Low Energy Skyrmion-Skyrmion Scattering

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Abstract

We study the scattering of two Skyrmions at low energy and large separation. We use the method proposed by Manton for truncating the degrees of freedom of the system from infinite to a manageable finite number. This corresponds to identifying the manifold consisting of the union of the low energy critical points of the potential along with the gradient flow curves joining these together and by positing that the dynamics is restricted here. The kinetic energy provides an induced metric on this manifold while restricting the full potential energy to the manifold defines a potential. The low energy dynamics is now constrained to these finite number of degrees of freedom. For large separation of the two Skyrmions the manifold is parametrised by the variables of the product ansatz. We find the interaction between two Skyrmions coming from the induced metric, which was independently found by Schroers. We find that the static potential is actually negligible in comparison to this interaction. Thus to lowest order, at large separation, the dynamics reduces to geodesic motion on the manifold. We consider the scattering to first order in the interaction using the perturbative method of Lagrange and find that the dynamics in the no spin or charge exchange sector reduces to the Kepler problem.
1. Introduction and Conclusion

Scattering of solitons in a non-integrable, non-linear classical or quantum field theory remains an intractable and difficult problem. It, however, concerns one of the most interesting aspects of the nature of the corresponding physics. Numerical methods have given reasonable ideas on how the scattering proceeds but they are still unsatisfactory for uncovering the detailed dynamics governing the scattering.

A method has been proposed by Manton\cite{1} for truncating the degrees of freedom from the original infinite number to a relevant finite number of variables for the case of low energy scattering. One first considers theories of the Bogomolnyi type. In these theories there is typically a topological charge (an integer $N$) which characterizes the soliton sector giving in some sense the number of single solitons in the sector. In each topological sector there exist exact static solutions with a fixed number of arbitrary parameters, the moduli space, which describe these solutions. The number of such parameters is equal to the number of parameters for the one soliton solution multiplied by the topological charge (the number of solitons). There exist asymptotic solutions which are easily identifiable with $N$ single solitons with large mutual separations, but for small separations the individual solitons are subject to strong deformations and they lose their identity. As these are static solutions at essentially arbitrary separations between the solitons, clearly there can be no static forces between them. Otherwise the solutions would not exist; the solitons would move towards or away from one another. Thus in each soliton sector the sub-manifold of solutions corresponds to an equi-potential surface, the minimal energy surface. The kinetic energy can be properly interpreted as corresponding to a metric on the space of all configurations. Typically
we have

\[ T = \int d^3x \frac{1}{2} g_{ij}(\phi_k) \dot{\phi}_i \dot{\phi}_j. \]  

(0)

In many instances \( g_{ij} \) is just flat, but it can be non-trivial, for example, for non-linear sigma models such as the Skyrme model. In any case, in general, the induced metric on the sub-manifold of static solutions is non-trivial. If one considers a time evolution with initial conditions corresponding to a point on the sub-manifold, with initial velocity tangent to the sub-manifold and arbitrarily small, it is intuitively evident that the evolution will remain on the sub-manifold and correspond simply to geodesic motion.

It is a difficult task to prove that such a truncation of the degrees of freedom is a sensible, mathematically rigorous, perturbative scheme. The non-linearity of the theory means that the degrees of freedom tangential to the equi-potential surface are coupled to all other degrees of freedom at higher order. We are assuming that these couplings are negligible. Gibbons and Manton applied this program with remarkable success to the case of magnetic monopoles in the BPS limit and it has also been applied to many other situations, including vortex scattering, Kaluza Klein monopoles, black holes, lumps in \( CP^N \) models, etc. (We refer the reader to the article of Samols for detailed references).

