THE SCALAR FIELD POTENTIAL IN INFLATIONARY MODELS:
RECONSTRUCTION AND FURTHER CONSTRAINTS

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Abstract

In this paper, we present quantitative constraints on the scalar field potential for a general class of inflationary models. (1) We first consider the reconstruction of the inflationary potential for given primordial density fluctuation spectra. Our work differs from previous work on reconstruction in that we find a semi-analytic solution for the potential for the case of density fluctuations with power-law spectra. In addition, for the case of more general spectra, we show how constraints on the density fluctuation spectra imply corresponding constraints on the potential. We present a series of figures which show how the shape of the potential depends on the shape of the perturbation spectrum and on the relative contribution of tensor modes. (2) We show that the average ratio $\langle R \rangle$ of the amplitude of tensor perturbations (gravity wave perturbations) to scalar density perturbations is bounded from above: $\langle R \rangle \leq 1.6$. We also show that the ratio $\langle R \rangle$ is proportional to the change $\Delta \phi$ in the field: $\langle R \rangle \approx 0.42 \Delta \phi / M_{\text{pl}}$. Thus, if tensor perturbations are important for the formation of structure, then the width $\Delta \phi$ must be comparable to the Planck mass. (3) We constrain the change $\Delta V$ of the potential and the change $\Delta \phi$ of the inflation field during the portion of inflation when cosmological structure is produced. We find both upper and lower bounds for $\Delta \phi$ and for $\Delta V$. In addition, these constraints are then used to derive a bound on the scale $\Lambda$, which is the scale of the height of the potential during the portion of inflation when cosmological perturbations are produced; we find $\Lambda \leq 10^{-2} M_{\text{pl}}$. Thus, the last $\sim 60$ e-foldings of inflation must take place after the GUT epoch. This bound on $\Lambda$, although comparable to those found previously, is found here using different methods. (4) In an earlier paper, we defined a fine-tuning parameter $\lambda_{FT} \equiv \Delta V / (\Delta \phi)^4$ and found an upper bound for $\lambda_{FT}$. In this paper, we find a lower bound on $\lambda_{FT}$. The fine-tuning parameter is thus constrained to lie in the range $6 \times 10^{-11} (\Lambda / 10^{17} \text{GeV})^8 \leq \lambda_{FT} \leq 10^{-7}$. (5) Finally, we consider the effects of requiring a non-scale-invariant spectrum of perturbations (i.e., with spectral index $n \neq 1$) on the fine-tuning parameter $\lambda_{FT}$. For spectral indices $n$ less than unity, the upper bound on the fine-tuning parameter $\lambda_{FT}$ becomes more restrictive than the $n = 0$ case by a factor $\mathcal{F} \sim 2 - 5$.

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I. INTRODUCTION

The inflationary universe model [1] provides an elegant means of solving several cosmological problems, including the horizon problem, the flatness problem, and the monopole problem. In addition, quantum fluctuations produced during the inflationary epoch may provide the initial conditions required for the formation of structure in the universe. During the inflationary epoch, the energy density of the universe is dominated by a (nearly constant) vacuum energy term $\rho \simeq \rho_{vac}$, and the scale factor $R$ of the universe expands superluminally (i.e., $\dot{R} > 0$). If the time interval of accelerated expansion satisfies $\Delta t \geq 60 \dot{R}/\dot{R}$, a small causally connected region of the universe grows sufficiently to explain the observed homogeneity and isotropy of the universe, to dilute any overdensity of magnetic monopoles, and to flatten the spatial hypersurfaces (i.e., $\Omega \rightarrow 1$). In most models, the vacuum energy term is provided by the potential of a scalar field. In this paper, we present constraints on this scalar field potential for a general class of inflationary models. This present work extends the results of a previous paper [2] where we quantified the degree of fine tuning required for successful inflationary scenarios.

In the original model of inflation [1], now referred to as Old inflation, the universe supercools to a temperature $T \ll T_C$ during a first-order phase transition with critical temperature $T_C$. The nucleation rate for bubbles of true vacuum must be slow enough that the Universe remains in the metastable false vacuum long enough for the required $\sim 60$ e-foldings of the scale factor. Unfortunately, the Old inflationary scenario has been shown to fail [3] because the interiors of expanding spherical bubbles of true vacuum cannot thermalize properly and produce a homogeneous radiation-dominated universe after the inflationary epoch.

In order to overcome the "reheating problem" of the Old inflationary scenario, a new class of inflationary models was developed [4, 5, 6]. In this class of models, the diagram of the effective potential (or free energy) of the inflation field $\phi$ has a very flat plateau and the field evolves sufficiently slowly for inflation to take place (i.e., the field evolves by "slowly rolling" off the plateau). In these models, the phase transition can be second order or weakly first order. Many inflationary models which are currently under study are of this latter type, e.g., New Inflation [4, 5], Chaotic Inflation [6], and Natural Inflation [7]. The evolution of the field $\phi$ is determined by the equation of motion

$$\ddot{\phi} + 3H \dot{\phi} + \Gamma \dot{\phi} + \frac{dV}{d\phi} = 0,$$

where $H$ is the Hubble parameter and $V$ is the potential. The $\Gamma \dot{\phi}$ term determines the decay rate of the $\phi$ field at the end of inflation (see, e.g., Ref. [8]). In this equation of motion, spatial gradient terms have been neglected (gradients are exponentially suppressed during the inflationary epoch).

In most studies of inflation, the field $\phi$ is assumed to be "slowly rolling" during most of the inflationary epoch. The slowly rolling approximation means that the motion of the inflation field is overdamped, $\ddot{\phi} = 0$, so that equation [1] becomes a first order equation; the $\Gamma \dot{\phi}$ term is also generally negligible during this part of inflation. Thus, the motion is controlled entirely by the force term $(dV/d\phi)$ and the viscous damping term
(3H\dot{\phi})$ due to the expansion of the universe. Near the end of the inflationary epoch, the field approaches the minimum of the potential (i.e., the true vacuum) and then oscillates about it, while the $\Gamma \phi$ term gives rise to particle and entropy production. In this manner, a “graceful exit” to inflation is achieved.

For completeness, we note that workable models of inflation which use a first order phase transition have been proposed, notably Extended Inflation [9] and Double Field Inflation [10]. However, these models require an additional “slowly rolling” field in order to complete the phase transition.

All known versions of inflation with slowly rolling fields produce density fluctuations, which tend to be overly large unless the potential for the slowly rolling field is very flat. In particular, these models produce (scalar) density fluctuations [11] with amplitudes given by

$$\left.\frac{\delta \rho}{\rho}\right|_{\text{scalar}} \approx \frac{1}{10} \frac{H^2}{\dot{\phi}},$$

where $(\delta \rho/\rho)_{\text{scalar}}$ is the amplitude of a density perturbation when its wavelength crosses back inside the horizon (more precisely, the Hubble length) after inflation, and the right hand side is evaluated at the time when the fluctuation crossed outside the Hubble length during inflation. The expression [1.2] applies to any inflationary model which has a slowly rolling field $\phi$. The quantum fluctuations in the motion of the field $\phi$ cause the hypersurface of the phase transition to be nonuniform and result in density perturbations with magnitude given by Eq. [1.2]. For the case of Extended or Double Field Inflation, these perturbations are superimposed with the perturbations caused by the collisions of bubbles.

In addition to the scalar perturbations described above, inflationary models can also produce tensor perturbations (gravity wave perturbations — see Ref. [12]). The amplitude of these perturbations is given by

$$\left.\frac{\delta \rho}{\rho}\right|_{\text{GW}} = \frac{\sqrt{32\pi}}{30} \frac{H}{M_{\text{pl}}},$$

where the right hand side is again evaluated at the time when the fluctuation crossed outside the Hubble length during inflation (and where $M_{\text{pl}}$ is the Planck mass).

The allowable amplitude of these perturbations is highly constrained by measurements of the isotropy of the microwave background. On scales of cosmological interest, these measurements [13] indicate that

$$\left.\frac{\delta \rho}{\rho}\right|_{\text{hor}} \leq \delta \approx 2 \times 10^{-5}.$$

In this expression, the left hand side represents the total amplitude of perturbations produced by inflation, i.e., both scalar perturbations (Eq. [1.2]) and tensor perturbations (Eq. [1.3]); note that the two types of perturbations add in quadrature. The right hand side of equation [1.4] represents the experimental measurements (both detections
and limits) of the cosmic microwave background. In general, these measurements are a function of the observed size scale (or angular scale). All measurements (to date) on different size scales are roughly consistent with an approximate value $\delta \approx 2 \times 10^{-5}$; details of scale dependence will be considered later.

For the general class of inflationary models with slowly rolling fields, the coupled constraints that the universe must inflate sufficiently and that the density perturbations must be sufficiently small require the potential $V(\phi)$ to be very flat [2, 14]. In a previous paper [2], we derived upper bounds on a “fine-tuning parameter” $\lambda_{FT}$, defined by

$$
\lambda_{FT} \equiv \frac{\Delta V}{(\Delta \phi)^4},
$$

where $\Delta V$ is the decrease in the potential $V(\phi)$ during a given portion of the inflationary epoch and $\Delta \phi$ is the change in the value of the field $\phi$ over the same period. In this paper, we define $\Delta V$ and $\Delta \phi$ over the portion of inflation where cosmic structure is produced; as discussed below, this portion of inflation corresponds to the $N \approx 8$ e-foldings which begin roughly 60 e-foldings before the end of inflation. The parameter $\lambda_{FT}$ is the ratio of the height of the potential to its (width)$^4$ for the part of the potential involved in the specified time period; $\lambda_{FT}$ thus measures the required degree of flatness of the potential. In Ref. [2], we found that $\lambda_{FT}$ is constrained to be very small for all inflationary models which satisfy the density perturbation constraint and which exhibit overdamped motion; in particular, we obtained the bound

$$
\lambda_{FT} \leq \frac{2025}{8} \delta^2 \approx 10^{-7}.
$$

(1.6a)

We also showed that if the potential is a quartic polynomial with the quartic term in the Lagrangian written as $\frac{1}{4} \lambda_q \phi^4$, then a bound on $\lambda_{FT}$ implies a corresponding bound on $\lambda_q$; specifically,

$$
|\lambda_q| \leq 36 \lambda_{FT}.
$$

(1.6b)

Thus, the bound of equation [1.6a] implies that the quartic coupling constant must be extremely small.

