General relations of heavy quark-antiquark potentials

induced by reparameterization invariance*

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Abstract

A set of general relations between the spin-independent and spin-dependent potentials of heavy quark and anti-quark interactions are derived from reparameterization invariance in the Heavy Quark Effective Theory. It covers the Gromes relation and includes some new interesting relations which are useful in understanding the spin-independent and spin-dependent relativistic corrections to the leading order nonrelativistic potential.

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Understanding the heavy quark-antiquark static potential, including the spin-independent (SI) and spin-dependent (SD) parts, is of special interest in heavy quark physics because it provides useful knowledge bridging the first principles of QCD and the experimental data in the $c\bar{c}$ and $b\bar{b}$ systems in which nonperturbative QCD effects are significant. There have been two kinds of approaches to the SD potentials in the literatures. The first kind of approach is in the framework of $1/m$ expansion, where $m$ stands for the heavy-quark mass. The formulae for the SD potentials in this approach were first given by Eichten and Feinberg [1], in which the potentials were expressed in terms of certain correlation functions of color-electric and color-magnetic fields weighted by a Wilson loop factor. Assuming color-electric confinement, they evaluated the correlation functions containing color-magnetic fields to leading order in QCD perturbation. Based on an intuitive color-electric flux tube picture of color-electric confinement, Buchmüller [2] pointed out that the long-range interaction in the spin-orbit coupling potential should be of opposite sign compared with the above evaluation. Later Gromes [3] derived an important relation between the SI and SD potentials from the Lorentz invariance of the total potential and the correlation function expressions given in Ref. [1]. This relation is more fundamental, and it implies that the simple short-range assumption of the correlation functions containing color-magnetic fields is not always adequate. It is shown that Buchmüller’s result is consistent with this relation. The other kind of approach is to calculate loop contributions to the SD potentials in the full perturbative QCD theory without making $1/m$ expansion [4,5]. In this kind of approach, logarithmic $m$-dependence of the potentials emerges from the loop diagrams. Pantaleone, Tye, and Ng [5] pointed out that Gromes relation is still satisfied by the corresponding potentials in this approach to 1-loop level, however, there are extra $\ln m$-dependent spin-orbit coupling terms which are not included in the first kind of approach. It will be interesting if one can understand the potentials from a more general stand point which can cover both of these two approaches.

In this paper, we shall show that the reparameterization invariance [6–9] in the Heavy Quark Effective Theory [10] (HQET) leads to a set of general relations between the SI and
SD potentials which do cover the two approaches. We shall see that the Gromes relation is included in these relations, and there is a new relation between the SI and SD potentials which covers the property of the extra spin-orbit coupling potentials in the second approach. Furthermore, there are two more new relations between the SI potentials of different orders in the $1/m$ expansion, which are useful for understanding the spin-independent relativistic corrections to the leading order static potential.

The conventional way of studying the interaction potential between heavy quark and antiquark is to extract it from the scattering amplitude. Consider a heavy quark $Q_1$ with mass $m_1$ and a heavy antiquark $\bar{Q}_2$ with mass $m_2$. Let $p_1$ ($p_2$) and $p'_1$ ($p'_2$) be the initial-and final-state momenta of $Q_1$ and $\bar{Q}_2$, respectively. The on-shell conditions are

$$p_1^2 = p_2^2 = m_1^2, \quad p'_2 = p'_2 = m_2^2.$$  \hspace{1cm} (1)

Let $v$ be a four-velocity satisfying $v^2 = 1$, which can be $v = (1, 0, 0, 0)$ but not necessarily. As what is usually done in the HQET, we parameterize the momenta $p_1$, $p_2$, $p'_1$, $p'_2$ by

$$p_1 = m_1 v + k_1, \quad p_2 = m_2 v + k_2, \quad p'_1 = m_1 v + k'_1, \quad p'_2 = m_2 v + k'_2,$$  \hspace{1cm} (2)