The generalization to the more common situation where the set of static solutions do not include solitons at essentially arbitrary separations can also be developed. Here the forces between the solitons do not exactly cancel, but these are assumed to be weak. The moduli space of the minimum energy critical point is of a smaller dimension than before and does not ever represent many isolated single solitons. Thus truncation of the dynamics to the sub-manifold of such configurations is inadequate to describe the scattering of the solitons. It is, however, evident that the moduli space of the asymptotic critical point of \( N \) infinitely sep-
arated single solitons will in fact be $N \times k$ dimensional, where $k$ is the dimension of the moduli space for one soliton. It is argued by Manton, that the low energy scattering will be restricted to or will lie close to the part of the configuration space corresponding to the union of the gradient flow curves which start at the asymptotic critical point and reach other critical points. They will reach the minimal critical point or, in fact, other critical points, which may actually have energy more or less than the starting point. We find this idea also intuitively reasonable. As long as the gradients involved are not too steep very little radiation should be incurred. For the Skyrme model, numerically it is known in the two soliton sector that the difference in energy between the “deuteron” bound state and the asymptotic, infinitely separated, configuration is rather small, thus the gradients involved should not be large.

In this paper we identify the sub-manifold corresponding to the gradient flow curves for the Skyrme model, in the two Skyrmion sector for large separation between the Skyrmions, as the set of configurations parametrized by the product ansatz (undeformed). The force between two Skyrmions at large distance is governed by their far field behaviour. For the product ansatz this gives rise to an interaction which falls off like $1/d^3$, where $d$ is the separation. Localized deformations of each Skyrmion will not affect the behaviour of the interaction. Modification of the far field behaviour of each Skyrmion will only weaken their mutual interactions if the self energy of the modified Skyrmions is to be kept finite. Modifications which sharpen the far field fall off are permitted while those which weaken the fall off cause the self energy to diverge. We are not concerned here with intermediate range modifications and interactions. Therefore it is clear that the product ansatz with undeformed Skyrmions at large separation gives a good parametrization of the manifold of gradient flow curves up to order $1/d^3$. 
This fact is confirmed by numerical calculations where it is observed that the undeformed product ansatz is surprisingly accurate even down to separations of the order of the size of a Skyrmion\textsuperscript{1}.

We compute the induced metric on this manifold coming from the kinetic energy. We actually find that the metric dominates the interactions between the Skyrmions, the static potential is relatively negligible. To first order in the inverse separation the free metric is modified by a term which behaves like $1/d$, which is much greater than the potential which behaves like $1/d^3$. The characteristic scale is set by $f_\pi$, by $\epsilon$ and the details of the Skyrmion configuration. It is reasonable to expect that it is given by the size of the Skyrmion. We remark that this scale has nothing to do with the scale given by a light massive pion. We have considered a massless pion, adding a small mass does not drastically modify the Skyrmion size. At large distance of course the pion mass cuts off all interactions with the usual Yukawa exponential. Thus our results are valid for the separation in the range that is large compared with the Skyrmion size but small compared with the Compton wavelength of the pion.

The scattering, to first order, is again described by geodesic motion on the manifold parametrized by the product ansatz variables. Even though this is a great simplification from an infinite number of degrees of freedom to just 12 collective coordinates, the problem remains intractable. We calculate 2 conserved “angular” momenta coming from invariance under right iso-rotation coupled with spatial rotation and single left iso-rotation. Then we use the Lagrange perturbative method to obtain approximate equations of motion which correspond to a systematic expansion in the inverse separation. It is actually quite reasonable to make this further approximation since we have already dropped terms of second order in the induced metric. The equations are still rather complicated and we
expect a rich structure in the scattering cross section. We show that the system reduces simply to the Kepler problem for Skyrmions synchronously rotating in a direction orthogonal to the scattering plane.

2. The Skyrme model

The Skyrme model is described by the Lagrangean density,

\[ \mathcal{L}_{sk} = -\frac{f_\pi^2}{4} \text{tr}(U^\dagger \partial_\mu U U^\dagger \partial^\mu U) + \frac{1}{32\epsilon^2} \text{tr}([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2) \]  

(1)

where \( U(x) \) is a unitary matrix valued field. We take

\[ U(x) \in SU(2). \]

The Skyrme Lagrangean corresponds to the first two terms of a systematic expansion in derivatives of the effective Lagrangean describing low energy interaction of pions. It should be derivable from QCD hence \( f_\pi \) and \( \epsilon \) are in principle calculable parameters. These calculations are actually unfeasable and we take \( f_\pi \) and \( \epsilon \) from phenomenological fits. What is even more surprising is that the Skyrme model includes the baryons as well. These arise as topological soliton solutions of the equations of motion. The original proposal of this by Skyrme in the 60's was put on solid footing by Witten in the 80's. For further references and details see the review article by Wambach and Walhout.