In this paper we continue a quantitative study of the constraints on the scalar-field potential for models of inflation that have a slowly rolling field. In the first part of this paper, we consider the reconstruction of the inflationary potential for given primordial density fluctuation spectra. This reconstruction process has already been considered by many recent papers [15]. In this paper, we show that for the case of density fluctuations with power-law spectra, the reconstruction of the inflationary potential can be done semi-analytically and we find the corresponding semi-analytic solutions (see Eq. [3.12]). For the more general case, we show how constraints on the density fluctuation spectra imply corresponding constraints on the potential.

Our results show how the shape of the potential depends on the perturbation spectrum and on the relative contribution of tensor modes and scalar perturbations (see Figures 1 – 5). For the case in which tensor perturbations produce a substantial contribution to the total (e.g., in Figure 1), the potentials $V(\phi)$ are concave upward for
all of the spectral indices \( n = 0.5 - 1 \) considered here. For the opposite case in which tensor modes are negligible (e.g., in Figure 4), the potentials are concave downward and somewhat like the cosine potential used in models of Natural Inflation [7]. Figure 5 shows a cosine potential which has been fit to the reconstructed potential for a particular case with little contribution from tensor modes (see Sec. III). Thus, for perturbation spectra with little contribution from tensor modes (and moderate departures from scale invariance), the reconstructed potential looks very much like a cosine potential.

In the next part of this paper, we present further constraints on the inflationary potential. In particular, we constrain both \( \Delta V \) and \( \Delta \phi \) individually. We show that both upper and lower bounds exist for \( \Delta \phi \) and for \( \Delta V \) (see equations [4.14], [4.15], and [4.33]). In addition, these constraints are used to derive a bound on the scale \( \Lambda \), i.e., the scale of the height of the potential during the portion of inflation when cosmological perturbations are produced; we obtain the bound \( \Lambda \leq 10^{-2} M_{\text{pl}} \). Thus, the final \( \sim 60 \) e-foldings of inflation must take place after the GUT epoch. Although found by different methods, this bound on \( \Lambda \) is comparable to those found previously [16, 17, 18, 19].

Next, we show that the average ratio \( \langle \mathcal{R} \rangle \) of the amplitude of tensor perturbations (gravitational wave perturbations) to scalar density perturbations is bounded from above: \( \langle \mathcal{R} \rangle \leq 1.6 \). Thus, tensor perturbations cannot be larger than scalar perturbations by an arbitrarily large factor. We also show that the ratio \( \langle \mathcal{R} \rangle \) is proportional to the change \( \Delta \phi \) in the field; in particular, we find that \( \langle \mathcal{R} \rangle \approx 0.42 \Delta \phi / M_{\text{pl}} \). Thus, if tensor perturbations are important for the formation of cosmological structure, then the width \( \Delta \phi \) must be comparable to the Planck mass.

Finally, we consider bounds on the fine-tuning parameter \( \lambda_{FT} \). We find a lower bound on \( \lambda_{FT} \) (see equation [5.1]). We also consider the effects of requiring a non-scale-invariant spectrum of perturbations (i.e., with spectral index \( n \neq 1 \)) on the fine-tuning parameter \( \lambda_{FT} \). We show that for \( n < 1 \), the bound on the fine-tuning parameter \( \lambda_{FT} \) becomes more restrictive than the \( n = 0 \) case (which is effectively the case considered in Ref. [2]).

The constraints presented in this paper apply to inflationary models involving one or more scalar fields that are minimally coupled to gravity, and which satisfy three conditions. First, we require that the evolution during the relevant time period satisfies the density perturbation constraint, which can be written in the form

\[
H^2 \frac{d\phi}{dt} \leq 10 \delta .
\]

Second, we assume that during the early stages of inflation, the evolution of the field \( \phi \) is overdamped so that the \( \ddot{\phi} \) term of Eq. [1.1] is negligible (along with the \( \Gamma \dot{\phi} \) term). This assumption leads to the simplified equation of motion

\[
3H \frac{d\phi}{dt} = - \frac{dV}{d\phi} .
\]

The consistency of neglecting the \( \ddot{\phi} \) term implies a constraint on the potential of the form

\[
\left| \frac{d}{dt} \left( \frac{1}{3H} \frac{dV}{d\phi} \right) \right| \leq \left| \frac{dV}{d\phi} \right| .
\]
which we refer to as the overdamping constraint. This constraint is often called the “slowly-rolling” condition, but we follow Ref. [2] and avoid this phrase because it suggests a constraint on $\dot{\phi}$ (see Eq. [1.10] below) rather than $\ddot{\phi}$. Notice that this constraint is a necessary but not a sufficient condition for the $\ddot{\phi}$ term to be neglected. Third, we also require that the $\phi$ field rolls slowly enough that its kinetic energy contribution to the energy density of the universe is small compared to that of the vacuum. Thus, the following constraint must be satisfied during the inflationary period:

$$\frac{1}{2}\ddot{\phi}^2 \leq V_{TOT},$$

(1.10)

where $V_{TOT}$ is the total vacuum energy density of the universe. Notice that additional fields (i.e., in addition to the inflation field $\phi$) can be present during the inflationary epoch. Thus, the total vacuum energy density $V_{TOT}$ can, in general, include contributions from other scalar field potentials in addition to $V(\phi)$. Notice also that the constraint [1.10] was not explicitly used in our previous work [2].

We have introduced several different potentials and energy scales and it is important to maintain the distinctions between them. The quantity $V(\phi)$ is the potential of the inflationary field $\phi$ and varies with time as $\phi$ evolves. The quantity $V_{TOT}$ is the total vacuum energy density of the universe and also varies with time. The quantity $\Delta V$ is the change in the potential $V(\phi)$ over the portion of inflation when cosmological perturbations are produced; thus, $\Delta V$ is a given constant for a given inflationary scenario. Finally, we have defined $\Lambda$ to be the energy scale of inflation when cosmological perturbations are produced; to be specific, we define

$$\Lambda^4 \equiv V_{TOT}\bigg|_{60},$$

(1.11)

where the right hand side denotes that $V_{TOT}$ is evaluated when the present-day horizon scale left the horizon during inflation (this event generally occurs about 60 e-foldings before the end of inflation).

This paper is organized as follows. In Sec. II we formulate the problem for what we call standard inflationary models: models involving any number of scalar fields that are minimally coupled to gravity, and that obey the density perturbation and overdamping constraints. We define notation and transform the problem into a mathematically convenient form. In Sec. III we reconstruct the inflationary potential for the case in which the total primordial spectrum of density perturbations (both scalar and tensor contributions) is a power-law. We also show how constraints on the primordial spectrum lead to corresponding constraints on the potential. In Sec. IV, we use our formulation to derive several additional constraints on the inflationary potential. In particular, we constrain $\Delta \phi$ and $\Delta V$ individually; we also derive a relationship between the width $\Delta \phi$ and the average ratio of the amplitude of tensor perturbations to scalar perturbations. We derive further constraints on the fine-tuning parameter $\lambda_{FT}$ in Sec. V; we show that $\lambda_{FT}$ is also bounded from below and we show the effects of non-scale-invariant spectra of density perturbations. Finally, we conclude in Sec. VI with a summary and discussion of our results.
II. FORMULATION OF THE PROBLEM

In this paper, we derive a set of bounds on the properties of the scalar field potential for a fairly general class of inflationary scenarios which utilize a slowly rolling field $\phi$. We consider inflationary scenarios involving an arbitrary number of scalar fields that are minimally coupled to gravity, and which satisfy the density perturbation constraint of Eq. [1.7], the overdamping constraint of Eq. [1.9], and the inflation constraint of Eq. [1.10]. We assume that these constraints hold for a period of $N$ e-foldings, given by

$$N = \int H \, dt,$$

where the limits of integration correspond to some portion of the inflationary epoch. As discussed below, we generally take $N = 8$ and hence $N$ is not the total number ($\sim 60$) of e-foldings required for successful inflation.

The density perturbation constraint must apply for physical size scales (at the present epoch) in the range 3000 Mpc (the horizon size) down to about 1 Mpc (the size scale corresponding to a galactic mass). This range spans a factor of 3000 in physical size and corresponds to $N = \log(3000) \approx 8$ e-foldings of the inflationary epoch [8]. Thus, the density perturbation constraint only applies for about $N = 8$ e-foldings. In this paper, we are mostly interested in the structure producing portion of the entire inflationary epoch and we will take $N = 8$ as our “standard” value.

