Swimming of Microorganisms
Viewed from String and Membrane Theories

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Abstract

Swimming of microorganisms is studied from a viewpoint of extended objects (strings and membranes) swimming in the incompressible fluid of low Reynolds number. The flagellated motion is analyzed in two dimensional fluid, by using the method developed in the ciliated motion with the Joukowski transformation. Discussion is given on the conserved charges and the algebra which are associated with the area (volume)-preserving diffeomorphisms giving the swimming motion of microorganisms. It is also suggested that the $N$-point string- and membrane-like amplitudes are useful for studying the collective swimming motion of microorganisms when fluctuation of the vorticity distribution exists in the sticky or rubber-like fluid.
1 Introduction

In the high-energy physics, we usually study the world with the length scale of $10^{-12} \sim 10^{-29}\mu m$, where field theories of particles as well as strings and sometimes exotic membranes play the important roles in describing theoretically the phenomena of this tiny world. Unfortunately, we have never experimentally observed the dynamics of fission and fusion of strings and membranes, except for the old-aged hadronic strings. A lot of theoretical machineries have been so far developed to describe the dynamics of these extended objects, examples of which are conformal field theories based on Virasoro, Kac-Moody and W-algebras (area-preserving diffeomorphisms), the field theory of strings and so on. Therefore, it is by no means a waste of time to search for another region of physics where the knowledge of high-energy physics will probably be useful.

In this paper, we take up the swimming problem of microorganisms as such an interesting application, where we know the following fact on the ways of swimming [1]: There exist only three different universality classes of the swimming ways of microorganisms; (1) Swimming with cilia is adopted by the spherical organisms with the length scale of $20 \sim 2 \times 10^9\mu m$, an example of which is paramecium; (2) the smaller microorganisms with the size of $1 \sim 50\mu m$ swim with flagella, an example of which is the sperm; (3) the bacteria with the size of $0.2 \sim 5\mu m$ swim with bacterial flagella the motion of which resembles the screwing of the wine-opener. Why is it possible that such simple classification be realized in the swimming problem of microorganisms?

Recently, Shapere and Wilczek [2] have studied the swimming of microorganisms from the gauge theoretical viewpoint; Knowing one cycle of the swimming motion gives a closed path on the space of shapes, they have found that the net self-propulsion of translation and rotation in the low Reynolds liquid are expressed in terms of the Wilson operator on the closed path. Starting from their works, we will develop the ciliated and flagellated motions and study on our problem from the viewpoint of algebraic structure existing in the deformation operation of the microorganisms in the low Reynolds liquid, as well as from the swimming dynamics of a group of microorganisms, which may be connected to the $N$-point correlation function of string and membrane theories.

Let us recall the hydrodynamics. The Reynolds number $R$, giving the
ratio of the kinetic term over the viscosity term by \( R = \rho V L / \mu \), where \( \rho, V, L \), and \( \mu \) denote density, typical velocity, typical length scale and the coefficient of viscosity, respectively, satisfies

\[
R \leq 1 \times [L/\text{mm}],
\]

for the velocity of microorganisms \( V \leq 1\text{mm/s} \) with \( \rho = 1\text{g/cm}^3 \) and \( \mu = 10^{-3}\text{Pas} \) of water. Therefore, for the microorganisms with the length \( L \ll 1\text{mm} \), we have \( R \ll 1 \) (the low Reynolds number fluid) which leads to the following equations of motion for the incompressible fluid:

\[
\nabla \cdot \mathbf{v} = 0,
\]

and

\[
\Delta \mathbf{v} = \frac{1}{\mu} \nabla p,
\]

or equivalently

\[
\Delta (\nabla \times \mathbf{v}) = 0,
\]

where \( p \) is the pressure and \( \mathbf{v}(x) \) is the velocity field of the fluid. The surface of a microorganism swimming in the \( D = 3 \) dimensional fluid of the real world forms a closed membrane at a fixed time \( t \), the position of which can be parametrized by introducing \( (D - 1) \) parameters \( \xi^i \) \((i = 1, \cdots, D - 1)\) as \( X^\mu = X^\mu(t; \xi^1, \cdots, \xi^{D-1}) \). It is sometimes instructive to consider the \( D = 2 \) dimensional fluid. Then, the surface of a ciliate (flagellate) becomes a closed (open) string and its position can be described by a complex number

\[
Z = x^1 + ix^2 = Z(t; \theta),
\]

with \(-\pi \leq \theta \leq \pi\). In the sticky fluid of \( R = 0 \), there is no slipping between the surface of a microorganism and the fluid, namely, we have the matching condition

\[
\mathbf{v}(x = X(t; \xi)) = \dot{X}(t; \xi),
\]

or in the general coordinate system with the metric tensor \( g^{\mu\nu}(x) \)

\[
g^{\mu\nu}(x) v_\mu(x)|_{x = X(t; \xi)} = \dot{X}^\nu(t; \xi).
\]

Now we will exemplify the ciliated as well as flagellated motion of microorganisms in \( D = 2 \) dimensional fluid. The ciliated motion was attacked by Shapere and Wilczek several years ago, so that our main concern will be the flagellate motion, but we first discuss the ciliated motion from a bit different viewpoint by summarizing our notations.
2 The ciliated motion in the D=2 fluid

