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SAGA HE 51
5003403

SAGA-HE-51

October 1993

Nonstatic Effect on the Intersoliton Potential and the $1/N$ Expansion

Sakae SAITO

Department of Physics, Nagoya University, Nagoya 464-01

Masayuki UEHARA

Department of Physics, Saga University, Saga 840

Abstract

We examine, in the sine-Gordon laboratory, a nonstatic effect on constructing the classical intersoliton potential starting from the sum-ansatz configuration and the validity of the $1/N$ expansion in the two-soliton sector, where the $1/N$ expansion means the expansion in powers of the coupling constant or of the inverse of the soliton mass.

§1 Introduction

Recently the study of the nuclear force beyond the product ansatz in the Skyrme model has been performed by the iterative approximation^{1,2,3)} in order to remedy a shortage of the attractive spin-isospin independent nuclear force in the model. The method to use the Atiyah-Manton configuration⁴⁾ and the numerical approach⁵⁾ have also been examined.

The iterative approximation is considered to be the simplest prescription to go beyond the product ansatz. The first iterative correction is shown to be attractive, but seems to be too strong.^{1,2)} In order to reduce the size of the first iterative correction the authors have required that the first iterated configuration is orthogonal to the quasi zero-modes of the configuration of the product ansatz. In this paper we point out that a correct method to reduce the size of the first iteration is not to impose the orthogonality on the iterative configuration, but is to take into account of a nonstatic effect.

The $1/N$ expansion in the two-soliton sector seems not to be clear in contrast to the single-soliton sector, where the $1/N$ expansion is the expansion in powers of the coupling constant or of the inverse of the soliton mass. For example, it is not clear that the kinetic energy of the relative motion of the two solitons is of $O(N^{-1})$ or $O(N)$. If the momentum canonically conjugate to the spatial separation is to be simply of $O(N^0)$, the kinetic energy is of $O(N^{-1})$, and then the nonstatic effect is not crucial. We examine the $1/N$ expansion in the two-soliton sector in the sine-Gordon laboratory, and show that the nonstatic effect in the two-soliton system is important due to the fact that both the intersoliton potential and the kinetic energy of the relative motion are of the same order of the soliton mass.⁶⁾

The conventional iterative approximation is briefly reviewed in the next section. The nonstatic effect is discussed in section 3 and an "exact" potential is deduced from the exact soliton-soliton classical configuration by converting the time dependence into the dependence on the intersoliton distance. The numerical solution starting from the sum-ansatz is shown to give the "exact" potential. Section 4 is devoted to the issue of the

$1/N$ expansion in the two-soliton sector, and we show that the semiclassical momentum squared of the relative motion is almost of $O(N)$. The Dirac formalism of the two-soliton system consisting of the collective coordinate and the fluctuation around the classical two-soliton configuration is studied in section 5. Conclusions and discussion are given in the last section.

§2 Conventional iterative approximation

At first, we discuss the conventional iterative approximation starting from the sum-ansatz configuration: The two-soliton configuration of the sum ansatz, in which each soliton is centered at $r/2$ and $-r/2$, respectively, is written as

$$\phi(x; r) = \phi_0(x; r) = \phi_s(x - r/2) + \phi_s(x + r/2) \quad (1)$$

with the single-soliton static configuration $\phi_s(x) = 4m/\sqrt{\lambda} \cdot \tan^{-1} e^x$, where m and λ are the mass parameter and the coupling constant in the sine-Gordon theory. The classical configuration is of $O(N^{1/2})$. The classical potential for the sum ansatz is given as

$$\begin{aligned} V_0(r) &= \int dx \left(\frac{1}{2} \phi_0'^2 + U[\phi_0] \right) - 2M_s, \\ &= \frac{M_s}{\sinh^2(\frac{r}{2})} \left(1 - \frac{\tilde{r}}{2 \cosh(\frac{r}{2}) \sinh(\frac{r}{2})} \right), \end{aligned} \quad (2)$$

where $M_s = 8m^2/\lambda$ is the soliton mass and $\phi_0' = \partial\phi_0/\partial x$, and the potential is of $O(N)$ as is well known. It reduces to

$$4M_s e^{-\tilde{r}} (1 - 2\tilde{r} e^{-\tilde{r}}) \quad (3)$$

for a large r , the leading term of which has been given a long time ago.⁷⁾

Since ϕ_0 does not satisfy the static equation of motion, there appears the deviation from the equation of motion denoted as

$$S_0(x; r) = -\phi_0''(x, r) + U^{(1)}[\phi_0], \quad (4)$$

where $U^{(n)}[\phi] = \partial^n U[\phi]/\partial \phi^n$ with $U[\phi] = m^2/\lambda [1 - \cos(\sqrt{\lambda}/m \phi)]$ and S_0 is of $O(N^{1/2})$.