The relevant time variable for an inflationary epoch is the number of e-foldings since the beginning of the epoch. We therefore adopt a new time variable $x$ defined by

$$dx \equiv \frac{H \, dt}{N}.$$  \hfill (2.1)

The variable $x$ thus ranges from 0 to 1 during the relevant time period. The point $x = 0$ corresponds to the time during inflation when perturbations on the physical size scale of the horizon at the present epoch (i.e., 3000 Mpc) were produced. Keep in mind that many additional e-foldings of the scale factor could have taken place before $x = 0$. We also introduce the notation

$$F(x) \equiv -\frac{dV}{d\phi},$$  \hfill (2.2)

where $F$ represents a force.

In this newly defined notation, the overdamping constraint is written as

$$\left| H \frac{d}{dx} \left( \frac{F}{H} \right) \right| \leq 3NF,$$  \hfill (2.3)

and the density perturbation constraint is

$$3H^3/F \leq 10 \delta.$$  \hfill (2.4)

Using the equation of motion [1.8] and the relation

$$H^2 = \left( \frac{8\pi}{3} \right)V_{TOT}/M_{pl}^2,$$


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we can write the condition that the universe is dominated by potential energy rather than kinetic energy (Eq. [1.10]) in the form

$$ F < \left( \frac{27}{4\pi} \right)^{1/2} H^2 M_{\text{pl}} , $$

where $M_{\text{pl}}$ is the Planck mass. Furthermore, the quantities $\Delta V$ and $\Delta \phi$ can be written in the form

$$ \Delta V = \frac{N}{3} \int_0^1 (F^2/H^2) \, dx $$

$$ \Delta \phi = \frac{N}{3} \int_0^1 (F/H^2) \, dx . $$

We have chosen our sign convention so that $\Delta V$ is a positive quantity and so that $x = 0$ at the beginning of the constrained time period. Keep in mind that $\Delta V$ and $\Delta \phi$ are the changes in the potential and the inflation field during the $N = 8$ e-foldings during which cosmic structure is produced; they are not the total changes in these quantities over the entire inflationary epoch.

In Sections IV and V, we find general constraints on the inflationary potential. For these calculations, we introduce the formulation described below. For the sake of definiteness, in Sections IV and V we assume that the density perturbation constraint of equation [2.4] is saturated at the epoch $x = 0$, i.e., when the present-day horizon scale left the horizon during inflation (this assumption and the following definitions are not used in the reconstruction of the potential in Sec. III). Physically, this assumption means that scalar density perturbations are responsible for the observed fluctuations in the cosmic microwave background as measured by the COBE satellite. We thus have

$$ \frac{3H_B^3}{F_B} = 10\delta , $$

where the subscripts denote the epoch at which $x = 0$ and where we consider $\delta$ to be a known number ($\sim 2 \times 10^{-5}$). Equation [2.8a] will be used in Sections IV and V.

We note that in general tensor perturbations may produce some fraction of the total perturbations; in this case, one should replace $\delta$ in equation [2.8a] by the corresponding smaller value $\delta_S$ which denotes only the scalar contribution, i.e.,

$$ \frac{3H_B^3}{F_B} = 10\delta_S . $$

In this case, the general form of the results derived in Sections IV and V remain the same with $\delta$ replaced by $\delta_S$. For completeness, we also note that the maximum of the density perturbation constraint need not occur at $x = 0$; this complication is considered in Ref. [2] and will not significantly affect the results of this paper.

For convenience, we rescale the functions $F$ and $H$ by their starting values in order to obtain dimensionless quantities, i.e.,

$$ f(x) \equiv F(x)/F_B , $$

$$ h(x) \equiv H(x)/H_B . $$
Since we can use equation [2.8] to eliminate \( F_B \) from our equations, we are left with a single unknown parameter – namely \( H_B \). We choose to eliminate \( H_B \) in favor of the energy scale \( \Lambda \) at \( x = 0 \), i.e., we define

\[
H_B^2 \equiv \frac{8\pi}{3} \frac{\Lambda^4}{M_{pl}^2} = \frac{8\pi}{3} \frac{V_{TOT}(x = 0)}{M_{pl}^2} .
\]  

(2.10)

The quantity \( \Lambda^4 \) is equal to the value of the total vacuum energy density of the universe at \( x = 0 \) (which occurs \( \sim 60 \) e-foldings before the end of inflation).

For mathematical convenience, we also eliminate \( f(x) \) in favor of a new function \( p(x) \) defined by

\[
p(x) \equiv \sqrt{\frac{f(x)}{h(x)}} .
\]  

(2.11)

In terms of the dimensionless functions \( p(x) \) and \( h(x) \), the physical quantities of interest in this paper can be written

\[
\Delta V = \frac{16\pi^2}{75} \frac{N}{\delta^2} \frac{\Lambda^8}{M_{pl}^4} \int_0^1 p^4 \, dx ,
\]  

(2.12)

\[
\Delta \phi = \left( \frac{2\pi}{75} \right)^{1/2} \frac{N}{\delta} \frac{\Lambda^2}{M_{pl}} \int_0^1 (p^2/h) \, dx ,
\]  

(2.13)

\[
\lambda_{FT} = \frac{300\delta^2}{N^3} \, J[p, h] ,
\]  

(2.14)

where \( J \) is the functional defined by

\[
J[p, h] = \frac{\int_0^1 p^4 \, dx}{\left[ \int_0^1 (p^2/h) \, dx \right]^4} .
\]  

(2.15)

In terms of the new functions \( p \) and \( h \), the density perturbation constraint [2.4] can be written as

\[
p(x) \geq h(x)
\]  

(2.16)

and the overdamping constraint as

\[
\left| \frac{1}{p} \frac{dp}{dx} \right| \leq \frac{3}{2} N .
\]  

(2.17)

Using equations [2.8 – 2.11], the constraint of equation [2.5] becomes

\[
\frac{p^2}{h} \leq \frac{15}{2\sqrt{2\pi}} \frac{\delta}{\Lambda^2} \frac{M_{pl}^2}{\Lambda^2} \equiv \beta ,
\]  

(2.18)
where we have defined the right hand side of the inequality to be the dimensionless parameter $\beta$. In addition, the functions $p$ and $h$ are subject to the initial conditions
\[ p(0) = 1 = h(0) . \] (2.19)
Notice that the ratio $h/p$ is constrained to have a maximum value of unity, but the individual functions $h$ and $p$ can vary substantially.

This concludes the formulation of the problem. We want to bound the physical quantities defined by equations [2.12 – 2.15] subject to the constraints of equations [2.16 – 2.18] and the initial conditions [2.19].

III. RECONSTRUCTION OF INFLATIONARY POTENTIALS

In this section, we consider the problem of reconstructing the scalar field potential. As noted by many authors in the recent literature [15], knowledge of both the scalar perturbations and the tensor perturbations allows one to reconstruct a portion of the scalar field potential that gives rise to inflation. In this paper, we formulate the reconstruction problem in terms of the variables defined in the previous section. We then find a semi-analytic solution for the potential for the case of (total) perturbation spectra which are pure power-laws (see equation [3.12]).

To preview some of the most interesting results of this section, we refer the reader to Figures 1 – 5. There we show how the shape of the potential depends on the perturbation spectrum and on the relative contribution of tensor modes. For example, when tensor modes provide a significant fraction of the total, the potentials $V(\phi)$ are concave upward for all spectral indices considered in this paper $n = 0.5 – 1$ (e.g., Figure 1). For the opposite case where tensor modes provide a negligible contribution to the total, the potentials are concave downward (e.g., Figure 4). For this latter case, the potential shape is well approximated by a cosine (see Figure 5) as in the model of Natural Inflation [7].

For the rest of this section we show how these results are obtained. In addition, we comment on their usefulness for the case when the exact power law index is not known, but instead there is a range consistent with the existing status of observations. Although our knowledge of the true primordial spectrum of perturbations is not exact, constraints may be placed on the spectrum; we show how constraints on the primordial power spectrum produce corresponding constraints on the scalar field potential.

The scalar perturbations (see equation [1.2]) will generally be some function of the variable $x$ introduced in the previous section, i.e.,
\[ \left. \frac{\delta \rho}{\rho} \right|_{\text{scalar}} = \frac{1}{10} \frac{H^2}{\dot{\phi}} = \delta_S(x) . \] (3.1)
Similarly, the tensor perturbations (gravity wave perturbations) can be written in the form
\[ \left. \frac{\delta \rho}{\rho} \right|_{\text{GW}} = \frac{\sqrt{32\pi}}{30} \frac{H}{M_{\text{pl}}} = \delta_T(x) , \] (3.2)
where the right hand side is some function of \( x \). We note that the expressions used here are correct only to leading order in the “slow-roll” approximation. Although higher order corrections to these expressions have been calculated [20], the leading order terms are adequate for our purposes.

For the particular case in which inflation arises from a single scalar field \( \phi \) with a potential \( V(\phi) \), we can write the above expressions in terms of the potential. The tensor modes are related to the potential through equation [3.2], which can be written as the expression

\[
V(x) = \frac{675}{64\pi^2} M_{\text{pl}}^4 \delta_T^2(x). \tag{3.3}
\]

Similarly, the scalar modes are related to the potential through equation [3.1], which can be written in the form

\[
-\frac{1}{V^2} \frac{dV}{dx} = \frac{16\pi^2 N}{75} M_{\text{pl}}^{-4} \delta_S^{-2}(x). \tag{3.4}
\]

The combination of these two equations thus implies the following simple differential equation

\[
-\frac{1}{\delta_T^3} \frac{d\delta_T}{dx} = \frac{9N}{8} \delta_S^{-2} \equiv C \delta_S^{-2}, \tag{3.5}
\]

where we have defined the constant \( C = 9N/8 \) (see Ref. [25]).