Time evolution of the envelope of the cilia can be viewed as a small but
time-dependent deformation of a unit circle in a properly chosen scale,

\[ Z(t, \theta) = s + \alpha(t, s), \quad (7) \]

where \( s = e^{i\theta} \) and \( \alpha(t, s) \) is arbitrary temporally periodic function with period
\( T \) satisfying \( |\alpha(t, s)| \ll 1 \) with \( -\pi \leq \theta \leq \pi \). The complex representation of
the covariant velocity vector \( v_\mu \) can be denoted as

\[ 2v_x(z, \bar{z}) = (v_1 + i v_2)(z, \bar{z}) \quad (8) \]
\[ 2v_x(z, \bar{z}) = (v_1 - i v_2)(z, \bar{z}) \quad (9) \]

since their transformation property under the conformal transformation \( z = f(w) \) reads

\[ v_x(z, \bar{z}) = \frac{\partial w}{\partial \bar{z}} v_x(w, \bar{w}) \quad (10) \]
\[ v_z(z, \bar{z}) = \frac{\partial w}{\partial z} v_x(w, \bar{w}). \quad (11) \]

The matching condition (6) is then given by

\[ 2v_x \left( s + \alpha(t, s), s^{-1} + \bar{\alpha}(t, s) \right) = \dot{\alpha}(t, s). \quad (12) \]

The factors 2 in Eqs.(8), (9) and (12) come from the flat two dimensional
metric expressed in terms of \( z \) and \( \bar{z} \), that is, \( g^{\bar{z}z}(z, \bar{z}) = 2 \). Perturbatively, Eq.(12) can be solved as

\[ 2v_x(s) = \dot{\alpha}(t, s) - 2 \left[ \alpha(t, s) \partial_z v_x^{(0)} + \bar{\alpha}(t, \bar{s}) \partial_{\bar{z}} v_x^{(0)} \right] \bigg|_{z=s} \quad (13) \]

with the solution of the lowest order \( v_x^{(0)}(z, \bar{z}) \) satisfying

\[ 2v_x^{(0)}(z, \bar{z}) \bigg|_{z=s} = \dot{\alpha}(t, s). \quad (14) \]

The general solution of the velocity vector for the \( R \approx 0 \), incompressible
liquid is already known; the equations of motion (2) and (3) are then

\[ \partial_z v_x + \partial_{\bar{z}} v_x = 0 \quad (15) \]
and 

\[ 4 \partial_z \partial \bar{z} v_z = \frac{1}{\mu} \partial_z p \]  

or equivalently 

\[ \partial_z \partial \bar{z} (\partial_z v_z - \partial_{\bar{z}} v_z) = 0. \]

Introducing the stream function \( U(z, \bar{z}) \), Eq.(15) is solved as \( v_z = \partial_\bar{z} U \) and \( v_{\bar{z}} = -\partial_z U \), and Eq.(17) in terms of \( U \), \((\partial_z \partial \bar{z})^2 U(z, \bar{z}) = 0\), has a general solution with two arbitrary holomorphic functions \( \phi_1(z) \) and \( \phi_0(z) \) as

\[ U(z, \bar{z}) = z \phi_1(z) - \overline{\phi_1(z)} + \overline{\phi_0(z)} - \phi_0(z). \]

Now, we have the general solution for \( v_z \) as

\[ v_z(z, \bar{z}) = \phi_1(z) - z \phi_1'(z) + \phi_2(z), \]

where \( \phi_2(z) = \phi_0'(z) \). If we require the velocity field \( v_z \) be finite at spacial infinity, we have the expansion,

\[ \phi_1(z) = \sum_{k<0} a_k z^{k+1} \quad \text{and} \quad \phi_2(z) = \sum_{k<-1} b_k z^{k+1}. \]

Following Shapere and Wilczek, self-propulsion of translation and rotation against the fluid can be extracted as the counterflows of \(-a_{-1}\) and \(-\text{Im} b_{-2}\), since the translational and rotational flows, \( a_{-1} \) and \( \text{Im} b_{-2} \), surviving even at spacial infinity after solving Eq.(5) or (12) should be cancelled by the self-propulsion of the microorganism. Here we will find the following useful property; if the velocity vector is known on the unit circle as \( v(s) = \sum_{k=-\infty}^{+\infty} v_k s^{k+1} \), then the velocity vector at any place can be easily determined as

\[ v_z(z, \bar{z}) = v^{(-)}(z) + v^{(+)}(\bar{z}) + (z^{-1} - z) v^{(-)}(\bar{z}), \]

with the help of

\[ v^{(+)}(\bar{z}) \equiv \sum_{k \geq 0} v_k \bar{z}^{-k-1} \]

\[ v^{(-)}(z) \equiv \sum_{k < 0} v_k z^{k+1}. \]
Then from Eqs. (13), (14), and (21), we obtain a general expression of \( v_z \) caused by the time dependent small deformation of \( \alpha(t, s) \) as

\[
2v_z(s) = \dot{\alpha}(t, s) - \left\{ \alpha(t, s) \left( \partial_x \dot{\alpha}^{(+)}(t, s) - \frac{\partial_t}{t} \dot{\alpha}^{(-)}(t, s) \right) \right. \\
\left. + \frac{\alpha(t, s)}{t} \left( \partial_x \dot{\alpha}^{(+)}(t, s) - \frac{s^2 \partial_t}{t} \dot{\alpha}^{(-)}(t, s) \right) \right\}. \tag{24}
\]