The first iterative correction to the sum ansatz is put as^{1,2)}

$$\phi(x, t) \rightarrow \phi_{\text{cl}}(x, t; r) = \phi_0(x; r) + \eta_1(x, t; r), \quad (5)$$

as if η_1 is the fluctuation around ϕ_0 . Inserting ϕ_{cl} into the equation to ϕ , we get the equation to $\eta_1(x, t; r)$:

$$\ddot{\eta}_1 - \eta_1'' + U^{(2)}[\phi_0]\eta_1 = -S_0 + O(\eta_1^2). \quad (6)$$

In the above the intersoliton distance r is treated as a time-independent parameter in this section, and the dot denotes the time derivative. Since the inhomogeneous term $\tilde{S}_0$ is of $O(N^{1/2})$, η_1 is also of $O(N^{1/2})$, that is of the same order as the classical sum-ansatz configuration.

A general solution of the linearized inhomogeneous equation in Eq.(6) is composed of a particular solution ϕ_1 to the inhomogeneous equation and the general solution η_2 to the homogeneous equation,

$$\eta_1(x, t; r) = \phi_1(x; r) + \eta_2(x, t; r). \quad (7)$$

The particular solution ϕ_1 need not to have the proper time dependence, because the inhomogeneous term S_0 is time-independent. The linearized equation to $\phi_1(x; r)$ is written as

$$-\phi_1''(x; r) + U^{(2)}[\phi_0]\phi_1(x; r) = -S_0(x; r). \quad (8)$$

Here, we emphasize that ϕ_1 is a function of the same arguments (x, r) as $\phi_0(x, r)$, and that it does not introduce any new degrees of freedom in addition to ϕ_0 . If r is a dynamical variable, the momentum conjugate to r is given as

$$p = \dot{r} \int dx \{ \partial_r \phi_0(x; r) + \partial_r \phi_1(x; r) \}^2, \quad (9)$$

and we have no other momenta which introduce constraints. Thus, the role of ϕ_1 should not be regarded as a fluctuation having new degrees of freedom, but as a modification

of the form of the configuration. (See section 5 for the Dirac formalism of the pure fluctuation and the collective coordinates.)

The solution to Eq.(8) is written as

$$\phi_1(x; r) = - \int dy G(x, y; r) S_0(y; r), \quad (10)$$

where $G(x, y; r)$ is the full Green function. The asymptotic behavior of ϕ_1 is of $O(e^{-3/2r})$, as can be seen from that of S_0 .

When truncating nonlinear terms in ϕ_1 up to the quadratic ones, we have the familiar form of the first correction V_1 to V_0 :

$$V_1(r) = -\frac{1}{2} \int dx dy S_0(x; r) G(x, y; r) S_0(y; r), \quad (11)$$

which behaves as e^{-3r} asymptotically. This form of the first correction to the potential has been given for the instanton-instanton force,⁸⁾ and discussed in the Skyrme model^{1,2)} and in the sine-Gordon laboratory.²⁾ It was shown in the Skyrme model, by using the free Green function, that V_1 is strongly attractive, and that the resultant potential $V_0 + V_1$ is rather near to V_1 almost for the whole range of r except for at long distances. This is not considered to be favorable as a scheme of the iterative approximation.

In order to reduce the size of the first iterative correction, it was required that ϕ_1 is orthogonal to the quasi zero-mode $\partial\phi_0/\partial r$,¹⁾ and shown that its effect is quite large in reducing the size in the sine-Gordon laboratory.²⁾ The orthogonality is a gauge-fixing condition, which is required to kill the redundant degree of freedom between the collective coordinate r and the fluctuation which is the new degrees of freedom having the proper time dependence. However, $\phi_1(x; r)$ is not a new degree of freedom: There is no reason to impose such a gauge-fixing condition on ϕ_1 , as stated before. A correct method, referring to the exact two-soliton solution of the sine-Gordon theory, will be discussed in the next section.

If the $O(\eta_1^2)$ term is discarded in Eq.(6), η_2 is the solution to the homogeneous equation, which has the proper time dependence. Inserting ϕ_{ex} into Eq.(6) with the $O(\pi_1^2)$ term, however, we have another inhomogeneous term, which is proportional to ϕ_1^2 and is also of $O(N^{1/2})$. The residual configuration η_2 is then decomposed into $\phi_2(x; r) + \eta_2(x; t; r)$, where ϕ_2 is the second correction to the *classical* two-soliton configuration. The iterative series $\phi_0, \phi_1, \phi_2, \dots$ is similar to the series of the Newton method to obtain the root $f(x) = 0$ starting with a point x_0 .⁸⁾ It is, however, questionable whether the series converges, because the static configuration does not exist.