The topological solitons, called Skyrmions, correspond to non-trivial mappings of \( \mathbb{R}^3 \) plus the point at infinity into \( SU(2) \):

\[ U(x) : \mathbb{R}^3 + \infty \rightarrow SU(2) = S^3. \]  

(2)
But

$$\mathbb{R}^3 + \infty = S^3$$

(3)

classify the homotopy classes of mappings

$$U(x) : S^3 \to S^3$$

(4)

which define

$$\Pi_3(S^3) = \mathbb{Z}$$

(5)

classify the space of configurations.

The topological charge of each sector is given by

$$N = \frac{1}{24\pi^2} \int d^3 \vec{x} \epsilon^{ijk} \text{tr}(U^i \partial_i U U^j \partial_j U U^k \partial_k U)$$

(6)

which is integer and is identified with the baryon number. Thus for the scattering of two Skyrmions, we are looking at the sector of baryon number equal to 2. In this sector the minimum energy configuration should correspond to the bound state of two Skyrmions, which must represent the deuteron. The asymptotic critical point corresponds to two infinitely separated Skyrmions. There exist, known, non-minimal critical points, corresponding to a spherically symmetric configuration, the di-baryon solution. The energy of this configuration is about three times the energy of a single Skyrmion. There are also, possibly, other non-minimal critical points with energy less than two infinitely separated Skyrmions. The scattering of two Skyrmions will take place on the union of the paths of steepest descent which connect the various critical points.
3. Lagrangean of the Skyrmion-Skyrmion system

With the definitions

\[ \mathcal{L}_\mu^a(U) = -\frac{i}{2} \text{tr}[\tau^a \partial_\mu U U^\dagger] \]  
\[ \mathcal{R}_\mu^a(U) = -\frac{i}{2} \text{tr}[\tau^a U^\dagger \partial_\mu U] \]  
\[ D_{ab}(U) = \frac{1}{2} \text{tr}[\tau^a U \tau^b U^\dagger] \]

the Skyrme Lagrangean density becomes

\[ \mathcal{L}_{sk} = \frac{f^2}{2} \mathcal{L}(U) \cdot \mathcal{L}(U)^\mu \]
\[ - \frac{1}{4\epsilon^2} \left[ \mathcal{L}(U) \cdot \mathcal{L}(U)^\mu \mathcal{L}(U)_{\nu} \cdot \mathcal{L}(U)^\nu \mathcal{L}(U)_{\sigma} \cdot \mathcal{L}(U)^\sigma \right] \]

where the \( \cdot \) implies a contraction in isospace. We can separate this into a kinetic energy \( T \) which is the part quadratic in time derivatives and a potential energy \( V \) without any time derivatives:

\[ T = \frac{f^2}{2} \mathcal{L}_0 \cdot \mathcal{L}_0 - \frac{1}{2\epsilon^2} \left[ \mathcal{L}_0 \cdot \mathcal{L}_i \mathcal{L}_0 \cdot \mathcal{L}_i - \mathcal{L}_0 \cdot \mathcal{L}_i \mathcal{L}_i \cdot \mathcal{L}_i \right] \]
\[ V = -\frac{f^2}{2} \mathcal{L}_i \cdot \mathcal{L}_i - \frac{1}{4\epsilon^2} \left[ \mathcal{L}_i \cdot \mathcal{L}_j \mathcal{L}_j \cdot \mathcal{L}_j - \mathcal{L}_i \cdot \mathcal{L}_j \mathcal{L}_j \cdot \mathcal{L}_i \right] \]

We consider the scattering only for large separation, thus we do not have to know the structure of the complicated region where the two Skyrmions interact strongly and consequently are much deformed. In the region of large separation the product ansatz corresponds to

\[ U(\vec{x}) = U_1(\vec{x} - \vec{R}_1) \ U_2(\vec{x} - \vec{R}_2) \]
\[ = AU(\vec{x} - \vec{R}_1) A^\dagger \ BU(\vec{x} - \vec{R}_2) B^\dagger \]
where \(U(\vec{x} - \vec{R}_1)\) and \(U(\vec{x} - \vec{R}_2)\) correspond to the field of a single Skyrmion solution centered at \(\vec{R}_1\) and \(\vec{R}_2\) respectively. This particular ansatz gives a 12 dimensional configuration space. It can be seen that

\[
\mathcal{L}_\mu^a(U_1 U_2) = \mathcal{L}_\mu^a(U_1) + D_{ab}(U_1) \mathcal{L}_\mu^b(U_2)
\]  