The two types of perturbations add in quadrature, so that the total spectrum of primordial perturbations, which we denote as \( q(x) \), can be written as the sum

\[
q^2(x) = \delta_T^2(x) + \delta_S^2(x). \tag{3.6}
\]

If we assume that the total spectrum \( q(x) \) is a known function, we can then combine the equations [3.5] and [3.6] to obtain a single differential equation for \( \delta_T \):

\[
[\delta_T^{-2} - q^2] \frac{d\delta_T}{dx} = C \delta_T^3. \tag{3.7}
\]

Thus, if the primordial spectrum \( q(x) \) were known exactly, we could simply solve the above differential equation for \( \delta_T(x) \) and then solve for the scalar field potential \( V(x) \) [21]. Notice that we must also specify the initial condition \( \delta_T(0) \), i.e., the amplitude of the tensor modes at \( x = 0 \). Since \( \phi(x) \) is directly calculable from the equation of motion once we know \( V(x) \), the usual form of the potential \( V(\phi) \) as a function of the scalar field can also be obtained. This hypothetical “solution” for the potential is correct to leading order in the “slow roll” approximation (see also Ref. [15]).

One problem with the above discussion is that we do not know the true primordial spectrum \( q(x) \). However, the total spectrum of perturbations is often assumed to be a power-law in wavenumber \( k \), i.e., the amplitudes of the perturbations vary with physical length scale \( L \) according to the law

\[
q \sim \frac{\delta \rho}{\rho} \bigg|_{\text{hor}} \sim L^{(1-n)/2}, \tag{3.8}
\]

where \( n \) is the index of the power-law.
where the subscript denotes that $\delta \rho / \rho$ is evaluated at the time of horizon crossing. The parameter $n$ is the power-law index of the primordial power spectrum,

$$P(k) \sim |\delta k|^2 \sim k^n,$$

and $k$ is the wavenumber of the perturbation [22]. Notice that the left-hand-side of equation [3.8] is to be evaluated when the perturbation of lengthscale $L$ enters the horizon. Notice also that $n = 1$ corresponds to a scale-invariant spectrum and that $n < 1$ corresponds to spectra with more power on large length scales. We stress that a considerable amount of processing in required to convert the primordial spectrum into observable quantities and such work is now being vigorously pursued [26]; this transformation between the primordial spectrum and actual observed quantities is generally very complicated and model dependent.

For now we take the exponent $n$ as given and proceed to a reconstruction of the potential. Subsequently we will consider the situation where the primordial spectrum is not entirely known and not necessarily pure power law. We define a new function

$$v(x) \equiv \delta T^2 / q^2,$$

where $v(x) \leq 1$ by definition. For convenience we also define $\alpha = N(1 - n)/2$. Thus for the scale invariant case of $n = 1$ we have $\alpha = 0$, while for $n < 1$ we have $\alpha > 0$. Notice that for the case in which $\alpha = 0$ (corresponding to a scale-invariant perturbation spectrum), the function $v$ is proportional to the potential $V$ (see equation [3.3]). In terms of this new function $v$, the differential equation [3.7] becomes

$$\frac{(v - 1) dv}{2v} = C v + \alpha (v - 1).$$

For the case of $\alpha = constant$, equation [3.11] can be integrated to obtain the solution

$$\frac{-C}{C + \alpha} \log \left[ \frac{(C + \alpha) v - \alpha}{(C + \alpha) v_B - \alpha} \right] + \log(v/v_B) = 2\alpha x,$$

where $v_B$ denotes the function $v(x)$ evaluated at $x = 0$. Keep in mind that $v_B$ represents the ratio (squared) of the amplitude of tensor modes to the total amplitude of density fluctuations.

We can use the above results to reconstruct the inflationary potential as follows. Once the initial condition (i.e., $v_B$) is specified, equation [3.12] provides an implicit, but analytic, solution for $v(x)$. We can then use equation [3.10] to find $\delta T^2$ and then use equation [3.3] to find the potential as a function of $x$. Notice that we have found $V(x)$ and not $V(\phi)$. In order to make this conversion, we must also solve the equation of motion for $\phi(x)$; this equation is written in integral form in equation [2.7].

We have performed the reconstruction process outlined above for varying values of the initial ratio $v_B$ and for varying choices of the index $n$. The results are shown in Figures 1 – 4. For each choice of $v_B$ (which determines the relative amplitude of the
tensor modes), the figures show the resulting potentials $V(\phi)$ for $n=0.5, 0.6, 0.7, 0.8, 0.9, \text{ and } 1.0$. The open symbols represent $x=1$, i.e., the epoch at which galaxy sized perturbations left the horizon during inflation. Keep in mind that this reconstruction process only contains information about the potential during the $N=8$ e-foldings when structure-forming perturbations are produced. This procedure says nothing about the potential at subsequent epochs.

The results shown in Figures 1 - 4 show interesting general trends. For the case in which tensor perturbations produce a substantial contribution to the total (e.g., in Figure 1), the potentials $V(\phi)$ are concave upward. For the opposite case in which tensor modes are negligible (e.g., in Figure 4), the potentials are concave downward and somewhat reminiscent of a cosine potential. To follow up on this latter issue, we fit a cosine potential to the reconstructed potential for the specific case $n = 0.6$ and $v_B = 10^{-4}$. The result is shown in Figure 5. Thus, for perturbation spectra with moderate departures from scale invariance and little contribution from tensor modes, the reconstructed potential looks very much like a cosine. This type of potential is used in the model of Natural Inflation [7] and was first suggested for reasons of technical naturalness. In particular, the required small parameter $\lambda_{FT}$ (see equations [1.5] and [1.6]) occurs naturally in this model.

As mentioned above, the transformation between actual observed quantities such as the microwave anisotropy and the primordial spectrum is complicated and model dependent. Thus, a definitive prediction for $q(x)$ may be difficult to obtain in the near future. However, the observations can be used to imply constraints on the spectrum $q(x)$. For example, our analysis of the observations may imply that the true spectrum lies within some range of power laws. Then we can obtain the range of possibilities for the potential from the figures by restricting ourselves to those curves corresponding to that range of power laws.

For example, we might reasonably require that the amplitude of the perturbations does not change too much with varying lengthscale (wavenumber). In the present formulation, this statement takes the form

$$A \leq a(x) \leq B,$$

where we have defined

$$a(x) \equiv \frac{1}{q} \frac{dq}{dx}.$$  

For the special case in which the primordial spectrum is a pure power-law, the index $a(x)$ defined here is a constant independent of $x$ and is related to the spectral index $n$ through the identity $a = N(1-n)/2$. In general, the index $a$ will not be constant, but we expect that the function $a(x)$ will be a slowly varying function.

Constraints of the form [3.13] imply corresponding constraints on the potential (for a given set of initial conditions). If the index $a$ is constrained as in equation [3.13], then the amplitude $\delta_T$ is constrained to lie between the solutions found with $a = A$ and $a = B$. Since the potential is proportional to $\delta_T^2$, the potential will be similarly constrained (see Figure 1). In other words, the potential is allowed to be in the range of curves corresponding to the appropriate range of indices $n$ in the figures.
In this section, we have considered the reconstruction of the inflationary potential. Building on previous work by several groups [15], we found a semi-analytic solution for the potential for pure power-law spectra, and plotted our results in Figures 1 – 5.

IV. CONSTRAINTS ON THE HEIGHT AND WIDTH OF INFLATIONARY POTENTIALS

In this section we present a series of constraints on the scalar field potential. These bounds apply to all inflationary models which belong to the general class of models defined in Sec. II, i.e., models which obey the density perturbation constraint [2.16], the overdamping constraint [2.17], and the condition of vacuum energy domination [2.18].

A. Relationship Between the Width of the Potential and the Relative Amplitude of Tensor Perturbations

Tensor perturbations (i.e., gravity wave perturbations) can arise during the inflationary epoch. Many authors have explored their effects [12]. The ratio of the amplitude of tensor perturbations to the scalar perturbations can be written in the form

\[ \mathcal{R} = \frac{\sqrt{2\pi}}{3} \left| \frac{\dot{\phi}}{M_{pl} H} \right| . \]  

(4.1)

In terms of the functions defined in Sec. II of this paper, this ratio can be expressed in the form

\[ \mathcal{R}(x) = \frac{8\pi}{15\sqrt{3}} \frac{\Lambda^2}{\delta M_{pl}^2} \frac{p^2}{h}. \]  

(4.2)

If we now take the average value of \( \mathcal{R} \) over the portion of inflation when structure is produced, we obtain

\[ \langle \mathcal{R} \rangle = \frac{8\pi}{15\sqrt{3}} \frac{\Lambda^2}{\delta M_{pl}^2} \int_0^1 \frac{(p^2 / h)}{dx}, \]  

(4.3)

where \( \langle \mathcal{R} \rangle \) denotes the average value.

We must now derive an expression for the width of the potential \( \Delta \phi \) during the \( N \approx 8 \) e-foldings when cosmological structure can be produced. Using the functions defined in Sec. II and equation [2.13], we can write the width \( \Delta \phi \) as

\[ \Delta \phi = \left( \frac{2\pi}{75} \right)^{1/2} \frac{N}{\delta} \frac{\Lambda^2}{M_{pl}^2} K[p, h], \]  

(4.4)

where \( K \) is the functional defined by

\[ K[p, h] = \int_0^1 \frac{(p^2 / h)}{dx}. \]  

(4.5)

Comparing equation [4.3] for the ratio \( \langle \mathcal{R} \rangle \) with equation [4.4] for \( \Delta \phi \), we discover the simple relationship

\[ \langle \mathcal{R} \rangle = \frac{\sqrt{32\pi}}{3N} \frac{\Delta \phi}{M_{pl}}. \]  

(4.6)
Thus, for the \( N \approx 8 \) e-foldings where density fluctuations of cosmological interest can be produced, we obtain

\[
\langle R \rangle \approx 0.42 \Delta \phi / M_{\text{pl}}.
\]  

(4.7)

One important implication of this result can be stated as follows: If tensor perturbations play a major role in the formation of structure, then the width \( \Delta \phi \) during the appropriate part of inflation must be comparable to the Planck mass \( M_{\text{pl}} \). [Notice that since this argument applies only to the average value of the ratio \( R \), it is logically possible for tensor modes to be significant at some particular length scale, even though the average \( \langle R \rangle \) is small. However, the overdamping constraint prevents the potential (and hence \( R \)) from changing very quickly and all but eliminates this possibility.]