§3 Nonstatic effect and the “exact” potential

The sine-Gordon theory possesses the exact soliton-soliton configuration, which describes soliton-soliton scattering. We deduce an “exact” potential from the exact two-soliton configuration by relating the time t to the intersoliton distance r . This potential serves as a reference frame to examine the validity of the approximations we adopt. The exact two-soliton configuration with the velocity u is written as

$$\phi_{\text{exact}}(x, t) = 4 \frac{m}{\sqrt{\lambda}} \tan^{-1} \left[\frac{u \sinh(mx)}{\cosh(umt)} \right] \quad (12)$$

at the non-relativistic limit: $\sqrt{1 - u^2} \approx 1$. Here we stress that the velocity is of $O(N^0)$, irrespective to the relativistic or the nonrelativistic kinematics, because it is a parameter independent of the coupling constant λ .

The topological charge density $j_0(x, t)$, proportional to $\partial \phi_{\text{exact}} / \partial x$, has two peaks at least in far past and far future. In the scattering process the two peaks of the topological charge density get closer to each other, reach the minimum position, and then they leave from each other again by the repulsive force between them. We note that the relative distance r between the peaks satisfies the equation,

$$\cosh^2(ut) = u^2 (1 + \cosh^2(\frac{r}{2})), \quad (13)$$

where $\bar{t} = mt$ and $\bar{r} = mr$. This relation relates t to r one-to-one correspondingly. The time-dependent behavior of the exact two-soliton configuration is interpreted as the two-soliton configuration written in terms of the intersoliton distance r and the spatial variable x . The definition of the intersoliton distance is not unique,⁹⁾ but ours is intuitive and transparent.

The “exact” intersoliton potential is then given by the usual definition:

$$\begin{aligned} V_{\text{exact}}(r) &= \int dx \left(\frac{1}{2} \phi_{\text{exact}}'^2 + U[\phi_{\text{exact}}] \right) - 2M_s, \\ &= \frac{M_s}{\cosh^2(\frac{\bar{r}}{2})} \left[1 - \frac{\log(\cosh(\frac{\bar{r}}{2}) + \sqrt{1 + \cosh^2(\frac{\bar{r}}{2})})}{\cosh(\frac{\bar{r}}{2}) \sqrt{1 + \cosh^2(\frac{\bar{r}}{2})}} \right], \end{aligned} \quad (14)$$

where t in $\phi_{\text{exact}}(x, t)$ is replaced by r through Eq.(13). The asymptotic behavior of the “exact” potential is

$$V_{\text{exact}}(r) \xrightarrow{r \rightarrow \infty} 4M_s e^{-r} (1 - 2\bar{r} e^{-r}), \quad (15)$$

which is the same as that of the potential by the sum ansatz. When Eq.(13) is regarded as a trajectory of a classical particle with the reduced mass $M_s/2$, the same asymptotic potential $V(r) \sim 4M_s \exp(-mr)$ is consistently obtained for $u \approx 0$, and the size of the potential is determined by the Newton equation to a point particle in this case.

Here, we emphasize that the exact two-soliton configuration $\phi_{\text{exact}}(x; r)$ does not satisfy the static equation of motion either because of the time dependence:

$$\begin{aligned} -\phi_{\text{exact}}''(x; r) + U'[\phi_{\text{exact}}] &= -\ddot{\phi}_{\text{exact}}(x, t)|_{t=t(r)} \\ &\equiv S_{\text{time}}(x; r). \end{aligned} \quad (16)$$

If we dare to put

$$\phi_{\text{exact}}(x; r) = \phi_0(x; r) + \phi_1(x; r), \quad (17)$$

the linearized equation to ϕ_1 can be rewritten as

$$-\phi_1''(x; r) + U'[\phi_0]\phi_1(x; r) = S_{\text{time}}(x; r) - S_0(x; r) \quad (18)$$

$$\equiv -\tilde{S}_1(x; r), \quad (19)$$

where ϕ_0 and S_0 are the sum-ansatz configuration and its deviation from the equation of motion discussed in §2, respectively. The right-hand side of Eq.(18) shows that the inhomogeneous term involves a nonstatic effect. Therefore, the whole of $-S_0$ should not be regarded as the true deviation from the equation of motion. The asymptotic behavior of S_1 is of $O(e^{-9/2r})$, and its magnitude is seen to be much smaller than that of S_0 . The first correction V_1 is now given as

$$V_1(r) = - \int dx \left(S_0(x; r) - \frac{1}{2} S_1(x; r) \right) G(x, y; r) S_1(y; r), \quad (20)$$

the range of which is of $O(e^{-8r})$, and its magnitude can be seen to be much reduced. The contribution from higher order terms in ϕ_1 is almost negligible for the case of S_1 . The potentials V_{exact} , V_0 and $V_0 + V_1$ are shown in Fig.1.