From this expression, the net translationally swimming velocity \( v_T^{(\text{cilia})} \) of the ciliated microorganism reads

\[
2v_T^{(\text{cilia})} = -\dot{\alpha}_0(t) + \sum_{n\leq 1} n(\dot{\alpha}_n \alpha_{n+1} - \overline{\alpha}_n \overline{\alpha}_{n+1} - \overline{\alpha}_n \overline{\alpha}_{n+1}) - \sum_{n>1} n\omega \overline{\alpha}_n \overline{\alpha}_{n-1}, \tag{25}
\]

where \( \alpha_n(t) \) is defined by \( \alpha(t, s) = \sum_{n=-\infty}^{+\infty} \alpha_n(t) s^n \). On the other hand, the net angular momentum \( v_R^{(\text{cilia})} \) gained by the microorganism from the fluid becomes

\[
2v_R^{(\text{cilia})} = -\text{Im} \left\{ \dot{\alpha}_1(t) + \sum_{n \leq 1} n(\dot{\alpha}_n \alpha_{n+2} - \overline{\alpha}_n \overline{\alpha}_{n+2} - \overline{\alpha}_n \overline{\alpha}_{n+2}) + \sum_{n > 1} n\omega \overline{\alpha}_n \alpha_n \right\}. \tag{26}
\]

The net translation and rotation resulted after the period \( T \) come from \( O(\alpha^2) \) terms since the \( O(\alpha) \) terms cancell after the time integration over the period.

3 The flagellated motion in D=2 fluid

Microorganisms swimming using a single flagellum can be viewed as an open string with two end points, H and T, where H and T represent the head and the tail-end of a flagellum, respectively. Our discussion will be given by assuming that the distance between H and T is time-independent and is chosen to be 4 in a proper length scale. This assumption can be shown to be
valid for the flagellated motion by small deformations in the incompressible fluid[7]. Then, at anytime \( t \), we can take a complex plane of \( z \), where \( H \) and \( T \) are fixed on \( z = 2 \) and \(-2\), respectively. This coordinate system \( z \) can be viewed as that of the space of *standard shapes* of Shapere and Wiltzdek. Time dependent, but small deformation of the flagellate can be parametrized as

\[
Z(t, \theta) = 2(\cos \theta + i \sin \theta \alpha(t, \theta)),
\]

where the small deformation \( \alpha(t, \theta) \) can be taken to be a real number\(^1\) satisfying

\[
\alpha(t, \theta) = -\alpha(t, -\theta).
\]

Here, we parametrize the position of the flagellum twice, starting from the end point \( T \) at \( \theta = -\pi \), coming to the head \( H \) at \( \theta = 0 \), and returning to \( T \) again at \( \theta = \pi \). Motion of the two branches corresponding to \(-\pi \leq \theta \leq 0\) and \( \pi \geq \theta \geq 0 \) should move coincidentally, which requires the condition (28). The Joukowksi transformation \( z = z(w) = w + w^{-1} \), separates the two coincident branches in the \( z \) plane to form lower and upper parts of an unit circle in the \( w \) plane, outside domain of which we are able to study the swimming problem of the flagellate in a quite similar fashion to that of the ciliate. The parametrization of our microorganism in the \( w \) plane corresponding to Eq.(27) is now

\[
W(t, \theta) = e^{i\theta}(1 + \alpha(t, \theta)) + O(\alpha^2).
\]

The general solution of the velocity field \( v_z \) in the \( z \) plane can be transferred to that of \( v_w \) in the \( w \) plane, by using the transformation property (10), that is, we have generally

\[
v_z(w, z) = \psi_1(w) - z(w) \frac{\partial \bar{\psi}_1}{\partial z}(w) + \bar{\psi}_2(w),
\]

where the two arbitrary holomorphic functions have the following expansion

\[
\psi_1(w) = \sum_{k<0} a_k w^{k+1} \quad \text{and} \quad \psi_2(w) = \sum_{k<-1} b_k w^{k+1}.
\]

\(^1\)When \( \alpha \) is taken to be a complex number, the length of the flagellum is locally changeable at \( O(\alpha) \). For such an elastic flagellum, we have similar results to that of the ciliated motion. In case of real \( \alpha \), its length is locally preserved at \( O(\alpha) \), giving a non-elastic flagellum, which is the more realistic one.
In the study of the ciliated motion, the property (21) was extremely useful; the corresponding expression in the flagellated motion can also be obtained from a given velocity vector on the unit circle in the $w$ plane $v_w = \sum_{k=-\infty}^{\infty} v_k s^{k+1}$, by using the general solution of (30) and (31). In terms of velocity field of the real space, we have

\[
v_z(w, \bar{w}) = (1 - w^2)^{-1} \left( \alpha + \beta w^{-1} - w^{-2} v^(-) (w) \right) + (1 - \bar{w}^{-2})^{-1} \left( \alpha \bar{w}^{-2} + \beta \bar{w}^{-1} + v^{(+)} (\bar{w}) \right) + (\bar{w} - \bar{w}^{-1})^{-3} \left\{ (w + w^{-1}) - (\bar{w} + \bar{w}^{-1}) \right\} \times \left\{ \bar{\beta}(\bar{w} + \bar{w}^{-1}) + 2\bar{w} - 2v^(-)(w) + (\bar{w} - \bar{w}^{-1})v^{(-)}(w) \right\},
\]

where $v^{(\pm)}$ are defined by

\[
v^{(+)}(\bar{w}) \equiv \sum_{k=0} v_k \bar{w}^{-k-1} \quad \text{(34)}
\]

\[
v^{(-)}(w) \equiv \sum_{k=0} v_k w^{k+1}. \quad \text{(35)}
\]