where $k_1$, $k'_1$, $k_2$, and $k'_2$ are residual momenta. We consider here a nonrelativistic scattering process in which $k_i, k'_i \ll m_i v$ ($i = 1, 2$), so that $1/m$ expansion makes sense. Note that the momentum transfer

$$q \equiv p'_1 - p_1 = p_2 - p'_2$$  \hspace{1cm} (3)

is independent of the parameter $v$. In the HQET, the heavy quark field $h_{v+}(x)$ and the heavy antiquark field $h'_{v-}(x)$ are related to the heavy quark field $\psi(x)$ in the full theory by [10]

$$h_{v+}(x) \equiv P_+ h_v(x) = P_+ e^{imv \cdot x} \psi(x),$$

$$h'_{v-}(x) \equiv P_- h'_v(x) = P_- e^{-imv \cdot x} \psi(x).$$  \hspace{1cm} (4)
where $P_{\pm} \equiv \frac{1 \pm \beta}{2}$. Let $|v, +, k, s \rangle$ and $|v, -, k, s \rangle$ be, respectively, the state-vectors for a heavy quark and an antiquark in the Hilbert space. The wave functions $\mathcal{U}_{v,s}(k)$ and $\mathcal{V}_{v,s}(k)$ for the heavy quark and antiquark are defined by [10]

\begin{align}
\langle 0 | h_{v+} (x) | v, +, k, s \rangle &= \sqrt{\frac{m}{E}} \mathcal{U}_{v,s}(k) e^{-ik \cdot x}, \\
\langle 0 | h'_{v-} (x) | v, -, k, s \rangle &= \sqrt{\frac{m}{E}} \mathcal{V}_{v,s}(k) e^{ik \cdot x},
\end{align}

and the heavy quark polarization vector $s_v$ in the $v$ parameterized effective theory is related to the spin vector $s$ by [7, 9]

\begin{equation}
{s_v}^\mu = s^\mu - \frac{p^\mu + m \nu^\mu}{m + p \cdot v} s \cdot v.
\end{equation}

There are three independent momenta among $p_1$, $p'_1$, $p_2$, $p'_2$ due to momentum conservation. Introduce

\begin{equation}
\kappa_1 \equiv \frac{k_1 + k'_1}{2}, \quad \kappa_2 \equiv \frac{k_2 + k'_2}{2}.
\end{equation}

It is convenient to use $\kappa_1$, $\kappa_2$, and $q$ (cf. (3)) as three independent momentum parameters. To order-$1/m^2$, the general form of the Lorentz- and C-, P-, T-invariant Bethe-Salpeter irreducible scattering amplitude [4] can be written (in the Pauli form) as
\[ A_{\xi_1, \xi_2} (v; k_1', k_2') = \left[ U_0(q) + \frac{1}{m_1} U_1(q) v \cdot \kappa_1 + \frac{1}{m_2} U_2(q) v \cdot \kappa_2 + \frac{1}{m_1^2} U_3(q) \kappa_1^2 \right. \\
+ \frac{1}{m_1 m_2} U_4(q) \kappa_1 \cdot \kappa_2 + \frac{1}{m_2^2} U_5(q) \kappa_2^2 \left. \right] \bar{U}_{v, s_1} (k_1') U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) V_{v, s_2} (k_2') \]

\[ - \frac{i}{m_1} U_6(q) \bar{U}_{v, s_1} (k_1') \sigma_{\mu \nu} q^\mu \kappa_1^\nu U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) V_{v, s_2} (k_2') \]

\[ - \frac{i}{m_2^2} U_7(q) \bar{U}_{v, s_1} (k_1') U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) \sigma_{\mu \nu} q^\mu \kappa_2^\nu V_{v, s_2} (k_2') \]

\[ + \frac{i}{m_1 m_2} U_8(q) \bar{U}_{v, s_1} (k_1') \sigma_{\mu \nu} q^\mu \kappa_1^\nu U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) V_{v, s_2} (k_2') \]

\[ + \frac{i}{m_1 m_2} U_9(q) \bar{U}_{v, s_1} (k_1') U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) \sigma_{\mu \nu} q^\mu \kappa_1^\nu V_{v, s_2} (k_2') \]