Fig.1

Since the first correction ϕ_1 is defined through the linearized equation such as Eqs.(8) and (18) ignoring non-linear terms, the same procedure can be applied to the second, third corrections and so on. Since the source of the second correction is proportional to $O(\phi_1^2)$, it behaves as $e^{-9/2r}$ for the case of S_0 , but does as $e^{-15/2r}$ for S_1 . The magnitude and the range of the second correction are thus reduced by the nonstatic effect. The iterative series is expected to converge for the case of S_1 because of the nonstatic effect.

It is considered that the nonstatic effect also plays an important role in studying the interaction between two Skyrmions if we start from the product ansatz. The configuration in the product ansatz does not satisfy the equation of motion, and the interaction between the Skyrmions is of $O(N_c)$; therefore, this is similar to the case of the sine-Gordon theory starting from the sum ansatz. Unfortunately, it is hard, however, to obtain the source function with the nonstatic effect for the Skyrme model as in the sine-Gordon theory.

In order to study such a nonstatic effect, one of the most useful approaches will be to solve the full equation of motion numerically, starting from the initial product-ansatz configuration at a large distance. Such a numerical study has already been examined by Verbaarschot et al.¹¹⁾, but further studies will be required to obtain results for the interactions between the two Skyrmions in various channels.

In order to see how such a numerical study works in the sine-Gordon laboratory, we tried to solve the full equation of motion. The discretized equation on a spatial lattice was solved, starting from the sum-ansatz configuration, where each soliton is centered at $r_0/2$ and $-r_0/2$, respectively:

$$\phi_0(x; t) = \phi_s(\gamma(\tilde{x} - \frac{\tilde{r}_0}{2} + ut)) + \phi_s(\gamma(\tilde{x} + \frac{\tilde{r}_0}{2} - ut)) \quad (21)$$

with $\gamma = 1/\sqrt{1-u^2}$, which is introduced for ϕ_0 to satisfy the equation of motion as exactly as possible in the asymptotic region, since we take $u = 0.4$, that is not so small that we can ignore γ .

We set $mr_0/2 = 10.0$ and $u = 0.4$ as the initial condition. The spatial lattice-size is taken to be $(-20, 20)$, and the spacing is 0.1 in unit of $1/m$. The time step is 0.125×10^{-3} . Much more rough time steps and smaller lattice-size give almost similar results, but we need the small step and the large size to obtain the smooth topological charge distribution, from which we estimate the position of the soliton. The minimum distance between the two solitons is restricted up to about $r = 2.94$ for $u = 0.4$ from the nonrelativistic relation Eq.(13). We calculated the intersoliton potential by transforming the time-dependence into the intersoliton distance through the topological charge distribution, which is shown in Fig.2a. Since we cannot ignore γ , we modify the definition of the potential as

$$V(\tilde{r}) = \frac{1}{2} \int d\tilde{x} \left(\frac{\partial \phi(x, t)}{\partial \tilde{x}} \right)^2 + \int d\tilde{x} U[\phi] - 2M, \quad (22)$$

which reduces to the "exact" potential in terms of $\tilde{r}$ for $\phi = \phi_{\text{exact}}$, where $\tilde{x} = \gamma x$ and $\tilde{t} = \gamma t$.

Fig.2a and 2b

We see that the numerical approach to solve the full equation of motion works well in the sine-Gordon laboratory. The crucial point is to solve the configuration time-dependently. The results are shown in Fig.2b.

§4 The $1/N$ expansion in the two-soliton sector

In the soliton-soliton scattering process both of the solitons are initially put sufficiently far from each other and move opposite to each other with the velocity u and $-u$ in the center of mass system. The velocity u is a parameter of $O(N^0)$ independent from the coupling constant λ , irrespectively to the relativistic or nonrelativistic kinematics as stated below Eq.(12).

Inserting the exact soliton-soliton solution ϕ_{exact} with the velocity u into the Hamiltonian, we have the total energy as

$$E_{\text{tot}} = \frac{2M_s}{\sqrt{1-u^2}} \quad (23)$$

$$= 2M_s + M_s u^2 + O(u^3), \quad (24)$$

where the second term of Eq.(24) is just the nonrelativistic kinetic energy of the relative motion of the solitons. The total relative energy must be the same in the intermediate region, where the intersoliton potential changes the motion of the solitons:

$$E_{\text{rel}} = M_s u^2 = K + V(r), \quad (25)$$

where K is the kinetic energy of the relative motion of the solitons in the nonrelativistic kinematics. Since the soliton-soliton potential $V(r)$ is repulsive in the sine-Gordon theory and is of $O(N)$, E_{rel} should be of $O(N)$, because no cancellation occurs between K and V . The kinetic energy K is also of $nc\ell$, because $E_{\text{rel}} \sim K$ in the asymptotic region, where the intersoliton potential fades away. This is consistent with the N -independence of the velocity as we have already insisted before.

