 \int_0^{+\infty} \frac{d\omega}{\pi} \coth \epsilon(\omega) \cos \omega (s_1 - s_2)
\]  
(A.1b)

\[
[cc] = \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 \cos \Omega s_1 \cos \Omega s_2 \ e^{-\frac{1}{2} \gamma_0 (s_1 + s_2)} \\
\quad \times \gamma_0 \int_0^{+\infty} \frac{d\omega}{\pi} \coth \epsilon(\omega) \cos \omega (s_1 - s_2)
\]  
(A.1c)

then

\[
a_{11}(t) = \frac{1}{\sin^2 \Omega t} \ e^{\gamma_0 t} [s,s]
\]  
(A.2a)

\[
a_{12}(t) + a_{21}(t) = 2 \frac{1}{\sin^2 \Omega t} \ e^{\frac{1}{2} \gamma_0 t} \left\{ \sin \Omega t \ [sc] - \cos \Omega t \ [s,s] \right\}
\]  
(A.2b)

\[
a_{22}(t) = \frac{1}{\sin^2 \Omega t} \left\{ \cos^2 \Omega t \ [s,s] - \sin 2 \Omega t \ [sc] + \sin^2 \Omega t \ [cc] \right\}
\]  
(A.2c)

By switching the orders of the integrations in (A.1a), (A.1b) and (A.1c), and by first performing the integrations of \(s_1\) and \(s_2\), we obtain
\[ [ss] = \frac{\Omega^2}{2\Omega_0^2} \left\{ I_{\gamma_e}(0) + \alpha I_{\omega e}(0) \right\} \]
\[ - \frac{\Omega^2}{2\Omega_0^2} \left\{ \cos \Omega t + \alpha \sin \Omega t \right\} I_{\gamma_e}(t) + \frac{\Omega^2}{\Omega_0^2} I_{\gamma_e}(t) \right\} e^{-\frac{1}{2} \gamma_0 t} \]
\[ + \alpha \left[ \cos 2\Omega t + \alpha \sin 2\Omega t \right] I_{\omega e}(0) \right\} e^{-\gamma_0 t} \]

\[ [sc] = \frac{\Omega^2}{2\Omega_0^2} \alpha \left\{ I_{\gamma_e}(0) + \alpha I_{\omega e}(0) \right\} \]
\[ - \frac{\Omega^2}{2\Omega_0^2} \alpha \left\{ \cos \Omega t + \alpha \sin \Omega t \right\} I_{\gamma_e}(t) + \left[ \alpha \cos \Omega t - \sin \Omega t \right] I_{\omega e}(t) \right\} e^{-\frac{1}{2} \gamma_0 t} \]
\[ + \frac{\Omega^2}{2\Omega_0^2} \alpha \left\{ \cos 2\Omega t + \alpha \sin 2\Omega t \right\} I_{\gamma_e}(0) \right\} e^{-\gamma_0 t} \]

\[ [cc] = \frac{\Omega^2}{2\Omega_0^2} \left\{ \left[ 1 + \frac{\gamma_0^2}{2\Omega_0^2} \right] I_{\gamma_e}(0) - \alpha I_{\omega e}(0) \right\} \]
\[ + \frac{\Omega^2}{\Omega_0^2} \left\{ \alpha \sin \Omega t - \left( 1 + \frac{\gamma_0^2}{2\Omega_0^2} \right) I_{\gamma_e}(t) + \alpha \left( \cos \Omega t + \alpha \sin \Omega t \right) I_{\omega e}(t) \right\} \]
\[ - \frac{\Omega^2}{\Omega_0^2} \sin \Omega t I_{\gamma_e}(t) \right\} e^{-\frac{1}{2} \gamma_0 t} \]
\[ + \frac{\Omega^2}{2\Omega_0^2} \left\{ \left[ \frac{\Omega_0^2}{\Omega_0^2} + \alpha^2 \cos 2\Omega t - \alpha \sin 2\Omega t \right] I_{\gamma_e}(0) \right\} \]
\[ + \alpha \left[ - \cos 2\Omega t - \alpha \sin 2\Omega t \right] I_{\omega e}(0) \right\} e^{-\gamma_0 t} \]

where

\[ I_{\gamma_e}(t) = \frac{1}{2} \int_0^{+\infty} \frac{d\omega}{\pi} \cot \theta(\omega) \left[ \gamma_- + \gamma_+ \right] \cos \omega t \] (A.4a)

\[ I_{\gamma_e}(t) = \frac{1}{2} \int_0^{+\infty} \frac{d\omega}{\pi} \cot \theta(\omega) \left[ \gamma_- - \gamma_+ \right] \sin \omega t \] (A.4b)

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\[
I_{\omega c}(t) = \frac{1}{2} \int_{0}^{+\infty} \frac{d\omega}{\pi} \cosh(\omega) \left[ \omega_- + \omega_+ \right] \cos \omega t
\]  

(A.4c)

and

\[
\gamma_{\pm} = \frac{\frac{1}{2} \gamma_0}{(\frac{1}{2} \gamma_0)^2 + (\Omega \pm \omega)^2}
\]  

(A.5a)

\[
\omega_{\pm} = \frac{\Omega \pm \omega}{(\frac{1}{2} \gamma_0)^2 + (\Omega \pm \omega)^2}
\]  

(A.5b)

(a) In the high temperature limit

\[
I_{\gamma c}(t) \approx \frac{kT}{\hbar} \int_{0}^{+\infty} \frac{d\omega}{\pi} \left[ \gamma_- + \gamma_+ \right] \cos \omega t = \frac{kT}{\hbar} e^{-\frac{1}{2} \gamma_0 t} \cos \bar{\Omega} t
\]  