\[ + \frac{q^2}{2 m_1 m_2} \left[ U_{10}(q) q^2 + U_{11}(q) \kappa_1 \cdot \kappa_2 \right] \bar{U}_{v, s_1} (k_1') \sigma_{\mu \nu} U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) \sigma_{\mu \nu} V_{v, s_2} (k_2') \]

\[ - \frac{1}{m_1} U_{12}(q) \bar{U}_{v, s_1} (k_1') \sigma_{\mu \nu} q^\mu U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) \sigma_{\lambda \nu} q^\lambda V_{v, s_2} (k_2') \]

\[ + \frac{1}{m_1} U_{13}(q) \bar{U}_{v, s_1} (k_1') \sigma_{\mu \nu} \kappa_1^\nu U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) \sigma_{\lambda \nu} \kappa_2^\lambda V_{v, s_2} (k_2') \]

\[ + \frac{1}{m_1} U_{14}(q) \bar{U}_{v, s_1} (k_1') \sigma_{\mu \nu} \kappa_1^\nu U_{v, s_2} (k_1) \bar{V}_{v, s_2} (k_2) \sigma_{\lambda \nu} \kappa_1^\lambda V_{v, s_2} (k_2') \]

where \( U_0(q) - U_5(q) \) are SI potentials, and \( U_6(q) - U_{14}(q) \) are SD potentials. We have explicitly extracted the heavy quark mass dependence in each term according to the following rules:

(a) for the SI terms, every small momentum \( \kappa_i \) is associated with a factor \( 1/m_i \); (b) for the SD terms, every \( \kappa_i \) is associated with a factor \( 1/m_i \), while every \( \sigma_{\mu \nu} q^\mu \) is associated with a factor \( 1/m_1 \), or \( 1/m_2 \) depending on whether \( \sigma_{\mu \nu} q^\mu \) is sandwiched in the quark wave functions or the antiquark wave functions. These rules are consistent with the spirit of \( 1/m \) expansion in the HQET. In general, the above rules may be invalid in the following cases:

i) the fundamental or the effective theory contains an unusual large tensor interaction which is not suppressed by a factor of \( 1/m \) [11];

ii) the fundamental theory contains an axial-vector couplings which is not suppressed by a factor of \( 1/m \) [11]. (in this case, \( P_\pm \gamma_5 \gamma_\mu P_\pm = \pm P_\pm i \gamma_5 \sigma_{\mu \nu} v^\nu P_\pm = \pm \frac{i}{2} P_\pm \epsilon_{\mu \nu \rho \sigma} \sigma^\rho v^\sigma P_\pm \equiv \pm P_\pm \sigma_\mu^\nu P_\pm \), where \( \sigma_\mu^\nu \) is twice of the spin operator in the HQET so that there is a leading order spin-spin interaction which is not \( 1/m \)-suppressed);

iii) the coupling is proportional to the heavy quark mass like the Yukawa coupling in the
electroweak theory;

iv) theories with pseudoscalar particle exchange, which does not have leading order static potentials [11].

In QCD none of the above cases happens, therefore these rules do apply to (8). Note that there is no order-$1/m$ SD terms in (8). This is due to the identities

$$P_\pm \sigma_{\mu\nu} v^\mu P_\pm = 0, \quad P_\pm \gamma_\mu P_\pm = \pm v_\mu$$

in the HQET.

As we have mentioned, the form (2) is only a kind of parameterization of the momenta $p_1, p_2, p'_1, p'_2$. For given $p_1, p_2, p'_1$, and $p'_2$, different sets of $v$ and $k_i, k'_i$ correspond to different parameterizations which should give the same physical predictions. In other words, (8) should be invariant against the change of the parameterization. This is the so-called reparameterization invariance in the HQET, and it has proved to be very powerful in the study of heavy flavor physics [6–9]. Let us consider an infinitesimal change of $v$ in (2),

$$v \rightarrow v' = v + \Delta v. \quad \text{(10)}$$

The constraint $v^2 = 1$ implies that

$$\Delta v \cdot v = 0. \quad \text{(11)}$$

To keep the physical momenta $p_1, p_2, p'_1, p'_2$ unchanged, the corresponding changes of $k_i$ and $k'_i$ should be

$$\Delta k_1 = \Delta k'_1 = -m_1 \Delta v, \quad \Delta k_2 = \Delta k'_2 = -m_2 \Delta v. \quad \text{(12)}$$