Witten⁶⁾ and Coleman.¹²⁾ stated that the kinetic energy should be of $O(N)$ in order that a baryon goes near to and then goes away from another baryon against the potential of $O(N)$, and that the fixed velocity and angle should be used, instead of using the fixed momentum and momentum transfer, for baryon-baryon scattering.

We examine here the N -dependence of the kinetic energy in the quantum mechanics of the soliton-antisoliton bound states in the $1/N$ expansion. We note that the doublet states have the mass given by

$$M_n = 2M_s \sin\left(\frac{\lambda}{16m^2}n\right) \quad (26)$$

with $n = 0, 1, \dots, < 8m^2\pi/\lambda$.¹⁴⁾ The relative energy of the system is then written as

$$E_n = M_n - 2M_s = -2M_s \left\{1 - \sin\left(\frac{\lambda}{16m^2}n\right)\right\}, \quad (27)$$

which is of $O(N)$ for the deep bound states with $n \approx 0$ and is of $O(N^{-1})$ for the shallow bound states with $n \approx 8m^2\pi/\lambda$. The attractive potential is the same for both the deep and shallow bound states and is of $O(N)$, if the approximate description by the potential is allowed.

It must be due to the cancellation between the kinetic and the potential energies that the binding energies for the shallow bound states are of $O(N^{-1})$. This means that the kinetic energy is also of $O(N)$, because the kinetic energy cancels the potential of $O(N)$.

On the other hand, the binding energy for the deep bound states are of $O(N)$, and the cancellation need not occur. This can be seen by examining the solution of the Schrödinger equation with the potential

$$V(r) = -\frac{M_s}{4} \frac{v_0}{\cosh^2 \pi r/2}, \quad (28)$$

which is the same as the nonrelativistic potential used by Korepin¹³⁻¹⁵⁾ to give the semi-classical scattering amplitude, where v_0 is a parameter of $O(N^0)$. For the $n = 0$ bound state solution of the Schrödinger equation,¹⁶⁾ we can easily see that the expectation

value of the relative momentum squared $< p^2 >_0$ turns out to be of $O(N)$, and then $K = < p^2 >_0 / M_s \sim O(N^0)$:

$$\begin{aligned} < p^2 >_0 &= - \int dr \psi_0^*(r) \frac{d^2 \psi_0}{dr^2} \\ &= \frac{\pi^2}{4} \left[\frac{1}{2} \sqrt{v + \frac{1}{4}} - \frac{1}{2} + \frac{1}{8\sqrt{v + \frac{1}{4}}} \right] \sim O(N) \end{aligned} \quad (29)$$

with $v = M_s^2 v_0 / m^2$.

Since the quantum mechanical momentum satisfies the canonical commutation relation,

$$[r, p] = i\hbar, \quad (30)$$

the operator p seems to be of $O(N^0)$. Nevertheless, it should be noticed that the squared momentum averaged by the bound state wave function is not of $O(N^0)$. Thus, the non-static effect should be taken into account from the first in the two-soliton sector.

We now proceed to the problem of the large N behavior of the S matrix in soliton-soliton scattering. Witten⁶⁾ and Coleman¹²⁾ argued that the soliton-soliton scattering amplitude has a smooth large N limit. The argument by them depends on the observation that the Hamiltonian is of $O(N)$ in the Hartree approximation of the N quarks, and that it is written as $H = N\hat{H}$, where $\hat{H}$ is the reduced Hamiltonian of $O(N^0)$, and then the wave function describing baryon-baryon scattering does not depend on N . This means that the amplitude is of $O(N^0)$.

On the other hand the kinetic term in the soliton-soliton system,

$$\frac{p^2}{M_s} = - \frac{1}{M_s} \frac{\partial^2}{\partial r^2}, \quad (31)$$

is not simply of $O(N)$. Thus, factorizing N from the Hamiltonian is impossible for the soliton description, and then the scattering amplitude is not necessarily of $O(N^0)$. In fact, the S matrix expressed as

$$S = \exp[2i\delta_{ss}(u)] \quad (32)$$

does not have a smooth large N limit, because the phase shift δ_{ss} is not of $O(N^0)$. The semiclassical phase shift of soliton-soliton scattering derived from the exact two-soliton solution is given by¹⁷⁾

$$\delta_{ss}(u) = \frac{2M_s}{m} \int_0^u \frac{\log x}{1-x^2}, \quad (33)$$

which reduces to the small u expression,

$$\delta_{ss}(u) = \frac{2M_s}{m} \left(u \log u - u + O(u^2) \right), \quad (34)$$

which is of $O(N)$ because the velocity is N -independent. Thus, the S matrix in the soliton model does not have a smooth large N limit.