As we show below, the allowed width \( \Delta \phi \) is also bounded from above (see equation [4.13]). Thus, we also obtain an upper limit on the ratio \( \langle R \rangle \), i.e.,

\[
\langle R \rangle \leq 1.6.
\]  

(4.8)

Although scalar perturbations can be larger than tensor modes by an arbitrarily large factor, the converse is not true: tensor modes can be at most a factor of \( \sim 1.6 \) larger than the scalar contribution.

**B. Constraints on the Width of the Potential**

In this section we find both lower (equation [4.11]) and upper (equation [4.13]) bounds on the width of the potential. These bounds apply to the portion of the inflationary epoch when cosmologically interesting perturbations are produced.

In order to find bounds on \( \Delta \phi \), we must find bounds on the functional \( K \) in equation [4.5]. This functional can be minimized by saturating the density perturbation constraint [2.16]; we thus obtain

\[
K \geq \int_0^1 p \, dx.
\]  

(4.9)

A lower limit on the remaining integral can be found by using the overdamping constraint [2.17]. Since we have fixed \( p(0) \) and we want to find the smallest possible value for the integral in equation [4.9], we must choose the sign of the derivative \( dp/dx \) to be negative and as large as possible given the overdamping constraint. Our limit thus becomes

\[
K \geq \frac{2}{3N} \left\{ 1 - e^{-3N/2} \right\} \approx \frac{2}{3N},
\]  

where the last approximate equality has a relative error less than \( 10^{-5} \) for \( N = 8 \). As we show in Appendix A, the lower bound of equation [4.10] is in fact the greatest lower bound for this problem. Putting the above results together, we obtain

\[
\frac{\Delta \phi}{M_{\text{pl}}} \geq \frac{2}{15} \left( \frac{2\pi}{3} \right)^{1/2} \delta^{-1} \frac{\Lambda^2}{M_{\text{pl}}^2};
\]  

(4.11)

this is the desired lower bound.
We now derive an upper limit on the allowed width $\Delta \phi$. In this case, we use the constraint of equation [2.18] which implies that the kinetic energy of the rolling field does not dominate the vacuum energy density. Using this constraint, we see immediately that the functional $K$ of equation [4.5] is bounded by

$$K[p, h] \leq \beta,$$  

(4.12)

where $\beta$ is the dimensionless parameter defined in equation [2.18]. Combining this bound with expression [4.4] for $\Delta \phi$, we find

$$\frac{\Delta \phi}{M_{pl}} \leq N \sqrt{\frac{3}{4\pi}}.$$  

(4.13)

For the standard choice $N = 8$, this limit implies $\Delta \phi / M_{pl} \leq 3.9$. We note that this upper limit follows directly from the definition of $\Delta \phi$ and the condition [1.10] which must be met in order for inflation to take place. In particular, this bound is independent of the density perturbation constraint.

Putting all of the results of this subsection together, we find that the change $\Delta \phi$ in the field is constrained to lie in the range

$$\frac{2}{15} \left(\frac{2\pi}{3}\right)^{1/2} \delta^{-1} \frac{\Lambda}{M_{pl}^2} \leq \frac{\Delta \phi}{M_{pl}} \leq N \sqrt{\frac{3}{4\pi}}.$$  

(4.14)

Another way to write this constraint is in terms of the Hubble parameter $H_B$ at the epoch $x = 0$, i.e.,

$$\frac{H_B}{15 \delta} \leq \Delta \phi \leq N \sqrt{\frac{3}{4\pi}} M_{pl}.$$  

(4.15)

The above bounds suggest that the change $\Delta \phi$ in the scalar field is rather constrained. For all cases, $\Delta \phi$ cannot be much larger than the Planck scale $M_{pl}$. The lower bound shows that the change in the field $\Delta \phi$ must be at least a factor of $\sim 10^3$ larger than the Hubble parameter. Notice that for an inflationary energy scale $\Lambda$ comparable to the GUT scale, the change $\Delta \phi$ in the inflaton field must be larger than $\sim M_{pl}$. In the following section, we calculate the width $\Delta \phi$ for three “standard” inflationary potentials and show that the condition $\Delta \phi \sim M_{pl}$ is in fact typical.

**C. Width of the Potential for Examples**

In this section, we calculate the width $\Delta \phi$ for several standard inflationary models, including monomial potentials (such as in the original version of Chaotic Inflation [6]), exponential potentials, and cosine potentials (such as in Natural Inflation [7]). Here, we write the number $N$ of e-foldings as

$$N = \int \frac{H \, dt}{M_{pl}^2} = \frac{8\pi}{M_{pl}^2} \int \frac{V \, d\phi}{dV/d\phi},$$  

(4.16)
where we have used the slowly rolling version of the equation of motion to obtain the second equality. In the integral in equation [4.16], the range of integration corresponds to the range $\Delta \phi$ of interest. For the cases of monomial potentials and exponential potentials, we only consider the portion of inflation during which density fluctuations of cosmologically interesting sizes are produced (i.e., we take $N = 8$ as usual). For the case of Natural Inflation (cosine potentials), we consider the entire overdamped phase of inflation (i.e., we take $N \approx 60$ or so).

We first consider the monomial potential of the form

$$V(\phi) = \lambda_j \phi^j,$$

where $j$ is an integer. For this class of models, the number of e-foldings is given by

$$N = \frac{4\pi}{j} M_{\text{pl}}^{-2} \left[ \phi_1^2 - \phi_2^2 \right],$$

(4.18)

where $\phi_1$ and $\phi_2$ are the initial and final values of the field. Without loss of generality, we take $\phi_1 > \phi_2$ [27]. Solving for the width $\Delta \phi$, we find

$$\frac{\Delta \phi}{M_{\text{pl}}} = \frac{\phi_1 - \phi_2}{M_{\text{pl}}} = \left[ N_j / 4\pi + (\phi_2 / M_{\text{pl}})^2 \right]^{1/2} - \phi_2 / M_{\text{pl}},$$

(4.19)

where we have eliminated $\phi_1$ using equation [4.18]. We examine this expression in two limits, $\phi_2 \ll M_{\text{pl}}$ and $\phi_2 \geq M_{\text{pl}}$. In the first case, $\Delta \phi / M_{\text{pl}} \sim (N_j / 4\pi)^{1/2} \sim 1$ for $N = 8$. The second possibility is that the final value of the field is in the regime $\phi_2 \geq M_{\text{pl}}$. Thus, either $\Delta \phi$ is comparable to the Planck scale or $\phi_2$ is larger than the Planck scale. In either case, an energy scale comparable to or larger than the Planck scale must be present in the problem.

As the next example, we consider an exponential potential of the form

$$V(\phi) = V_0 \exp[-\phi/\sigma],$$

(4.20)

where $\sigma$ is the energy scale that characterizes the fall-off of the potential. We note that this form is often used as an approximation to the true potential and is valid for only part of the inflationary epoch. However, as long as the form [4.20] holds for a few e-foldings of the scale factor, the following argument is valid. Using the definition [4.16], we obtain

$$N = 8\pi \frac{\Delta \phi}{M_{\text{pl}}} \frac{\sigma}{M_{\text{pl}}},$$

(4.21)

Solving for $\Delta \phi$, we find

$$\frac{\Delta \phi}{M_{\text{pl}}} = \frac{N}{8\pi} \frac{M_{\text{pl}}}{\sigma} \approx 0.32 \frac{M_{\text{pl}}}{\sigma},$$

(4.22)

where we have used $N = 8$ to obtain the final approximate equality. Equation [4.22] shows that either the width $\Delta \phi$ must be comparable to the Planck scale $M_{\text{pl}}$, or, the
fall-off scale $\sigma$ must be much larger than $M_{\text{pl}}$. Once again, an energy scale comparable to or larger than the Planck scale must be present.

Finally we consider a cosine potential, i.e.,

$$V(\phi) = \Lambda^4 \left[ 1 + \cos(\phi/f) \right], \tag{4.23}$$

such as that found in Natural Inflation [7]. For this case, we find

$$N_{T\text{OT}} = \frac{16\pi f^2}{M_{\text{pl}}^2} \log \left\{ \frac{\sin(\phi_2/2f)}{\sin(\phi_1/2f)} \right\}, \tag{4.24}$$

where we have denoted the number of e-foldings as $N_{T\text{OT}} \approx 60$ to emphasize that we do not use $N = 8$ for this case. This potential has a definite width, namely $f$. Thus, in this case, we have

$$\Delta \phi_{T\text{OT}} \sim f \sim M_{\text{pl}} \left( \frac{N_{T\text{OT}}}{16\pi} \right)^{1/2} \left\{ \log[2f/\phi_1] \right\}^{-1/2}, \tag{4.25}$$

where $\Delta \phi_{T\text{OT}}$ is the width of the potential over 60 e-foldings (rather than merely the 8 of structure formation). In this equation we have used the fact that $\phi_2 \sim f$ (more precisely, we assume that $\log[\sin(\phi_2/2f)]$ is of order unity). Thus, unless the remaining logarithmic factor in equation [4.25] becomes very far from unity, this potential has a width which is of order the Planck scale $M_{\text{pl}}$. A detailed treatment of the conditions for sufficient inflation with this potential (see Ref. [7]) confirms that $f$ and $\Delta \phi$ must be near the Planck scale $M_{\text{pl}}$ for this model.