The general form of the infinitesimal change of the heavy quark wave functions corresponding to (10)-(12) has been given in Ref. [9]. To the order-$1/m$, it is [7,9]

$$\Delta U_{v,s}(k) = \frac{N_c}{2} \left( 1 + \frac{k}{2m} \right) U_{v,s}(k)$$

$$\Delta V_{v,s}(k) = -\frac{N_c}{2} \left( 1 - \frac{k}{2m} \right) V_{v,s}(k) \quad \text{(13)}$$
The potentials $U_0(q), \cdots, U_{14}(q)$ in (8) are physical quantities which should be invariant under the reparameterization transformation (10)-(12). Thus, the infinitesimal of the scattering amplitude $A_{z',s_1s_2s_2}(v; k_1', k_1, k_2', k_2)$ can be easily worked out from (10)-(13). To order-1/$m$, it reads

$$\Delta A_{z',s_1s_2s_2}(v; k_1', k_1, k_2', k_2)$$

$$= \left\{ [U_0(q) + 2U_1(q) - 4U_3(q) - 2U_4(q)] \frac{\kappa_1 \cdot \Delta v}{2m_1} + [U_0(q) + 2U_1(q) - 4U_3(q) - 2U_4(q)] \frac{\kappa_2 \cdot \Delta v}{2m_2} \right\} \tilde{U}_{v,s_1}(k_1) \tilde{U}_{v,s_2}(k_1) \tilde{U}_{v,s_2}(k_2) V_{v,s_2}(k_2)$$

$$+ \frac{i}{4m_1} [U_0(q) + 4U_0(q) - 4U_6(q)] \tilde{U}_{v,s_1}(k_1') \sigma_{\mu
u} q^\mu \Delta u^\nu U_{v,s_1}(k_1) \tilde{U}_{v,s_2}(k_2) V_{v,s_2}(k_2)$$

$$+ \frac{i}{4m_2} [U_0(q) + 4U_7(q) - 4U_5(q)] \tilde{U}_{v,s_1}(k_1') \sigma_{\mu
u} q^\mu \Delta u^\nu U_{v,s_1}(k_1) \tilde{U}_{v,s_2}(k_2) V_{v,s_2}(k_2)$$

$$- \left( \frac{\kappa_1 \cdot \Delta v}{2m_1} + \frac{\kappa_2 \cdot \Delta v}{2m_2} \right) U_{11}(q) \tilde{U}_{v,s_1}(k_1') \sigma_{\mu
u} U_{v,s_1}(k_1) \tilde{U}_{v,s_2}(k_2) V_{v,s_2}(k_2)$$

$$- \frac{1}{m_2} U_{13}(q) \tilde{U}_{v,s_1}(k_1') \sigma_{\mu
u} U_{v,s_1}(k_1) \tilde{U}_{v,s_2}(k_2) V_{v,s_2}(k_2)$$

$$- \frac{1}{m_1} V_{13}(q) \tilde{U}_{v,s_1}(k_1') \sigma_{\mu
u} \Delta u^\nu U_{v,s_1}(k_1) \tilde{U}_{v,s_2}(k_2) V_{v,s_2}(k_2)$$

$$- \frac{1}{m_1} U_{14}(q) \tilde{U}_{v,s_1}(k_1') \sigma_{\mu
u} \Delta u^\nu U_{v,s_1}(k_1) \tilde{U}_{v,s_2}(k_2) V_{v,s_2}(k_2)$$