(14)

so \(\mathcal{T}\) and \(\mathcal{V}\) each separate into three parts: one depending only on \(U_1\), another depending only on \(U_2\) and a third function of both \(U_1\) and \(U_2\) which describes the interaction between Skyrmions. This last term has already been investigated for the potential part \(\mathcal{V}\), and gives a contribution of the order \(1/d^3\) where \(d = ||\vec{R}_1 - \vec{R}_2||\) (For details see Ref. 9 and references therein). We shall concentrate here on the interaction part of the kinetic energy \(\mathcal{T}\) given by:

\[
\mathcal{T}_{int} = f_\pi^2 \mathcal{L}_0^1 \cdot D \cdot \mathcal{L}_0^2
\]

\[
+ \frac{1}{2\epsilon^2} \left[ \left( \mathcal{L}_0^1 \cdot \mathcal{L}_0^1 \mathcal{L}_i^2 \cdot \mathcal{L}_i^2 + 1 \leftrightarrow 2 \right) + 2 \left( \mathcal{L}_0^1 \cdot \mathcal{L}_0^1 + 1 \leftrightarrow 2 \right) \mathcal{L}_i^1 \cdot D \cdot \mathcal{L}_i^2 
+ 2 \left( \mathcal{L}_i^1 \cdot \mathcal{L}_i^1 + 1 \leftrightarrow 2 \right) \mathcal{L}_0^1 \cdot D \cdot \mathcal{L}_0^2 
+ 4 \mathcal{L}_0^1 \cdot D \cdot \mathcal{L}_0^1 \mathcal{L}_i^1 \cdot D \cdot \mathcal{L}_i^2 
- \left( \mathcal{L}_0^1 \cdot D \cdot \mathcal{L}_i^1 \right)^2 
- \left( \mathcal{L}_i^1 \cdot D \cdot \mathcal{L}_0^1 \right)^2 
- 2 \mathcal{L}_0^1 \cdot \mathcal{L}_i^1 \mathcal{L}_0^2 \cdot \mathcal{L}_i^2 
\right]
\]

(15)

where \(\mathcal{L}_\mu^1 \equiv \mathcal{L}_\mu^a(U_1)\), \(\mathcal{L}_\mu^2 \equiv \mathcal{L}_\mu^b(U_2)\) and \(D \equiv D_{ab}(U_1)\). Since the time dependance of \(U_1\) is contained in \(A\) and \(\vec{R}_1\), we find that

\[
\mathcal{L}_0^a(U_1) = \left( \delta^{ab} - D_{ab}(AU(\vec{x} - \vec{R}_1)A^\dagger) \right) \mathcal{L}_0^a(A) - D_{ab}(A) \hat{\vec{R}}_1 \cdot \mathcal{L}_0^b(U(\vec{x} - \vec{R}_1))
\]

(16)

and correspondingly for \(U_2\), \(B\) and \(\vec{R}_2\).

It can be seen that \((\delta^{ab} - D_{ab}(AU(\vec{x} - \vec{R}_1)A^\dagger))\) is of order \(1/r^2\) while \(\mathcal{L}_0^a(U(\vec{x} - \vec{R}_1))\) is of order \(1/r^3\) at large distance due to the behaviour of \(F\) which falls off like \(1/r^2\).
This actually implies below that the term containing $\mathcal{L}_b^b$ above is not important to the term of leading order in the kinetic energy.