The constants $\alpha$ and $\beta$ in Eq.(33) are fixed so that $v_z$ has no singularities on the special points of the head H and tail-end T. From this requirement, we have

\[
\alpha = \frac{1}{2} \left( v^{(-)}(1) + v^{(-)}(-1) \right) = \sum_{l<0} v_{2l+1} \quad \text{(36)}
\]

\[
\beta = \frac{1}{2} \left( v^{(-)}(1) - v^{(-)}(-1) \right) = \sum_{l>0} v_{2l} \quad \text{(37)}
\]

as well as

\[
v_z(w, \bar{w}) \big|_{w=\bar{w}=\pm 1} = 0, \quad \text{(38)}
\]

which is quite consistent to our requirement of fixing the head and tail-end position in $z$ and $w$ planes.

Next task is to solve the matching condition of (6) which is written in the $z$ plane as

\[
2v_z(W(t, \theta), \bar{W}(t, \theta)) = \dot{Z}(t, \theta). \quad \text{(39)}
\]
Perturbatively, the solution of (39) reads

\[ 2v_z(s) = (s - s^{-1}) \dot{\alpha}(t, \theta) - 2 \alpha(t, \theta) \frac{d}{dt} v_z^{(0)} \left|_{w=s} \right. \]

(40)

with the help of the lowest order solution \( v_z^{(0)}(w, \bar{w}) \) satisfying

\[ 2v_z^{(0)} \left|_{w=s} \right. = (1 - s^2)(s - s^{-1}) \dot{\alpha}(t, \theta) \]

(41)

on the unit circle \((s = e^{i\theta})\). Using the mode expansion satisfying Eq.(28),

\[ \alpha(t, \theta) = \sum_{n=1}^{\infty} \alpha_n(t) \sin n\theta, \]

(42)

the lowest order solution is obtained:

\[ 2iv_z^{(0)}(w, \bar{w}) \]

(43)

\[ = - \left\{ (w - w^{-1}) \sum_{n \geq 1} \dot{\alpha}_n w^{-n} + (\bar{w} - \bar{w}^{-1}) \sum_{n \geq 1} \dot{\alpha}_n \bar{w}^{-n} \right\} \]

\[ + \left\{ (w + w^{-1}) - (\bar{w} + \bar{w}^{-1}) \right\} \]

\[ \times \left\{ \sum_{n \geq 1} n \dot{\alpha}_n \bar{w}^{-n} - \frac{\bar{w} + \bar{w}^{-1}}{\bar{w} - \bar{w}^{-1}} \sum_{n \geq 1} \dot{\alpha}_n \bar{w}^{-n} \right\}. \]

(44)

Then, from Eqs.(40)~(44), we have derived the velocity of the fluid which incorporates perturbatively the small but time-dependent deformation of the flagellate as

\[ 2v_z \left|_{w=e^{i\theta}} \right. = 2i \sin \theta \sum_{n \geq 1} \dot{\alpha}_n(t) \sin n\theta \]

\[ - 4 \sum_{n, m \geq 1} \alpha_n(t) \dot{\alpha}_m(t) \sin n\theta \cos \theta \sin m\theta + m \sin \theta \cos m\theta. \]

(45)

Now, we are able to determine the net swimming velocity \( v_T^{(\text{flagella})} \) gained by the flagellate motion of microorganisms:

\[ 2v_T^{(\text{flagella})} = -i \dot{\alpha}_1 - \sum_{m \geq 1} m\alpha_m \dot{\alpha}_{m+1} + \sum_{m \geq 2} m\alpha_m \dot{\alpha}_{m-1}, \]

(46)
On the other hand, the angular momentum $v_R^{(flagella)}$ is given by 

$$2v_R^{(flagella)} = -\frac{1}{2}\Delta z.$$  \hspace{1cm} (47)

After the time integration over the period $T$, $v_R^{(flagella)}$ vanishes since in our first order approximation, the length of the flagellum is fixed in the incompressible fluid. Therefore the second order approximation is necessary for the non-vanishing $v_R^{(flagella)}$.

### 4 The selection rules and the symmetry of microorganisms’ swimming

Even though the results in Eqns.(25), (26), (46) and (47) are obtained perturbatively, we are able to read from them the characteristics of the microorganisms’ swimming; In order for the ciliates to swim or rotate, they need the coexistence of the two different Fourier modes of $n_1$, $n_2$. The selection rules for the allowed $(n_1, n_2)$ combinations are

1. $|n_1 - n_2| = 1$ or $|n_1 + n_2 - 2| = 1$ for the ciliated translation.
2. $n_1 = n_2$ or $|n_1 + n_2| = 2$ for the ciliated rotation.

The corresponding selection rules for the flagellate motion are

3. $|n_1 - n_2| = 1$ for the flagellate translation,

where the Fourier modes are $\sin n\theta$ in this case.