For arbitrary $\kappa_1$ and $\kappa_2$, reparameterization invariance requires

$$U_0(q) + 2U_1(q) - 4U_3(q) - 2U_4(q) = 0,$$  \hfill (15)

$$U_0(q) + 4U_2(q) - 4U_5(q) - 2U_4(q) = 0,$$  \hfill (16)

$$U_0(q) + 4U_6(q) - 4U_8(q) = 0,$$  \hfill (17)

$$U_0(q) + 4U_7(q) - 4U_9(q) = 0,$$  \hfill (18)

$$U_{11}(q) = 0, \quad U_{13}(q) = 0, \quad U_{14}(q) = 0.$$  \hfill (19)

These are the general relations between the potentials in the momentum representation that we obtain from the reparameterization invariance of the scattering amplitude. We see that reparameterization invariance does not give any constraints on $U_{10}(q)$ and $U_{12}(q)$.
From the general symmetry argument, there can be terms containing $U_{11}(q)$, $U_{13}(q)$, $U_{14}(q)$ in (8). However, (19) shows that these terms are not consistent with reparameterization invariance, and hence they should actually vanish. This explains why these terms never appear in the results obtained from specific dynamical calculations [1–5].

Eqs. (15) and (16) relate the leading order SI potential $U_0(q)$ to its SI relativistic corrections $U_1(q), U_2(q), U_3(q)$, $U_4(q)$, and $U_5(q)$. These relations have never been shown in the literatures. Although the two relations are still not enough to fix all the five potentials $U_1(q)$-$U_5(q)$, they are at least helpful in understanding the general properties of the SI relativistic corrections to $U_0(q)$, which is one of the difficult problems in heavy quark physics.

Eqs. (17) and (18) relate $U_0(q)$ to its SD relativistic corrections. They are related to the Gromes relation. To see this precisely, we make the Fourier transformation of Eq. (14) and derive the relations between the potentials in the space-time representation corresponding to (17) and (18). Let $U_j(r)$ be the Fourier transform of $U_j(q)$ with given $Q_1$-$Q_2$ separation $r$. The relations corresponding to (17) and (18) in terms of $U_j(r)$ are

$$\frac{d}{dr} [U_0(r) + 4U_6(r) - 4U_8(r)] = 0,$$  \hspace{1cm} (20)

$$\frac{d}{dr} [U_0(r) + 4U_7(r) - 4U_9(r)] = 0.$$  \hspace{1cm} (21)

To compare this with the standard formulae given in Refs. [1–5], we also make the Fourier transformation of (8). The relevant terms in the space-time representation potential in (8), when taking $v = (1, 0, 0, 0)$, are
\[ V(r) = U_0(r) + \left( \frac{S_1}{m_1^2} + \frac{S_2}{m_2^2} \right) \cdot L \frac{1}{r} \frac{d}{dr} [U_6(r) + U_7(r)] \]
\[ + \left( \frac{S_1 + S_2}{m_1 m_2} \right) \cdot L \frac{1}{r} \frac{d}{dr} [U_8(r) + U_9(r)] \]
\[ + \frac{4}{m_1 m_2} \frac{(S_1 \cdot r)(S_2 \cdot r) - \frac{1}{3} S_1 \cdot S_2 r^2}{r^2} \nabla^2 U_{12}(r) \]
\[ + \frac{4}{m_1 m_2} S_1 \cdot S_2 \nabla^2 \left[ U_{10}(r) - \frac{2}{3} U_{12}(r) \right] \]
\[ + \left( \frac{S_1}{m_1^2} - \frac{S_2}{m_2^2} \right) \cdot L \frac{1}{r} \frac{d}{dr} [U_6(r) - U_7(r)] \]
\[ + \left( \frac{S_1 - S_2}{m_1 m_2} \right) \cdot L \frac{1}{r} \frac{d}{dr} [U_8(r) - U_9(r)] + \cdots, \]

where \( S_1 \) and \( S_2 \) are, respectively, the spins of \( Q_1 \) and \( Q_2 \), and \( L \) is the relative orbital angular momentum. Let

\[
\begin{align*}
V_0(r) &\equiv U_0(r), & V_1(r) &\equiv U_6(r) + U_7(r) - \frac{1}{2} U_6(r), \\
V_2(r) &\equiv U_8(r) + U_9(r), & V_3(r) &\equiv 4 U_{12}(r), \\
V_4(r) &\equiv 12 \left[ U_{10}(r) - \frac{2}{3} U_{12}(r) \right], \\
V_5(r) &\equiv U_6(r) - U_7(r), & V_6(r) &\equiv U_8(r) - U_9(r).
\end{align*}
\]