(A.6a)

\[
I_{\gamma s}(t) \approx \frac{kT}{\hbar} \int_{0}^{+\infty} \frac{d\omega}{\pi} \left[ \gamma_- - \gamma_+ \right] \sin \omega t = \frac{kT}{\hbar} e^{-\frac{1}{2} \gamma_0 t} \sin \bar{\Omega} t
\]  

(A.6b)

\[
I_{\omega c}(t) \approx \frac{kT}{\hbar} \int_{0}^{+\infty} \frac{d\omega}{\pi} \left[ \omega_- + \omega_+ \right] \cos \omega t = \frac{kT}{\hbar} e^{-\frac{1}{2} \gamma_0 t} \sin \bar{\Omega} t
\]  

(A.6c)

In particular,

\[
I_{\gamma c}(0) \approx \frac{kT}{\hbar}
\]  

(A.7a)

\[
I_{\omega c}(0) \approx 0
\]  

(A.7b)

then

\[
[s s] \approx \frac{kT}{\hbar} \frac{\Omega^2}{2\Omega_0^2} \left\{ 1 - \left[ \frac{\Omega_0^2}{\Omega^2} - \alpha^2 \cos 2\bar{\Omega} t + \alpha \sin 2\bar{\Omega} t \right] e^{-\gamma_0 t} \right\}
\]  

(A.8a)
\[ [sc] \simeq \frac{kT \Omega^2}{\hbar 2\Omega_0^2} \alpha \left(1 - \left[\cos 2\Omega t + \alpha \sin 2\Omega t\right] e^{-\gamma_0 t}\right) \]  
\[ [cc] \simeq \frac{kT \Omega^2}{\hbar 2\Omega_0^2} \left[1 + \frac{\gamma_0^2}{2\Omega^2} - \frac{\Omega^2}{\Omega^2} + \alpha^2 \cos 2\Omega t - \alpha \sin 2\Omega t\right] e^{-\gamma_0 t} \]  
(A.8b)  
(A.8c)

Substituting Eq. (A.8a-c) into Eq. (3.1.17), (3.1.18) and (3.3.19), we obtain Eq. (3.3.1), (3.3.2) and (3.3.3).

(b) For weak damping,

If the damping constant is very small, i.e.,

\[ \alpha << 1 \]  
(A.9)

then

\[ \frac{1}{2} \gamma_0 \left(\frac{1}{2} \gamma_0\right)^2 + (\Omega - \omega)^2 \sim \delta (\Omega - \omega), \]  
(A.10)

which suggests that the major contributions to integrals (A.4a), (A.4b) and (A.4c) come from a small region near \( \omega = \Omega \). Thus we can make the following expansion:

\[ \frac{\omega}{\tanh \frac{\hbar \omega}{2k_B T}} \sim \frac{\Omega}{\tanh \epsilon_r} \left\{1 - \left[1 - \frac{\epsilon_r}{\sinh \epsilon_r}\right] \frac{\Omega - \omega}{\Omega}\right\} \]  
(A.11)

to get

\[ I_{\gamma_c}(t) \sim \frac{1}{2} \frac{\Omega}{\tanh \epsilon_r} \epsilon_r e^{-\frac{1}{2} \gamma_0 t} \cos \Omega t \left\{1 - \alpha \left[1 - \frac{\epsilon_r}{\sinh \epsilon_r}\right] \sin \Omega t\right\} \]  
(A.12a)

\[ I_{\gamma_s}(t) \sim \frac{1}{2} \frac{\Omega}{\tanh \epsilon_r} \epsilon_r e^{-\frac{1}{2} \gamma_0 t} \sin \Omega t \left\{1 + \alpha \left[1 - \frac{\epsilon_r}{\sinh \epsilon_r}\right] \frac{\cos \Omega t}{\sin \Omega t}\right\} \]  
(A.12b)

\[ = \frac{\sqrt{\pi}}{\tanh \epsilon_r} \left[1 - \frac{\epsilon_r}{\sinh \epsilon_r}\right] \sqrt{\Lambda} e^{-\frac{1}{2} \Lambda^2 t^2} \]

The second term drops out when a high frequency cutoff \( \Lambda \) is assumed.

\[ I_{\omega_c}(t) \sim \frac{1}{2} \frac{\Omega}{\tanh \epsilon_r} \epsilon_r e^{-\frac{1}{2} \gamma_0 t} \sin \Omega t \left\{1 - \alpha \left[1 - \frac{\epsilon_r}{\sinh \epsilon_r}\right] \sin \Omega t\right\} \]  
(A.12c)
\[
I_{1,0}(0) \sim \frac{1}{2} \Omega \frac{1}{\tanh \epsilon_r} \quad (A.13a)
\]
\[
I_{2,0}(0) \sim \frac{1}{2} \Omega \frac{1}{\tanh \epsilon_r} \alpha \left[ 1 - \frac{\epsilon_r}{\sinh \epsilon_r} \right] \quad (A.13b)
\]
Substituting (A.12a-c) and (A.13a-b) into (A.3a-c), we obtain
\[
[s.s] \simeq \frac{\Omega}{4} \coth \epsilon_r \left\{ 1 - \left[ 1 + \alpha \sin 2\Omega t \right] e^{-\gamma_0 t} \right\} \quad (A.14a)
\]
\[
[s.c] \simeq \frac{\Omega}{4} \coth \epsilon_r \alpha \left\{ 1 - \left[ \cos 2\Omega t + \alpha \sin 2\Omega t \right] e^{-\gamma_0 t} \right\} \quad (A.14b)
\]
\[
[c.c] \simeq \frac{\Omega}{4} \coth \epsilon_r \left\{ 1 - \left[ 1 - \alpha \sin 2\Omega t \right] e^{-\gamma_0 t} \right\} \quad (A.14c)
\]
References