Viewing these selection rules, we are tempted to elucidate the algebraic structure possibly existing in the background of the swimming mechanism. It is similar to the Virasoro algebra, but is different from it. For such a purpose, introduction of the “action” will be convenient. The “action” $S$ reproducing the classical equations of motion of the swimming of $N$ microorganisms in the incompressible liquid with low Reynolds number may be given by

$$S_N = \sum_{i=1}^{N} \int dt \int d^{D-1}\xi(t) f^{(i)}(t;\xi(t)) \left[\dot{X}^\mu(t;\xi(t)) - v^\mu(X(t;\xi(t)))\right]$$

$$+ \frac{1}{2\pi\alpha'} \int d^D x \sqrt{g(x)} \left[ -\frac{1}{\mu} p(x) \partial_\nu v^\nu(x) + \frac{1}{4} \omega_{\mu
u}(x) \omega^{\mu\nu}(x) \right].$$  \hspace{1cm} (51)
where the velocity field $\omega_{\mu\nu}(x)$ is given by

$$\omega_{\mu\nu}(x) = \partial_\mu v_\nu - \partial_\nu v_\mu,$$

whose $D = 2$ expression is $\omega_{z,\bar{z}}(z, \bar{z}) = \partial_z v_\bar{z} - \partial_{\bar{z}} v_z$.

We have introduced the parameter $\alpha'$ so as to make $S$ dimensionless, where $\alpha'$ has the dimension of $(\text{Length})^D(\text{Time})^{-2}$. The reason why we have used the notation $\alpha'$ familiar in the string theories to describe the Regge slope, will be understood later. The Lagrange multiplier fields of $P^{(i)}(t; \xi_{(i)})$ $(i = 1, \ldots, N, \mu = 1, \ldots, D)$ guarantee the matching condition of (5) or (6) for $i$-th microorganism at any time, and the pressure $p(x)$ is also such multiplier giving the incompressibility given in Eq.(2). The field equation (3) can be easily reproduced. In the action (51), time $t$ appears only in the first term of representing the matching conditions, that is, the time evolution is triggered only by the self-motion of the microorganisms, of which influence spreads instantaneously over the whole space and causes the change of the fluid velocity there. Because of the lacking of the kinetic term, we may call $S$ as the "action". The additional metric contribution such as $\sqrt{g(x)}$ is only relevant for the curved space, an example of which has appeared in the flagellate swimming on the $w$ plane. The later discussion is given for the flat metric.

Now, we will define the following local transformation at a fixed time $t$:

$$\delta X^\mu_{(i)} = \lambda^\mu(X_{(i)}(t; \xi_{(i)}))$$
$$\delta P^{(i)} = 0$$
$$\delta v^\mu(x) = \lambda^\mu(x)$$
$$\delta p(x) = \kappa(x),$$

where we have assumed that the transformation parameters $\lambda^\mu(x)$ and $\kappa(x)$ are restricted by the equations of motions,

$$\partial_\mu \lambda^\mu(x) = 0, \quad \text{and} \quad \partial_\mu \lambda^{\mu\nu}(x) = \frac{1}{\mu} \partial_\nu \kappa(x),$$

where $\lambda^{\mu\nu} \equiv \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$ is the vorticity for $\lambda^\mu$. Meaning of the transformations (53)~(56) are quite simple; the deformation of the microorganisms (53) triggers the increase of the velocity field (55) and of pressure (56) so that they can be consistent with the incompressible fluid dynamics of the
low Reynolds number. It is also important to note that the succession of these time-independent transformations result in the time evolution of our problem. Therefore, the transformations (53)∼(56) resemble the ordinary canonical transformation generated by the Hamiltonian.

Now, let us find the conserved current associated with the transformations (53)∼(56). If we denote the Lagrangian of the fluid itself as $\mathcal{L}_f$:

$$\mathcal{L}_f = -\frac{1}{\mu} p \partial_\mu v^\mu + \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu},$$

then we have for the transformations (53)∼(56),

$$\delta \mathcal{L}_f = \partial_\mu \left[ \left( \lambda^{\mu\nu} - \frac{1}{\mu} g^{\mu\nu} \right) v_\nu \right].$$

Following the usual method, the conserved current associated with the parameters $\lambda^\mu(x)$ and $\kappa(x)$ is obtained:

$$J^\mu_{[\kappa,\lambda^\nu]}(x) = \frac{\delta \mathcal{L}_f}{\delta (\partial_\mu v_\nu)} \delta v_\nu - \left( \lambda^{\mu\nu} - \frac{1}{\mu} g^{\mu\nu} \right) v_\nu = \left( \omega^{\mu\nu} - \frac{1}{\mu} g^{\mu\nu} p \right) \lambda_\nu - \left( \lambda^{\mu\nu} - \frac{1}{\mu} g^{\mu\nu} \right) v_\nu.$$  (60)

We should notice that the conservation of this current can be directly proved with the help of the equations of motion (Eqs. (2) and (3)), namely,

$$\partial^\mu v_\mu = 0$$  (61)

and

$$\partial_\mu \left( \omega^{\mu\nu} - \frac{1}{\mu} g^{\mu\nu} p \right) = 0,$$  (62)

as well as Eq.(57). Instead of Eq.(62), we can adopt the conservation of the stress tensor

$$\partial_\mu T^{\mu\nu} = 0,$$  (63)

with

$$T^{\mu\nu} = \mu (\partial_\mu v_\nu + \partial_\nu v_\mu) - g^{\mu\nu} p.$$  (64)

Therefore, the conservation of the current $J^\mu_{[\kappa,\lambda^\nu]}(x)$ is essentially the reflection of the incompressibility and the conservation of the stress tensor or the
balancing of the stress force of the fluid. Using the current conservation in the outer region of the microorganisms, Gauss theorem leads to a conservation law:

$$\sum_{i=1}^{N} Q_{(i)}^{[\epsilon, \lambda]} = 0,$$

(65)

where

$$Q_{(i)}^{[\epsilon, \lambda]} = \int_{S_{(i)}} d^{D-1}S_{\mu} J_{\mu}^{[\epsilon, \lambda]}(x)$$

(66)

is the integration over the surface $S_{(i)}$ of the $i$-th microorganism. It should be cautious about the flow at spacial infinity which corresponds to the net translation and rotation of the group motion of the microorganisms. To take into account of this effect, it is convenient to include an imaginary microorganism, say $i = N$, at spacial infinity: $S_{(N)}$ may be an infinity large envelope of the whole space, or in the compactified space it can be an ordinary microorganism located at the infinity point.