In the single-soliton sector we can always take the soliton-rest frame, in which the mesons as the quantum fluctuation interact with the soliton centered at r under the interaction of $O(N^0)$. The interaction cannot recoil the soliton in leading order, and only a meson scatters off the rest soliton without any recoil of the soliton at that order. The recoil of the soliton occurs by the next-to-leading interaction of $O(N^{-1})$ as shown by Hayashi et al.[18,19]. Since the recoil momentum of the soliton has to balance with the meson momentum of $O(N^0)$, the recoil momentum is also $O(N^0)$, and the recoil energy is then of $O(N^{-1})$. The Lorentz transformation of the whole system makes the soliton move with a relativistic velocity, but the *change* of the velocity by the recoil remains to be of $O(N^{-1})$, in contrast to the two-soliton and the soliton-antisoliton systems, where the interaction to change the velocity is of $O(N)$.

§5 Fluctuation around the two-soliton system

We discuss the Dirac formalism of the system consisting of the collective coordinate $r(t)$ and the fluctuation around the classical two-soliton configuration. Throughout this section we ignore the motion of the center of mass of the whole system.

The quantum fluctuation in the two-soliton sector was discussed in literature.^{1,2)} Since the fluctuation is introduced around the sum-ansatz configuration in their treatment, the

iterative corrections to the classical configuration of $O(N^{1/2})$ have to be separated from the whole fluctuation. However the complete separation is very hard, because the convergence of the iterative series is questionable.

We assume here that we could construct the *nonstatic* two-soliton configuration, which satisfies the equation of motion, as discussed in §3:

$$\phi_{\mathbf{s}}(x; r) = \sum_{\mathbf{n}} \phi_{\mathbf{n}}(x; r) \quad (35)$$

in the *center of mass frame*, where $\phi_{\mathbf{s}}$ is the two-soliton configuration involving the nonstatic effect, and $r(t)$ is the only one collective coordinate, here. The equation of motion is written as

$$\ddot{\phi}_{\mathbf{s}}(x, r) - \phi_{\mathbf{s}}''(x, r) + U'(\phi_{\mathbf{s}}) = 0. \quad (36)$$

Let us define the fluctuation field $\eta(x, t; r)$ as

$$\phi(x, t) = \phi_{\mathbf{s}}(x, r(t)) + \eta(x, t; r), \quad (37)$$

where η has an explicit r dependence. The first class constraint holds between the momenta p and $\Pi(x, t; r)$, conjugate to r and $\eta(x, t; r)$, respectively:

$$p - \int dx \Pi(x, t) (\partial_r \phi_{\mathbf{s}}(x, r) + \partial_r \eta(x, t; r)) = 0, \quad (38)$$

where $\partial_r = \partial/\partial r$. and $\partial_r \phi_{\mathbf{s}}$ is a solution of Eq.(36) differentiated with respect to r ; that is, the zero-mode wave function. We set the gauge fixing condition as

$$\int dx \eta(x, t; r) \partial_r \phi_{\mathbf{s}} = 0. \quad (39)$$

Transforming Π into the component parallel to and transverse to $\partial_r \phi_{\mathbf{s}}$, and setting the transverse momentum $\pi(x, t; r)$ orthogonal to $\partial_r \phi_{\mathbf{s}}$,

$$\int dx \pi(x, t; r) \partial_r \phi_{\mathbf{s}} = 0. \quad (40)$$

we have

$$\Pi(x, t; r) = \tilde{p} [\mu(r) - \Xi]^{-1} + \pi(x, t; r), \quad (41)$$

where

$$\tilde{p} = p - \int dx \partial_r \eta \pi, \quad (42)$$

$$\Xi(r) = - \int dx \partial_r \eta(x, t; r) \partial_r \phi_{\mathbf{s}}, \quad (43)$$

and the r -dependent reduced mass μ of $O(N)$ is given as

$$\mu(r) = \int dx (\partial_r \phi_{\mathbf{s}})^2, \quad (44)$$

which reduces to $M_s/2$ at a large r and vanishes at $r = 0$.