We now outline the computation of the kinetic energy $T_{int}$ obtained after integrating $T_{int}$ over all space. The procedure is similar to the one used in Ref. 9 to calculate the interaction from the potential. We divide space into three regions I, II an III. II is a spherical region centered on the first Skyrmion, of a radius much smaller than $d$ but large enough so that outside region II the asymptotic behaviour of $F$ is valid. Region III is the same for the second Skyrmion. Region I is of course the remaining space and the field coming from both Skyrmions behave asymptotically here. We find the leading contribution to $T_{int}$ behaves as $1/d$ and comes from the first term in $T_{int}$ evaluated in region I. The contributions coming from the Skyrme term are of higher order due to the behaviour of $\mathcal{L}_b^b$. The leading contributions from region II and III are of order $1/d^2$ and hence negligible. The interaction over region II is actually computed by extending the integrand to the whole of space. This is justified since it only modifies the contribution at higher order in $1/d$.

This simplifies the evaluation of the integral and we find

$$
T_{int} = \int d^3x f_r^2 \mathcal{L}_0^a(U_1) \mathcal{L}_0^b(U_2) + O(1/d^2)
= \frac{\Delta}{d} \epsilon^{ia} \epsilon^{bd} \mathcal{R}^a(A) \mathcal{R}^b(B) \left( \delta^{ij} - \tilde{\delta}^{ij} \right) D_{ab}(A^1 B) + O(1/d^2)
$$

(17)

where $\Delta = 2\pi \kappa^2 f_\pi^2$, $F(r) \sim \kappa/r^2$ at large $r$, $\mathcal{R}^a(A) \equiv \mathcal{R}_0^a(A)$ and $\tilde{\delta} = \tilde{\delta}/d$. We have used the relation $\mathcal{R}^a(A) = D_{ab}(A) \mathcal{L}^b(A)$. The free part of the kinetic energy is well known$^7$ and we can finally write the Lagrangean $L$ for the $N = 2$ sector of the Skyrme model to leading order of the dynamics of the variables of the product
ansatz:

\[
L = -2M + \frac{1}{4}M\dot{\vec{d}}^2 + 2\Lambda \left( \mathcal{L}^a(A) \mathcal{L}^a(A) + \mathcal{L}^a(B) \mathcal{L}^a(B) \right)
+ \frac{\Delta}{d} \varepsilon^{iabc} \varepsilon^{ijkd} \mathcal{R}^a(A) \mathcal{R}^d(B) \left( \delta^{ij} - \dot{d}^i \dot{d}^j \right) D_{ab}(A^i B) + O(1/d^2)
\]

where

\[
M = 4\pi \int_0^\infty r^2 \mathrm{d}r \times \left\{ \frac{1}{8} f_\pi^2 \left[ \left( \frac{\partial F}{\partial r} \right)^2 + 2 \frac{\sin^2 F}{r^2} \right] + \frac{1}{2} \frac{\sin^2 F}{r^2} \right\} \]  

is the mass of a Skyrmion and

\[
\Lambda = (ef_\pi)^3 \int r^2 \mathrm{d}r \sin^2 F \left[ 1 + \frac{4}{(ef_\pi)^2} \left( F''^2 + \frac{\sin^2 F}{r^2} \right) \right]
\]

is its inertia momentum. We have replaced \( \vec{R}_1 - \vec{R}_2 \) by \( \vec{d} \) placing us in the center of mass reference frame and reducing the number of degrees of freedom of the system to 9. Equation (18) is the main result of our calculation.

The metric can easily be obtained from this expression by choosing local coordinates on the product ansatz manifold and extracting the quadratic form relating their time derivatives. With the potential part absent from the Lagrangean of the system (to first approximation) the solution of the problem now resides in finding the geodesics of the metric on the product ansatz manifold.

The result of equation (18) was also independently obtained by Schroers\(^1\). He found a leading contribution which behaves as \( 1/d \) and even calculated sub-leading spin-orbit coupling terms. The only other comparable calculation to our knowledge has been done by Wallout and Wambach\(^7\) for the case of massive pions. The limit as \( m_\pi \to 0 \) of their expression, however, does not leave a term which behaves as \( 1/d \) and hence does not reproduce our result. We believe that this contribution should come also from their evaluation of the integral giving the
induced kinetic energy in the far field region (region I) and then recovering our result as \( m_\pi \to 0 \). Such a contribution would also be proportional to \( (e^{-m_\pi d})^2 \), what they call “two pion exchange”. We believe that they have not computed the contribution from this region.