The transformation also generates the deformation of the shapes of microorganism (53). The generator of this deformation per unit time can be written as

$$\hat{L}_{(\lambda)} = \int d^{D-1}\xi \lambda^\mu (X(t; \xi)) \frac{\delta}{\delta X^\mu(t; \xi)},$$

(67)

which gives the volume (area for $D = 2$) preserving diffeomorphisms [3] owing solely to the incompressibility condition in Eq.(57). The second condition in Eq.(57) adds the further restriction on $\hat{L}_{(\lambda)}$: By the help of the stream function $\sigma_3(x)$, the incompressibility condition is automatically satisfied through $\lambda^\mu = \epsilon^{\mu\nu} \partial_\nu \sigma_3$, so that the Eq.(57) becomes the constraint on the stream function

$$\Delta (\Delta g_{\mu\nu} - \partial_\mu \partial_\nu) \sigma^\epsilon (x) = 0,$$

(68)

or in $D = 2$, in terms of the only non-vanishing component $\sigma = \sigma_3$

$$\Delta^2 \sigma(x) = 0.$$

(69)

This constraint has been already solved generally in $D = 3$ and $D = 2$ fluid.

Here we will discuss a little more on the nature of the charge $Q^{[\epsilon, \lambda]}$ in Eq.(65) as well as the algebraic structure of the generator in Eq.(67), taking up the simple case of $D=2$. Then, we can use the knowledge given in
Eqs.(15)~(19). The current $J^\mu(x)$ given in Eq.(60) reads in $D=2$

$$J^z_{[\kappa, \lambda]}(z, \bar{z}) = 2 \left( 2\omega_{\bar{z}z} - \frac{1}{\mu} \right) \lambda_z - 2 \left( 2\lambda_{\bar{z}z} - \frac{1}{\mu} \right) v_z,$$

(70)

where the factors of 2 come from the metric $g^{z\bar{z}} = 2$ needed for raising and lowering the indices. The general expression of the parameters $\kappa$ and $\lambda$ are obtained as

$$\lambda_z = \epsilon_1(z) - z\epsilon_1'(z) + \epsilon_2(z),$$

(71)

$$\lambda_{\bar{z}z} = 2(\epsilon_1'(z) - \epsilon_1''(z))$$

(72)

and

$$\kappa = -4\mu(\epsilon_1'(z) + \epsilon_1''(z)),$$

(73)

where we take the following most general expansion

$$\epsilon_1(z) = \frac{dL}{2}\ln z + \sum_{k=-\infty}^{+\infty} c_k z^{k+1},$$

(74)

and

$$\epsilon_2(z) = \frac{dL}{2}\ln z + \sum_{k=-\infty}^{+\infty} d_k z^{k+1},$$

(75)

admitting the locally-defined but single-valued transformations. The fields $v_z$, $\omega_{\bar{z}z}$, and $p$ have of course the expressions similar to (71)~(73)’s, where $\epsilon_1$ and $\epsilon_2$ are replaced by $\phi_1$ and $\phi_2$ in Eq.(20). Now we have

$$J^z_{[\epsilon_1, \epsilon_2]}(z, \bar{z}) = 16 \left[ \overline{\phi_1'}(\epsilon_1 + \epsilon_2) - \overline{\phi_1'}(\phi_1 + \phi_2) \right].$$

(76)

The corresponding charge defined in Eq.(66) on the circle, $|z| = r$, takes the following form;

$$Q^{[\epsilon_1, \epsilon_2]}_{[\epsilon_1, \epsilon_2]} = \frac{1}{2} \left\{ \oint_{|z|=r} \frac{dz}{2\pi i} J^z + \oint_{|z|=r} \frac{dz}{2\pi i} J^z \right\}$$

$$= 8 \left\{ -(a_{-1}d_L + \overline{a}_{-1}\overline{d}_L) + \sum_{k=-1}^{+\infty} (k + 1)(a_k d_{-k-2} + \overline{a}_k d_{-k-2} + b_k c_{-k-2} + \overline{b}_k c_{-k-2}) \right\}.$$

(77)
This means that there are infinite number of conserved charges which are just
equal to the coefficients \( \{ a_k \}_{k<0} \) and \( \{ b_k \}_{k<1} \) of \( \phi_1(z) \) and \( \phi_2(z) \). In other
words, two functions \( \phi_1(z) \) and \( \phi_2(z) \) give a solution of the fluid dynamics in
the \( D = 2 \) domain, so that if \( r \) is considered as the evolution parameter, the
functional form of them is conserved.