The Hamiltonian is written as

$$\begin{aligned} H &= \frac{\tilde{p}^2}{2\mu(r)} \left(1 - \frac{\Xi}{\mu}\right)^{-2} + 2M_s + V(r) \\ &+ \int dx \{ -\phi_{\mathbf{s}}'' + U'[\phi_{\mathbf{s}}] \} \eta(x, t; r) \\ &+ \frac{1}{2} \int dx \{ \pi^2 + \eta'^2 + U''(\phi_{\mathbf{s}}) \eta^2 \} + O(\eta^2), \end{aligned} \quad (45)$$

where the $O(\eta^2)$ is of $O(N^0)$, and we ignored the ordering problem between $\tilde{p}$ and Ξ . The linear term in η of $O(N^{1/2})$ in the Hamiltonian is given as

$$\int dx \left\{ \frac{\tilde{p}^2}{\mu^2} \partial_r^2 \phi_{\mathbf{s}} - \phi_{\mathbf{s}}'' + U'(\phi_{\mathbf{s}}) \right\} \eta(x, t), \quad (46)$$

which can be proved to vanish by using the constraint (39) and $\tilde{p}/\mu = \dot{r} + O(\eta)$ from Eq.(38), where the $O(\eta)$ term is of $O(N^{-1/2})$. Thus, the Hamiltonian expanded in powers of η does not include the term linear in η . The soliton part acts on the wave function as

$$- \frac{1}{2\mu(r)} \frac{\partial^2 \psi(r)}{\partial r^2} + V(r) \psi(r) = M_s u^2 \psi(r). \quad (47)$$

It should be noted that the leading order coupling of a virtual meson to the soliton is of $O(N^0)$ in contrast to the coupling of the on-mass-shell meson as stressed in ref.20. Thus,

quantum corrections to the potential are of $O(N^0)$ at most, and since these are the loop-corrections like the self-energy to the two-soliton system, all of them do not necessarily correspond to the one- or two-pion exchange potential in the quantum meaning.

We emphasize that if the fluctuation field η is introduced around ϕ_{as} , it is left to be of $O(N^0)$, because the unwanted term linear in η of $O(N^{1/2})$, Eq.(46), vanishes by the equation of motion to ϕ_{as} . The nonstatic effect on the classical configuration of the two-soliton sector is so important that the fluctuation around the classical configuration remains small.

§6 Conclusions and discussion

Conclusions in this paper are summarized as follows: The nonstatic effect is very important to calculate the iterative corrections to the sum-ansatz configuration, because the two solitons cannot remain rest owing to the interaction of $O(N)$. If the nonstatic effect is appropriately taken into account, even the first iterative correction reproduces the “exact” potential very well.

The iterative correction cannot be regarded as the fluctuation having new degrees of freedom different from the sum-ansatz configuration, and we need not impose any gauge-fixing conditions on $\phi_n(x; \tau)$.

The numerical approach as the initial value problem works well to reproduce the exact configuration and the “exact” potential, starting from the sum-ansatz configuration.

The meaning of the $1/N$ expansion in the two-soliton system is clarified. The momentum of the soliton is not automatically of $O(N^0)$: The semiclassical momentum such as $< p^2 >_\psi$ depends both on order of the interaction acting on the soliton and on the quantum state ψ , even if $[p, \tau] = i$ holds. The soliton-soliton scattering amplitude does not have a smooth large N limit, because the factor N cannot be factorized from the Hamiltonian in the soliton model and the phase shift is not of $O(N^0)$ in the sine-Gordon laboratory.

If the quantum fluctuation is introduced around the classical configuration involving the nonstatic effect properly, the fluctuation remains a small quantity of $O(N^0)$.

Here, we comment that the size of the potential in Eq.(28). From the analytic solution^{15,13-15)} of the Schrödinger equation with the repulsive potential of the same form as Eq.(28), we have

$$\delta(u) = \frac{2M_2}{m} \left(u \log u - u + \frac{1}{2} u \log \left(\frac{4}{v_0} \right) + O(u^2) \right) \quad (48)$$

for reflection scattering. This is the same as Eq.(34), when $v_0 = 4$. The size $v_0 = 4$ is the same as the potential by Rajaraman⁷⁾ and ours, Eqs.(14) and (15), but is different from that by Korepin and others,¹³⁻¹⁵⁾ who have taken the size, $v_0 = \pi^2$. The reason why they take such a size is to reproduce the exact number of the soliton-antisoliton bound states, which is the threshold value of the phase shift in the soliton-antisoliton system. The exact number of the bound states depends not only on the size of the potential but also on the shape sensitively. On the other hand the low velocity behavior of the phase shift is determined by the asymptotic form of the potential. Thus, we favor $v_0 = 4$ rather than $v_0 = \pi^2$ due to the restriction of the approximation.