We thus conclude that for these particular examples, the inflationary potentials contain energy scales which are comparable to (or larger than) the Planck scale $M_{\text{pl}}$. While all of the models considered here have $\Delta \phi \sim M_{\text{pl}}$, we note that constraint of equation [4.14] is much less restrictive for small values of $\Lambda$ (the energy scale of inflation); for example, if $\Lambda = 10^{12}$ GeV, the bound becomes very weak, $\Delta \phi \gtrsim 10^{-10} M_{\text{pl}}$. This apparent discrepancy is easy to understand. For simple “well-behaved” potentials (such as the examples considered here), the integral in equation [4.16] $\sim (\Delta \phi)^2$, and equation [4.16] reduces to

$$N \sim 8\pi \frac{(\Delta \phi)^2}{M_{\text{pl}}^2}. \tag{4.26}$$

We thus naively expect that any sufficiently well-behaved potential will have $\Delta \phi \sim M_{\text{pl}}$. However, in the general bound of equation [4.14], we allow the potential to take any form, provided only that the density perturbation constraint and the overdamping constraint are satisfied. This considerable extra freedom leads to the appreciably weaker bound.

**D. Constraints on the Change in Height of the Potential**

In this section, we constrain the allowed change in height $\Delta V$ of the potential during the overdamped phase of inflation when cosmological structure may be produced. We
find both upper and lower bounds on $\Delta V$ and show that these results imply an upper limit on the energy scale of inflation (during this phase).

The change in potential $\Delta V$ is given by equation [2.12]. In order to constrain $\Delta V$, we must constrain the functional

$$L[p] = \int_0^1 p^4 \, dx,$$

subject to the constraints of equations [2.16–2.19]. Perhaps counterintuitively, the upper limit on the energy scale $\Lambda$ arises from a lower limit on the change in height $\Delta V$. We thus consider this limit first. Using the overdamping constraint [2.17], we find

$$L \geq \frac{1}{6N} \left\{ 1 - e^{-6N} \right\} \approx \frac{1}{6N}.$$  

(4.28)

We note that this bound is the greatest lower bound for this constraint problem. Putting in the dimensionful quantities, we obtain the limit

$$\Delta V \geq \frac{8\pi^2}{225} \delta^{-2} M_{\text{pl}}^{-4} \Lambda^8.$$  

(4.29)

Since the vacuum contribution to the energy density cannot become negative [28], we must also require

$$\Lambda^4 \geq \Delta V.$$  

(4.30)

Combining these two limits and solving for the scale $\Lambda$, we obtain the desired limit on the energy scale $\Lambda$,

$$\frac{\Lambda}{M_{\text{pl}}} \leq \delta^{1/2} \left( \frac{15 \sqrt{2}}{4\pi} \right)^{1/2} \approx 6 \times 10^{-3}.$$  

(4.31)

This constraint implies that the structure producing portion of inflation must take place at an energy scale roughly comparable to (or less than) the GUT scale. Although found by slightly different methods, this constraint is equivalent to that derived earlier by Lyth [16]. Notice also that this constraint is comparable to that obtained in Ref. [19] by requiring that tensor perturbations are not in conflict with the COBE measurement. Finally, we note that this constraint is equivalent to the requirement $\beta \geq 1$.

We note that an upper bound for $\Delta V$ also exists. This bound can be obtained by finding an upper bound for the functional $L$ (see Eq. [4.27]). Using the overdamping constraint with the opposite sign, we find

$$L[p] \leq \frac{1}{6N} \left\{ e^{6N} - 1 \right\} \approx \frac{e^{6N}}{6N},$$  

(4.32)

and the corresponding bound on $\Delta V$ becomes

$$\Delta V \leq \frac{8\pi^2 e^{6N}}{225} \delta^{-2} M_{\text{pl}}^{-4} \Lambda^8.$$  

(4.33)
At first glance, the bound of equation [4.33] may not seem significant. However, at sufficiently small energy scales $\Lambda$, this bound becomes very severe. Let us define a new dimensionless parameter

$$
\eta \equiv \frac{\Delta V}{V_B} \leq \frac{8\pi^2 e^{6N}}{225} \delta^{-2} \left( \frac{\Lambda}{M_{pl}} \right)^4 \sim 6 \times 10^{29} \left( \frac{\Lambda}{M_{pl}} \right)^4,
$$

(4.34)

where $V_B = \Lambda^4$ is the value of the total vacuum energy density at $x = 0$. For example, if we consider inflationary models at low energies such as $\Lambda = 1$ TeV, we obtain the bound $\eta \leq 10^{-34}$.

It is easy to see why the above result makes inflation at low energies problematic. During the $N = 8$ e-foldings of inflation where structure is produced, we must have $\eta = \Delta V/V_B \ll 1$. However, the vacuum energy density must be essentially zero at the end of the entire inflationary epoch; thus, during the following 52 e-foldings of inflation, we must have $\Delta V/V_B \sim 1$ [29]. It seems unlikely that particle physics models will produce a scalar field potential with such extreme curvature.

Before leaving this section, we note that the above argument defines a suggestive lower bound for the energy scale of inflation. Arguing very roughly, we expect that models of inflation with the parameter $\eta$ very much smaller than unity are difficult to obtain. As shown by equation [4.34], the parameter $\eta$ decreases with the energy scale $\Lambda$ of inflation. We thus obtain a suggestive lower bound for $\Lambda$ by requiring that $\eta$ be larger than some “not too unnaturally small number”, say 1/10. The requirement that $\eta > 1/10$ implies that the energy scale of inflation to obey the constraint

$$
\Lambda \geq M_{pl} \delta^{1/2} e^{-3N/2} \left[ \frac{2250}{8\pi^2} \right]^{1/4} \approx 10^{-7} M_{pl} \approx 10^{12} \text{GeV},
$$

(4.35)

where the numerical value was obtained using $N = 8$. We stress that this bound is not a firm lower limit on the energy scale $\Lambda$, but it is suggestive. In particular, for energy scales $\Lambda$ much less than about $10^{12}$ GeV, the parameter $\eta$ becomes very small compared to unity.

V. CONSTRAINTS ON THE FINE-TUNING PARAMETER

In this section, we constrain the fine-tuning parameter $\lambda_{FT}$ as defined by equation [1.5]. In our previous paper [2], we found a firm upper limit on the parameter $\lambda_{FT}$. In this paper, we first complete the argument by finding a lower limit on $\lambda_{FT}$. Next, we show how density perturbation spectra which are not scale-invariant can place slightly tighter bounds on $\lambda_{FT}$.

A. Lower Bound on the Fine-Tuning Parameter

In order to bound $\lambda_{FT}$, we need bounds on both the height $\Delta V$ and the width $\Delta \phi$. We have already shown that $\Delta V$ is bounded from below by equation [4.29] and that $\Delta \phi$
is bounded from above by equation [4.13]. Combining these two results thus gives us a lower bound on the ratio $\lambda_{FT} = \Delta V / (\Delta \phi)^4$, i.e.,

$$\lambda_{FT} \geq \frac{128}{2025} \frac{\pi^4}{N^4} \frac{\delta^2}{2} \left( \frac{\Lambda}{M_{pl}} \right)^8. \quad (5.1)$$

Combining this result with the general bound of Ref. [2], we find that $\lambda_{FT}$ is confined to the range

$$\frac{128}{2025} \frac{\pi^4}{N^4} \delta^2 \left( \frac{\Lambda}{M_{pl}} \right)^8 \leq \lambda_{FT} \leq \frac{2025}{8} \delta^2. \quad (5.2)$$

For example, if we use representative values of $\Lambda/M_{pl} \sim 10^{-3}$, $N = 8$, and $\delta \sim 2 \times 10^{-5}$, the allowed range for the parameter $\lambda_{FT}$ becomes

$$6 \times 10^{-19} \leq \lambda_{FT} \leq 10^{-7}. \quad (5.3)$$

The bound on $\Delta \phi$ was obtained independently of the density perturbation constraint; the bound on $\Delta V$ was obtained by requiring $\delta \rho / \rho \leq \delta$.

**B. Effects of Departures from Scale Invariance**

For the limits presented thus far (see Sec. IV), we have used the density perturbation constraint in the form of equation [1.7], which assumes that the amplitude of the density perturbations produced by inflation must be less than a constant value (i.e., the constraint is the same for all perturbation wavelengths). However, one way to explain current cosmological data is with density perturbations with a non-scale-invariant spectrum [7, 29]. In this case the departures from scale invariance imply that our universe has density perturbations which exhibit *more power on large scales*. In terms of our constraint [1.7], this result implies that we should replace the constant parameter $\delta$ with some function $\delta(x)$ which is a *decreasing* function of time (and hence a decreasing function of $x$) during the structure producing portion of inflation. Keep in mind that in this present discussion $\delta$ represents the upper bound on the density fluctuations and not the amplitude of the fluctuations themselves.