Next, we will study the algebraic structure of the operator in Eq.(67). Let’s introduce the local version of its operator

\[
L_{[\lambda^\nu]} \equiv \lambda^\mu(x) \frac{\partial}{\partial x^\mu},
\]

which becomes in \( D = 2 \) as

\[
L_\sigma = 2(\lambda_z \partial_z + \lambda_\bar{z} \partial_{\bar{z}}) = 2(\partial_\sigma_1 \partial_{\bar{z}} - \partial_\sigma_1 \partial_{\bar{z}})
\]

when it is written in terms of the stream function \( \sigma(z, \bar{z}) \) having the following general form:

\[
\sigma(z, \bar{z}) = z \epsilon_1 - z \epsilon_1(z) + \int z \epsilon_2(z) - \int \bar{z} \epsilon_2(z).
\]

Then, \( L_\sigma \) can be considered as a Liouville operator of a dynamical system
moving in the phase space of \( (z, \bar{z}) \), having \( -\sigma(z, \bar{z}) \) as its Hamiltonian. Invariance of the phase volume during the temporal evolution of the dynamical
system shows that \( L_\sigma \) is really the area preserving diffeomorphism. The commutation relation is simple, namely

\[
[L_{\sigma_1}, L_{\sigma_2}] = -L_{[\sigma_1, \sigma_2]},
\]

where the \( \{ \sigma_1, \sigma_2 \} \) is the Poisson bracket defined by

\[
\{ \sigma_1, \sigma_2 \} = \partial_\sigma_1 \sigma_2 \partial_{\bar{z}} - \partial_\sigma_1 \partial_{\bar{z}} \sigma_2.
\]

The characteristic of our area-preserving diffeomorphism is summarized in
Eq.(80) which can be expanded as

\[
\sigma(z, \bar{z}) = \sum_k c_k z \bar{z}^{k+1} - \sum_k c_k z \bar{z}^{k+1} + d_{-2} \ln z - d_{-2} \ln \bar{z} + d_L \bar{z} (\ln z - 1) - d_L z (\ln \bar{z} - 1)
+ \sum_{k \neq -2} \frac{1}{k+2} \bar{z}^{k+1} \left( \frac{k+1}{k} \right) - \sum_{k \neq -2} \frac{1}{k+2} \bar{z}^{k+1}.
\]
The generators corresponding to each term of the expansion may be denoted as

\[ L_{ck} = L_{z^{k+1}z}, \quad L_{d2} = L_{inz}, \quad \text{and} \quad L_{d1} = L_{znz-1}, \quad \text{etc.} \quad \cdots \cdots (84) \]

Most of the generators are associated with the monomial \( z^{m_1}z^{m_2} \) which can be denoted as \( L_{(m_1,m_2)} \). Then, the following commutation relation is easily understood.

\[
\left[ L_{(m_1,m_2)}, L_{(n_1,n_2)} \right] = -(m_1n_2 - m_2n_1)L_{(m_1+n_1-1,m_2+n_2-1)}, \quad \cdots \cdots (85) \]

which defines the so-called \( W_{1+\infty} \)-algebra, having \( L_{(0,0)} \) as its U(1) operator. The overall translations and rotation are generated by \( L_{c1}, L_{\tau_1} \) and \( L_{d1} + L_{\tau_1} \), respectively. Virasoro-like algebras are included among the generators;

\[
\begin{align*}
[L_{c_m}, L_{c_n}] &= -(m-n)L_{c_{m+n}}, \quad \cdots \cdots (86) \\
[L_{\tau_m}, L_{\tau_n}] &= (m-n)L_{\tau_{m+n}}, \quad \cdots \cdots (87)
\end{align*}
\]

but

\[
\left[ L_{c_m}, L_{\tau_n} \right] = -(mn + m + n)L_{(m+1,n+1)} \quad \cdots \cdots (88)
\]

results in the appearance of a new generator \( L_{(m+1,n+1)} \). It is quite interesting to understand that net translation and rotation for the ciliated microorganisms are generated by the generator \( L_{(m+1,n+1)} \), where \( |n \pm m| = 1 \) corresponds to translation (49) and \( n = m \) or \( |n + m| = 2 \) corresponds to rotation [7]. As yet, the role of the generators \{ \( L_{\tau} \) \} in the swimming problem is not manifestly clear but the algebra can depend on the shape of the microorganisms that really happens in the case of area-preserving diffeomorphisms a part of which is included in our algebra. We could guess that the study on the classification of possible area (volume)-preserving algebras of such kind may lead to the understanding of three universality classes of microorganisms' swimming discussed in the beginning.

5 Collective motion of microorganisms and N-point correlation function

Finally, starting from the “action” (51) of a group of microorganisms, we will study a possibility of using \( N \)-point correlation functions similar to those of string and membrane in the study of swimming motion in the \( D = 2 \)
and $D = 3$ fluid, respectively. Consider the situation where the vortices are created and annihilated, so that the probability of having the vortex distribution (field) $\omega^{\mu\nu}(x)$ is given by

$$P[\omega^{\mu\nu}(x)] \sim \exp \left\{ -\frac{1}{2\pi i\alpha'} \int d^D x \sqrt{g(x)} \frac{1}{4} \omega^{\mu\nu}(x) \omega^{\mu\nu}(x) \right\},$$