We point out that our method cannot straight be applicable to antisoliton scattering off a soliton. The classical soliton-antisoliton configuration in the sine-Gordon theory becomes flat at $\sinh(\tau/2) = 1$, and immediately after that the soliton and antisoliton reappear, interchanging their positions instantaneously. The transmission occurs instantaneously, when we observe the motion of the peaks of the topological charge density. Thus, we cannot define the classical potential for $m\tau/2 < \sinh^{-1} 1$. We note that the reduced mass diverges at the point.

Although the sine-Gordon theory is a toy model for the three dimensional physics, we have much experiences that the results in the sine-Gordon laboratory work well in the three dimensional soliton physics such as the Skyrme model. We believe that the results obtained in this paper are instructive for the study of the Skyrme model.

References

- 1) H. Yamagishi and I. Zahed, Phys. Rev. **D43**(1991), 391; V. Thorsson and I. Zahed, Phys. Rev. **D45**(1992), 965.
- 2) S. Saito, Prog.Theor.Phys.Suppl. **91**(1987), 181; T. Otofujii, Y. Kondo, S. Saito and R. Seki, Phys. Rev. **D45**(1992), 2528.
- 3) R.D. Amado, H. Liu, M. Oka and E. Wong, Phys. Rev. Lett. **63**(1989), 852 ; M. Oka, H. Liu and R.D. Amado, Phys. Rev. **C39**(1989), 2317; M. Oka, Phys. Rev. Lett. **66**(1991), 1019.
- 4) M. F. Atiyah and N. S. Manton, Phys. Lett. **222B**(1989), 438; A. Hosaka, S. M. Griffies, M. Oka and R. D. Amado, Phys. Lett. **251B**(1990), 1; A. Hosaka, M. Oka and R. D. Amado, Nucl. Phys. **A530**(1991), 507.
- 5) T. S. Walhout and J. Wambach, Phys. Rev. Lett. **67**(1991), 314.
- 6) E. Witten, Nucl. Phys. **B160**(1979), 57.
- 7) R. Rajaraman, Phys. Rev. **D15**(1977), 2866.
- 8) E. Bogomolny, Phys. Lett. **91B**(1980), 431.
- 9) J. K. Perring and T. H. R. Skyrme, Nucl. Phys. **31**(1962), 550 .
- 10) J.J.M. Verbaarschot, T.S. Walhout, J. Wambach and H.W. Wylde, Nucl. Phys. **A468**(1987), 520; J.J.M. Verbaarschot, Phys. Lett. **195B**(1987), 177; E. Braaten and L. Carson, Phys. Rev. Lett. **56**(1986), 1897; Phys. Rev. **D38**(1988), 3525; T. Kurihara, H. Kanada, T. Otofujii and S. Saito, Prog. Theor. Phys. **81**(1989), 858.
- 11) J.J.M. Verbaarschot, T.S. Walhout, J. Wambach and H.W. Wylde, Nucl. Phys. **A461**(1987), 603. See also A.E. Alder, S.E. Koonin, R. Seki and H.M. Sommermann, Phys. Rev. Lett. **59**(1989), 2836 for scattering and W.Y. Crutchfield, N.J. Snyderman and V.R. Brown, Phys. Rev. Lett. **68**(1992), 1660 for a bound state.
- 12) S. Coleman, *Aspect of Symmetry* (Cambridge University Press, Cambridge 1985) 1/N p.351.
- 13) V. E. Korepin, JETP Lett. **23**(1976), 201.
- 14) L. D. Fadeev and V. E. Korepin, Phys. Reports **42**(1978), 1.
- 15) A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. **120**(1979), 253.
- 16) P. H. Morse and H. Feshbach, *Methods in Theoretical Physics* (McGraw-Hill, New York 1953).
- 17) R. Jackiw and G. Woo, Phys. Rev. **D12**(1975), 1643.
- 18) A. Hayashi, S. Saito and M. Uehara, Phys. Rev. **D43**(1991), 1520.
- 19) M. Uehara, A. Hayashi and S. Saito, Nucl. Phys. **A543**(1991), 680.
- 20) A. Hayashi, S. Saito and M. Uehara, Phys. Rev. **D46**(1992), 4856.

Figure caption

Fig.1 Intersoliton potentials in unit of M_s . V_{exact} and V_0 are drawn by the solid and dotted line, respectively. $V_0 + V_1$ and V_1 with S_1 by the dashed lines and those with S_0 by the dash-dotted lines.

Fig.2a Topological charge distributions by the numerical calculation. t_n denotes the time ordering from the start, which corresponds the potential given by the fore lozenges in Fig.2b.

Fig.2b Potential by the numerical calculation described with the lozenges.

Fig.1 Potentials

Fig.2a Topological charge distribution

Fig.2b Numerical results of the potential