For this discussion, we take the spectrum of density fluctuations to be a simple power-law (see equations [3.8] and [3.9]). We can incorporate this scale dependence into our density perturbation constraint by writing it in the form

$$\frac{1}{10} \frac{H^2}{\dot{\phi}} \leq \delta(x) = \delta_0 \exp[-\alpha x], \quad (5.4)$$

where $\delta_0$ represents the size of the allowed perturbations at the largest size scale (the present-day horizon scale) and where we have defined

$$\alpha = N (1 - n)/2. \quad (5.5)$$

As before, the scale invariant spectrum $n = 1$ corresponds to $\alpha = 0$ while a spectrum with more power on large scales $n < 1$ corresponds to $\alpha > 0$. Notice that we have written
equation [5.4] as an inequality; we assume that the perturbations produced during inflation (left-hand-side of the equation) are smaller than (or equal to) the actual primordial perturbations (right-hand-side of the equation). In terms of the dimensionless functions introduced in Sec. II, the density perturbation constraint becomes

\[
\frac{p(x)}{h(x)} \geq \exp[\alpha x].
\] (5.6)

In previous work [2], we assumed that \(\alpha = 0\) (i.e., that \(\delta\) is constant). For positive values of \(\alpha\) (i.e., for \(n < 1\)), our new constraint is more restrictive than that used previously and hence leads to tighter constraints.

We now show how this more restrictive constraint affects our upper bound on the fine-tuning parameter \(\lambda_{FT}\). To obtain a bound on \(\lambda_{FT}\), we must find an upper limit to the functional \(J\) defined in equation [2.15]. This functional is maximized by choosing \(h\) so as to saturate the density perturbation constraint, which now takes the form of equation [5.6]; we thus obtain

\[
J[p, h] \leq \frac{\int_0^1 p^4 \, dx}{\left[\int_0^1 p \exp[\alpha x] \, dx\right]^4} \equiv J_\alpha[p],
\] (5.7)

where the function \(p(x)\) appearing in the functional \(J_\alpha\) is subject to the same constraints as before. For the case of \(\alpha > 0\) (\(n < 1\)), the denominator in equation [5.7] is clearly larger than for \(\alpha = 0\) (\(n = 1\)); thus it is immediately clear that \(J_\alpha < J_0\) for any nonzero value of \(\alpha\), where \(J_0\) is the “old” functional with \(\alpha = 0\). Furthermore, our intuition tells us that \(J_\alpha\) reaches its maximum value when the function \(p(x)\) is as nonuniform as possible; in other words, we suspect that the least upper bound for \(J_\alpha[p]\) occurs for a function \(p\) that saturates the overdamping constraint. We show in Appendix B that this conjecture is in fact correct. The maximum for \(J_\alpha\) occurs for the function \(p(x)\) which decreases as fast as possible given the overdamping constraint. We thus obtain the bound

\[
J_\alpha \leq \frac{81N^3}{32} \left[1 - 2\alpha/3N\right]^4 \frac{1 - e^{-6N}}{\left[1 - e^{-3N/2+\alpha}\right]^4} \approx \frac{81N^3}{32} \left[1 - 2\alpha/3N\right]^4,
\] (5.8)

where the final approximate equality holds for most cases of interest since the exponentials are small. For \(\alpha \neq 0\), this bound is tighter than that obtained previously in Ref. [2]; however, the factor \(\mathcal{F}\) by which the bound is tighter is rather small,

\[
\mathcal{F} \approx \left[1 - 2\alpha/3N\right]^{-4} = \left(\frac{3}{2+n}\right)^4,
\] (5.9)

where we have used the definition of \(\alpha\) in the second expression. Thus, for the largest expected departures from scale-invariance, \(n \sim 1/2\), we find \(\mathcal{F} \approx 2\). Even for the rather extreme departure from scale-invariance of \(n = 0\), we obtain only a modest increase in the bound with \(\mathcal{F} = 81/16 \approx 5\). We therefore conclude that departures from scale-invariance lead to moderately tighter constraints on the fine-tuning parameter \(\lambda_{FT}\).
VI. SUMMARY AND DISCUSSION

In this paper, we have found constraints on the scalar field potential for a general class of inflationary models which have slowly rolling fields. These constraints apply to all models of inflation which exhibit overdamped motion of the scalar field and which obey the density perturbation constraint. This work thus extends that of Ref. [2].

[1] We have studied the reconstruction of the inflationary potential by considering both scalar and tensor modes. The simultaneous consideration of both types of perturbations leads to a differential equation which could be solved to find the potential if the total primordial spectrum of perturbations were known (see also Ref. [15]). We showed how constraints on this spectrum imply corresponding constraints on the reconstructed potential $V(\phi)$. Figures 1 – 4 show the reconstructed potentials for the expected range of parameter space. For the case of density perturbation spectra with moderate departures from scale invariance (e.g., $n = 0.6$) and little contribution from tensor modes, the reconstructed potential is very similar to a cosine (see Figure 5) such as in the model of Natural Inflation [7].

[2] We have derived a relationship between the amplitude of tensor perturbations and the width of the scalar field potential (see equation [3.21]). In particular, the average ratio $\langle R \rangle$ of tensor to scalar perturbations is comparable to the dimensionless width $\Delta \phi / M_{pl}$ of the potential. Thus, if tensor perturbations are important, then the width of the potential must be comparable to the Planck mass. As we discuss in item [3] below, the width $\Delta \phi$ is bounded from above; as a result, the average ratio $\langle R \rangle$ is also bounded from above. This result implies that while scalar perturbations can dominate over tensor perturbations by an arbitrarily large factor, the converse is not true; tensor perturbations can be at most a factor of $\sim 1.6$ larger than scalar perturbations.

[3] We have found both upper and lower limits on the change $\Delta \phi$ of the scalar field during the phase of inflation which produces cosmic structure (see equation [4.14]). These limits can be summarized by the relation

$$0.4 \left( \frac{\Lambda}{10^{15} \text{GeV}} \right)^2 \leq \frac{\Delta \phi}{M_{pl}} \leq 3.9 \left( \frac{N}{8} \right).$$

The lower limit depends on the energy scale at which inflation takes place. For energy scales larger than the GUT scale, the width $\Delta \phi$ must be larger than the Planck scale $M_{pl}$. The upper limit implies that the change in the scalar field during the $N = 8$ e-foldings of structure-forming perturbations cannot be larger than $\sim 4 M_{pl}$.

[4] We have found both upper and lower bounds on the change $\Delta V$ of the potential during inflation. These bounds can be used to find an upper limit on the energy scale $\Lambda$ of the part of inflation when cosmological structure is produced (see equation [4.31]),

$$\frac{\Lambda}{M_{pl}} \leq 6 \times 10^{-3} \left( \frac{\delta}{2 \times 10^{-5}} \right)^{1/2},$$
where $\delta$ is the maximum allowed amplitude of density perturbations (see equation [1.4]). This limit shows that the epoch of structure-forming perturbations must take place at an energy scale less than about the GUT scale. This bound is almost identical to those found earlier from the consideration of scalar perturbations [16, 18]. The bound is comparable to that obtained from the consideration of tensor perturbations [19].

[5] We have also presented a very rough argument which indicates that inflation at very low energy scales will encounter some difficulty: the fractional change in the height of the potential during the $N = 8$ e-foldings of structure formation is very small when the energy scale $\Lambda$ is small, i.e.,

$$\eta = \frac{\Delta V}{V_B} \sim 6 \times 10^{29} \left( \frac{\Lambda}{M_{\text{pl}}} \right)^4 \sim \frac{1}{10} \left( \frac{\Lambda}{10^{12} \text{GeV}} \right)^4.$$  

If $\Lambda$ is small compared to $\sim 10^{12}$ GeV, then $\eta \ll 1$ and it is difficult for the potential to drop to (roughly) zero in the remaining e-foldings for a normally shaped potential.

[6] We have found a lower bound on the fine-tuning parameter $\lambda_{FT}$. Our previous bound [2] showed that the parameter $\lambda_{FT}$ must be quite small ($\leq 10^{-7}$); this new bound shows that $\lambda_{FT}$ cannot be made arbitrarily small. These bounds thus confine the fine-tuning parameter to the range

$$6 \times 10^{-11} \left( \frac{\Lambda}{10^{17} \text{GeV}} \right)^8 \leq \lambda_{FT} \leq 10^{-7}.$$  

[7] We have explored the effects of non-scale-invariance of density perturbations on the fine-tuning parameter $\lambda_{FT}$ of Ref. [2]. If the density perturbations are required to be non-scale-invariant, then we obtain a stronger bound on $\lambda_{FT}$. However, for the departures from scale-invariance proposed as an explanation of recent observations of cosmological data on large scales (e.g., $n \approx 0.6$; see, e.g., Ref. [7]), the bound is improved by a rather modest factor ($\mathcal{F} \sim 2$).
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APPENDIX A: GLOBAL BOUND ON $\Delta \phi$

In this appendix, we show that the lower bound of equation [4.10] obtained for the functional $K[p, h]$ is in fact the greatest lower bound. So far we have shown that

$$K[p, h] \geq K_*$$

where $K_*$ is given by $K$ evaluated using $h(x) = p(x)$, which saturates the density perturbation constraint, and

$$p(x) = p_*(x) \equiv \exp[-3Nx/2],$$

which saturates the overdamping constraint.