We also add that our term is leading order in an expansion in inverse separation with respect to a scale which has nothing to do with the pion mass. This means that there are two length scales for nucleon-nucleon interaction predicted by the Skyrme model. It would be interesting to see if this can be phenomenologically or experimentally justified.

4. Approximate Euler-Lagrange equations

Unfortunately the two body Lagrangean has a very complicated structure. Simply finding the expression of the Euler-Lagrange equations is a long and tedious undertaking (let alone finding solutions to these equations). We do not record the equations here. The Lagrangean possesses numerous symetries and corresponding conserved quantities. There are two interesting symetries apart from translational invariance giving conserved charges.

From left iso-rotation

\[
A \to CA \\
B \to CB
\]  

(21)
where $C$ is a constant $SU(2)$ matrix we find the conserved “total isospin”:

$$J_L^k = 4\Lambda [\mathcal{L}^k(A) + \mathcal{L}^k(B)]$$

$$+ \frac{\Delta}{d} \left[ \epsilon^{knc} D_{ni}(A)(\delta^{ij} - \ddot{d}^i \ddot{d}^j) \mathcal{R}^d(A) \mathcal{R}^d(B) + \epsilon^{knc} D_{nj}(B) \mathcal{R}^d(A) \mathcal{R}^d(B) \right] \varepsilon_{i}^{ac} \mathcal{R}^c(A)(\delta^{ij} - \ddot{d}^i \ddot{d}^j) \mathcal{R}^d(B) + \epsilon^{knc} D_{nj}(B) \mathcal{R}^d(A) \mathcal{R}^d(B) \right].$$

(22)

The Lagrangean is also invariant under right isospin rotation coupled with a spatial rotation

$$A \rightarrow AC$$

$$B \rightarrow BC$$

$$d^a \rightarrow D_{ab}(C^\dagger) d^b$$

(23)

where $C$ is a constant $SU(2)$ matrix and $D_{ab}(C)$ is its representative for spin 1.

This gives the further conserved “total angular momentum”:

$$J_R^k = M(\ddot{d} \times \ddot{d})^k + 4\Lambda [\mathcal{R}^k(A) + \mathcal{R}^k(B)]$$

$$+ \frac{\Delta}{d} \left[ \epsilon^{ia} R^d(B)(\delta^{ij} - \ddot{d}^i \ddot{d}^j) \mathcal{R}^d(A) \mathcal{R}^d(B) + \epsilon^{ia} R^d(B) \mathcal{R}^d(A) \mathcal{R}^d(B) \right] \varepsilon_{i}^{ac} \mathcal{R}^c(A)(\delta^{ij} - \ddot{d}^i \ddot{d}^j) \mathcal{R}^d(B) + \epsilon^{ia} R^d(B) \mathcal{R}^d(A) \mathcal{R}^d(B) \right].$$

(24)

These conserved quantities do not help us greatly in solving the equations for the geodesics. In fact it should be noted that the peculiar form of the Lagrangean (not unlike one describing a pair of coupled rigid bodies floating in free space) does not even allow the usual separation of the rotational movement into a global and relative part. In order to go further we must resort to a supplementary approximation.

We use the perturbation method of Lagrange familiar in celestial mechanics. This approximation scheme neglects the changes induced on the free canonical momenta by the interaction term. To begin with we need the free Poisson brackets for the degrees of freedom of the system. We use the variables $(\ddot{d}, \ddot{d}, \mathcal{L}^a(A), \mathcal{L}^a(B), \mathcal{R}^a(A), \mathcal{R}^a(B))$. $\mathcal{L}^a$ is proportional to the isospin of a
Skyrmion and $R^a$ to its spin. We can easily compute the Poisson brackets for the free theory (the limit $d$ goes to infinity of $L$):

\[ \{d^i, \Pi^j\} = \delta^{ij} \]

\[ \{L^a(A), L^b(A)\} = -\frac{1}{2\Lambda}\epsilon_{abc}L^c(A) \]

\[ \{R^a(A), R^b(A)\} = \frac{1}{2\Lambda}\epsilon_{abc}R^c(A) \]

\[ \{L^a(A), R^b(A)\} = 0 \]