In terms of \( V_0(r), \cdots, V_6(r) \), (22) can be written in the standard form

\[
\begin{align*}
V(r) &= V_0(r) + \left( \frac{S_1}{m_1^2} + \frac{S_2}{m_2^2} \right) \cdot L \frac{1}{r} \left[ \frac{1}{2} \frac{dV_0(r)}{dr} + \frac{dV_1(r)}{dr} \right] + \left( \frac{S_1 + S_2}{m_1 m_2} \right) \cdot L \frac{1}{r} \frac{dV_2(r)}{dr} \\
&\quad + \frac{1}{m_1 m_2} \frac{(S_1 \cdot r)(S_2 \cdot r) - \frac{1}{3} S_1 \cdot S_2 r^2}{r^2} \nabla^2 V_3(r) + \frac{1}{3} \frac{1}{m_1 m_2} S_1 \cdot S_2 \nabla^2 V_4(r) \\
&\quad + \left( \frac{S_1}{m_1^2} - \frac{S_2}{m_2^2} \right) \cdot L \frac{1}{r} \frac{dV_5(r)}{dr} + \left( \frac{S_1 - S_2}{m_1 m_2} \right) \cdot L \frac{1}{r} \frac{dV_6(r)}{dr} + \cdots,
\end{align*}
\]

which covers the forms given in Refs. [1–5]. In terms of \( V_0(r), \cdots, V_6(r) \), the relations (20) (21) read

\[
\frac{d}{dr} [V_0(r) + V_1(r) - V_2(r) + V_5(r) - V_6(r)] = 0, \tag{25}
\]
\[
\frac{d}{dr} [V_0(r) + V_1(r) - V_2(r) - V_5(r) + V_6(r)] = 0, \tag{26}
\]

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and from which we obtain

\[
\frac{d}{dr} \left[ V_0(r) + V_1(r) - V_2(r) \right] = 0, \quad (27)
\]

\[
\frac{dV_5(r)}{dr} = \frac{dV_6(r)}{dr}. \quad (28)
\]

Eq. (27) is just the well-known Gromes relation, and Eq. (28) gives a new relation between the two extra spin-orbit coupling potentials \( \left( \frac{S_1}{m_1^2} - \frac{S_2}{m_2^2} \right) \cdot I L \frac{1}{r} \frac{dV_5(r)}{dr} \) and \( \left( \frac{S_1 - S_2}{m_1 m_2} \right) \cdot I L \frac{1}{r} \frac{dV_6(r)}{dr} \).

It is shown in Ref. [5] that the 1-loop perturbative QCD results in the second kind of approach do satisfy the relations (27) and (28). Now we see that (27) and (28) are general properties of the potentials, i.e. they are valid to all orders in perturbative QCD in the second kind of approach, and even beyond perturbation.

In summary, we have derived, from reparameterization invariance, a set of general relations between the SI and SD potentials of heavy quark and antiquark interactions, namely eqs. (15)-(19). It includes the Gromes relation (27) and a new relation (28), which cover all the results of the two kinds of approaches to the SD potentials [1-5]. Eqs. (15) and (16) are two new relations between various SI potentials, which are useful for understanding the SI relativistic corrections to the leading order static potential \( V_0(r) \). Eq. (19) contains three general restrictions to the form of the total potential showing that the three terms containing \( U_{11}(q) \), \( U_{13}(q) \), and \( U_{14}(q) \) in (8) should actually vanish. Reparameterization invariance does not give any constraints to the hyperfine and the tensor potentials.
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