Suppose that $K_*$ is not the greatest lower bound. Then there exist functions $p(x)$ and $h(x)$ which satisfy the constraints (and the initial conditions) and for which $K[p, h] < K_*$. In other words,

$$\int_0^1 (p^2/h) \, dx < \int_0^1 \exp[-3Nx/2] \, dx.$$  \hspace{1cm} (A3)

Since both integrands are positive definite, it follows that for some point $x_0$, we must have

$$\left. \frac{p^2}{h} \right|_{x_0} < \exp[-3Nx_0/2].$$  \hspace{1cm} (A4)

Since the functions $p$ and $h$ satisfy the density perturbation constraint (by hypothesis), $p/h \geq 1$ for all $x$ and in particular for $x_0$. Equation [A4] implies that

$$p(x_0) < \exp[-3Nx_0/2].$$  \hspace{1cm} (A5)

This final inequality violates the overdamping constraint and thus leads to a contradiction. Hence, the bound of equation [4.10] is the greatest lower bound.
APPENDIX B: BOUND ON THE FUNCTIONAL \( J_\alpha \)

In this Appendix, we find the least upper bound on the functional \( J_\alpha[p] \) defined by equation [5.7] in the text. In particular, we show that the desired bound is given by the functional evaluated at \( p = p_\alpha(x) \), where \( p_\alpha(x) \) is the function which decreases as fast as possible while maintaining the overdamping constraint \( [2.17] \), i.e.,

\[
p_\alpha(x) = \exp[-3Nx/2], \tag{B1}
\]

where we have used the initial condition \( [2.19] \).

We present a proof by contradiction. Suppose that \( p_\alpha \) does not provide the least upper bound as claimed. Then there exists another function \( p(x) \) which satisfies the constraints of Sec. II and for which

\[
J_\alpha[p] \geq J_\alpha[p_\alpha]. \tag{B2}
\]

After introducing the notation

\[
\langle \ldots \rangle \equiv \int_0^1 (\ldots) \, dx, \tag{B3}
\]

we can write the condition \( [B2] \) in the form

\[
\frac{\langle p\rangle^4}{\langle p \exp[\alpha x] \rangle^4} \geq \frac{\langle p_\alpha \rangle^4}{\langle p_\alpha \exp[\alpha x] \rangle^4}. \tag{B4}
\]

However, we also know that \( p_\alpha \) does provide the least upper bound for the functional \( J \) (as shown in Ref. [2]). As a result, the functions \( p(x) \) and \( p_\alpha(x) \) must also obey the inequality

\[
\frac{\langle p\rangle^4}{\langle p \rangle^4} \leq \frac{\langle p_\alpha \rangle^4}{\langle p_\alpha \rangle^4}. \tag{B5}
\]

The combination of these two results can be put in the form

\[
\frac{\langle p \exp[\alpha x] \rangle^4}{\langle p_\alpha \exp[\alpha x] \rangle^4} \leq \frac{\langle p\rangle^4}{\langle p_\alpha \rangle^4} \leq \frac{\langle p \rangle^4}{\langle p_\alpha \rangle^4}. \tag{B6}
\]

Eliminating the middle portion of equation \( [B6] \) and taking the fourth root, we find

\[
\frac{\langle p \exp[\alpha x] \rangle}{\langle p_\alpha \exp[\alpha x] \rangle} \leq \frac{\langle p \rangle}{\langle p_\alpha \rangle}. \tag{B7}
\]

Next we define “weight functions” according to

\[
w(x) \equiv \frac{p(x)}{\langle p \rangle}, \quad \text{and} \quad w_\alpha(x) \equiv \frac{p_\alpha(x)}{\langle p_\alpha \rangle}, \tag{B8}
\]
where, by definition,

\[ \langle w \rangle = 1 = \langle w_* \rangle. \]  

(B9)

Using the weight functions, the inequality [B7] becomes simply

\[ \langle w \exp[\alpha x] \rangle \leq \langle w_* \exp[\alpha x] \rangle. \]  

(B10)

In what follows, we show that this equation is false and hence leads to a contradiction.

Since both \( w \) and \( w_* \) have mean values of unity (Eq. [B9]) and since the two functions are different, we must have \( w > w_* \) for some values of \( x \) and \( w < w_* \) for other values of \( x \). However, by construction, the function \( w_* \) is monotonically decreasing and is decreasing as fast as possible given the overdamping constraint. Therefore, \( w = w_* \) at only one point [31], denoted here as \( x_0 \) and

\[
\begin{align*}
    w(x) &< w_*(x) \quad \text{for} \quad x < x_0, \\
    w(x) &> w_*(x) \quad \text{for} \quad x > x_0.
\end{align*}
\]  

(B11a)

(B11b)

Thus,

\[
\int_0^{x_0} (w_* - w) \, dx = C = \int_{x_0}^1 (w - w_*) \, dx.  
\]  

(B12)

Now, using the Mean Value Theorem, we find

\[
\int_0^{x_0} (w_* - w) \exp[\alpha x] \, dx = \exp[\alpha \xi] \int_0^{x_0} (w_* - w) \, dx = \exp[\alpha \xi] C,  
\]  

(B13)

where \( \xi \) is some number in the range \( 0 < \xi < x_0 \). Similarly, using the Mean Value Theorem a second time, we find

\[
\int_{x_0}^1 (w - w_*) \exp[\alpha x] \, dx = \exp[\alpha \eta] \int_{x_0}^1 (w - w_*) \, dx = \exp[\alpha \eta] C,  
\]  

(B14)

where \( \eta \) lies in the range \( x_0 < \eta < 1 \). Notice that for \( \alpha \neq 0 \), neither \( \xi = x_0 \) nor \( \eta = x_0 \) so that \( \eta > \xi \). Putting these results together, we find

\[
\int_0^{x_0} (w_* - w) \exp[\alpha x] \, dx = \exp[\alpha \xi] C < \exp[\alpha \eta] C = \int_{x_0}^1 (w - w_*) \exp[\alpha x] \, dx.  
\]  

(B15)

After a bit of rearrangement, this expression can be written in the form

\[
\langle w_* \exp[\alpha x] \rangle < \langle w \exp[\alpha x] \rangle,  
\]  

(B16)

which contradicts equation [B10] above. Thus, our supposition that \( p_* \) does not provide the least upper bound leads to a contradiction. The supposition must be false, i.e., the least upper bound of the functional \( J_*(p) \) is given by the functional evaluated at \( p = p_* \).
REFERENCES


[21] A similar differential equation is derived but not solved analytically in the paper by Copeland et al. (1993) in Ref. [15].

[22] For further discussion of density perturbations, see standard textbooks such as Refs. [23] and [8]; see also the recent review of Ref. [24].


[25] We note that different authors use different normalizations for $\delta_S$ and $\delta_T$ (see Refs. [15]), i.e., they use different numerical coefficients for the expressions in equations [3.1] and [3.2]. If we use these different normalizations, the constant $C$ will change by a factor of order unity. However, this change is sufficiently small that it will not affect the present discussion; in particular, the resulting shapes of the potential will not change.


[27] If we had the opposite case, $\phi_1 < \phi_2$, then we could simply perform a reflection $\phi \rightarrow -\phi$ without changing the physics.

[28] Strictly speaking, the energy density can become slightly negative; present day cosmological constraints allow a negative vacuum energy density roughly comparable to the critical density at the present epoch ($\sim 8h^2 \times 10^{-47}$ GeV$^4$). This correction will not affect the present argument in any substantial way.

[29] We note that the $\Delta V$ in the numerator refers to the change in the scalar field potential, while the $V$ in the denominator refers to the total vacuum energy density. For inflationary models with only one scalar field, the two potentials are the same.


[31] It is possible for $w = w_s$ to hold for a finite interval rather than for a single point. However, the same argument can be easily modified to accommodate this possibility.
FIGURE CAPTIONS

Figure 1. Reconstructed inflationary potential for the case where tensor perturbations provide 31% of the total at \( x = 0 \) (i.e., \( v_B = 10^{-1} \)). [Note that the parameter \( x \) characterizes the number of e-foldings subsequent to the epoch \( x = 0 \), which occurs \( \sim 60 \) e-foldings before the end of inflation when cosmological structure on the scale of our horizon was produced.] The various curves are for indices \( n = 0.5, 0.6, 0.7, 0.8, 0.9, \) and 1.0 (from bottom to top). The open symbols denote the epoch at which galaxy-sized perturbations leave the horizon during inflation.

Figure 2. Reconstructed inflationary potential for the case where tensor perturbations provide 10% of the total at \( x = 0 \) (i.e., \( v_B = 10^{-2} \)). The various curves are for indices \( n = 0.5, 0.6, 0.7, 0.8, 0.9, \) and 1.0 (from bottom to top). The open symbols denote the epoch at which galaxy-sized perturbations leave the horizon during inflation.

Figure 3. Reconstructed inflationary potential for the case where tensor perturbations provide 3.1% of the total at \( x = 0 \) (i.e., \( v_B = 10^{-3} \)). The various curves are for indices \( n = 0.5, 0.6, 0.7, 0.8, 0.9, \) and 1.0 (from bottom to top). The open symbols denote the epoch at which galaxy-sized perturbations leave the horizon during inflation.

Figure 4. Reconstructed inflationary potential for the case where tensor perturbations provide 1% of the total at \( x = 0 \) (i.e., \( v_B = 10^{-4} \)). The various curves are for indices \( n = 0.5, 0.6, 0.7, 0.8, 0.9, \) and 1.0 (from bottom to top). The open symbols denote the epoch at which galaxy-sized perturbations leave the horizon during inflation.

Figure 5. Comparison of reconstructed inflationary potential and a cosine potential. The reconstructed potential was obtained using \( n = 0.6 \) and \( v_B = 10^{-4} \) (tensor perturbations initially produce 1% of the total). The fit was obtained by constraining the cosine curve to agree with the reconstructed potential at the endpoints \( x = 0 \) and \( x = 1 \).