(89)

where $i\alpha'$ is the external parameter controlling the fluctuation of the vortex distribution. [The $\alpha' \to 0$ limit corresponds to the classical limit without the fluctuation.] In this situation we should sum over all the possible configurations of the velocity fields with Eq.(89) as their probability. The probability of having $N$ microorganisms with their surfaces located at $X_{(1)}, X_{(2)}, \cdots, X_{(N)}$, and with their time derivatives $\dot{X}_{(1)}, \dot{X}_{(2)}, \cdots, \dot{X}_{(N)}$, is given by the following $N$-point correlation function (amplitude)

$$G_N(X_{(1)}, \dot{X}_{(1)}; X_{(2)}, \dot{X}_{(2)}; \cdots; X_{(N)}, \dot{X}_{(N)}) = Z_N / Z_0.$$  (90)

Here the partition function $Z_N$ is defined by

$$Z_N = \int \mathcal{D}P^{(i)}(\xi) \int \mathcal{D}p(x) \int \mathcal{D}v(x) \exp \{ iS_N \},$$

(91)

with the “action” $S_N$ in Eq.(51). In the path integrations over $P^{(i)}(\xi)$ and $p(x)$ in Eq.(91) guarantee the matching condition (6) and the incompressibility of the fluid, respectively. Moving to the momentum representation naturally, we have

$$G_N(X_{(1)}, \dot{X}_{(1)}; \cdots; X_{(N)}, \dot{X}_{(N)})$$

$$= \int \mathcal{D}P^{(i)} \exp \left\{ i \sum_{i=1}^{N} \int dt \int d^{D-1} \dot{\xi}(i) P^{(i)}(x) \dot{X}^{\mu} \right\}$$

$$\times \tilde{G}_N(X_{(1)}, P^{(1)}; \cdots; X_{(N)}, P^{(N)}),$$

(92)

where

$$\tilde{G}_N(X_{(1)}, P^{(1)}; \cdots; X_{(N)}, P^{(N)})$$

$$= Z_0^{-1} \int \mathcal{D}v(x) \prod_x \delta(\partial v(x))$$

$$\times \exp \left\{ -i \sum_{i=1}^{N} \int dt \int d^{D-1} \dot{\xi}(i) P^{(i)}(x) v^{\mu}(X_{(i)}(\xi_{(i)})) \right\}.$$
\[
\times \exp \left\{ \frac{1}{2\pi i \alpha'} \int d^D x \frac{1}{2} v^\mu \Delta v_\mu \right\} 
\]  
(93)

\[
= \exp \left[ 2\pi i \alpha' \times \frac{1}{2} \sum_{i,j} \int dt_{(i)} \int dt_{(j)} \int d^{D-1} \xi_{(i)} \int d^{D-1} \xi_{(j)} \right.
\]  
\[
\times P^{(i)}_\mu(t_{(i)}; \xi_{(i)}) G_\perp^{\mu\nu} \left( X_{(i)}(\xi_{(i)}) - X_{(j)}(\xi_{(j)}) \right) P^{(j)}_\nu(t_{(j)}; \xi_{(j)}) \bigg] . 
\]  
(94)

Here the Green’s function \( G_\perp^{\mu\nu}(x - x') \) satisfies
\[
\Delta G_\perp^{\mu\nu}(x - x') = \delta^{(D)}(x - x'), 
\]  
(95)

and the transverse condition reflecting the incompressibility of the fluid, namely
\[
\partial_\mu G_\perp^{\mu\nu}(x - x') = \partial'_\nu G_\perp^{\mu\nu}(x - x') = 0. 
\]  
(96)

Now, we have obtained \( N \)-point correlation function for the collective swimming of \( N \) microorganisms. It is quite similar to the \( N \)-point function of strings [5] for \( D = 2 \) case and membranes [6] for \( D = 3 \) case. It is also related to the string field theory, since the incoming and outcoming strings are not point-like, but the Reggeons. More precisely, refinement is necessary on the treatment of functional measure and boundary condition in Eq.(93).

There is a possibility that the boundary condition terms changes \( \Delta g_{\mu\nu} \) into \( \Delta g_{\mu\nu} + S_\nu \partial_\mu - g_{\mu\nu} s^\lambda \partial_\lambda \) with
\[
S_\mu(x) = \sum_{i=1}^{N} \int d^{D-1} s^{(i)}_\mu \delta^{(D)}(x - X_{(i)}(t_{(i)})). 
\]

It is also interesting to consider the multi-loop amplitudes if the meaning of handle in the space of velocity field could be well understood.

Here we will discuss the use of \( N \)-point correlation function (92) in the collective swimming motion of microorganisms. If \( G_N \) represents the probability of having \( N \) microorganisms whose surfaces are located at \( X_{(i)} \) with velocity \( \dot{X}_{(i)} \) \( (i = 1, \ldots, N) \), then it can be viewed as the probability distribution of \( \dot{X}_N \) of the imaginary microorganism \( N \) located at spacial infinity under the given data of \( X_i \) and \( \dot{X}_{(i)} \) \( (i = 1, \ldots, N - 1) \). Following the usual strategy, the counterflow \( -\dot{X}_N \) can be indentified to the collective swimming motion of \( N - 1 \) microorganisms. Therefore, the averaged collective swimming motion over the fluctuation distribution is given by
\[
- \langle \dot{X}_N \rangle = - \sum_{\dot{X}_N} \dot{X}_N G_N(X_1, \dot{X}_1; \ldots; X_N, \dot{X}_N). 
\]  
(97)
It is also interesting to study the response of the $N$-point correlation function $G_N$, the Ward identity, for the deformation of the shapes of microorganisms in Eq.(53)∼(56)’s.

We hope much progress will come out along the strategies proposed in this article.

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