\[ \{L^a(A), D_{bc}(A)\} = -\frac{1}{2\Lambda}\epsilon^{ab}dD_{dc}(A) \]

\[ \{R^a(A), D_{bc}(A)\} = \frac{1}{2\Lambda}\epsilon^{acd}D_{db}(A) \]

where $\Pi^i$ is the conjugate momenta to $d^i$. Because of the symmetric nature of the free Hamiltonian, the same brackets are true if we replace $A$ by $B$ everywhere, and all the mixed brackets between $A$ and $B$ are zero. The free system in the center of mass reference frame possesses 5 vectorial conserved quantities: $d^i$, $L^a(A)$, $L^a(B)$, $R^a(A)$ and $R^a(B)$. Let $C^i$ be one of those quantities. Its time derivative is given by

\[ \frac{d}{dt}C^i = \{C^i, H\} \]

where $H = H_{free} + H_{int}$. The approximation comes by using the free Poisson brackets to compute $\{C^i, H_{int}\}_0$. This is of course only correct to first order. Since $C^i$ is conserved in the free system, $\{C^i, H_{free}\}_0 = 0$ and

\[ \frac{d}{dt}C^i \simeq \{C^i, H_{int}\}_0. \]

With the free Poisson brackets we obtain without difficulty the following coupled system of equations:
\[\ddot{e}^{k} + \frac{2\Delta}{Md^2} \left[ \delta^{ij} \ddot{e}^{k} + \delta^{jk} \ddot{e}^{i} + \delta^{ik} \ddot{e}^{j} - 3 \dot{e}^{i} \dot{e}^{j} \right] \epsilon^{ia} c^{\hbar} b d c^{\ell}(A) \mathcal{R}^{d}(B) D_{ab}(A^\dag B) = 0\]

\[\frac{d}{dt} \mathcal{L}^{k}(A) = \frac{\Delta}{2Md} \epsilon^{ia} c^{\hbar} b d c^{\ell}(A) \mathcal{R}^{d}(B) \left( \delta^{ij} - \ddot{e}^{i} \ddot{e}^{j} \right) \epsilon^{kcf} D_{fa}(A) D_{eb}(B)\]

\[\frac{d}{dt} \mathcal{L}^{k}(B) = \frac{\Delta}{2Md} \epsilon^{ia} c^{\hbar} b d c^{\ell}(A) \mathcal{R}^{d}(B) \left( \delta^{ij} - \ddot{e}^{i} \ddot{e}^{j} \right) \epsilon^{kcf} D_{fa}(A^\dag) D_{fb}(B)\]

\[\frac{d}{dt} \mathcal{R}^{k}(A) = - \frac{\Delta}{2Md} \epsilon^{iac} e^{\hbar} b d c^{\ell}(B) \left( \delta^{ij} - \ddot{e}^{i} \ddot{e}^{j} \right) \epsilon^{kcf} D_{fa}(A^\dag B) + \epsilon^{kaf} D_{fb}(A^\dag B) \mathcal{R}^{c}(A)\]  

\[\frac{d}{dt} \mathcal{R}^{k}(B) = - \frac{\Delta}{2Md} \epsilon^{iac} e^{\hbar} b d c^{\ell}(A) \left( \delta^{ij} - \ddot{e}^{i} \ddot{e}^{j} \right) \epsilon^{kdf} D_{fa}(A^\dag B) + \epsilon^{kbf} D_{fb}(A^\dag B) \mathcal{R}^{d}(B)\]  

Our approximation is reliable as long as the separation \(d\) between the particles is large enough for the conjugate momenta to stay close to their free values. As we have already worked with the undeformed product ansatz approximation, and neglected the potential, which are both valid for large \(d\), we feel confident that we have not lost any meaningful information by making this further approximation. If \(d\) is kept large we should then find geodesics similar (qualitatively at least) to those given by the exact equations of motion. The system of equations (28) is still quite complicated but some specialized solutions are easy to obtain and give simple trajectories as we shall now see.

5. Skyrmion-Skyrmion scattering

The simplest geodesic is obtained by taking \(B = A\) and replacing it in the
equations (28). We then get the following system of equations:

\[
\begin{align*}
\ddot{d}^k + \frac{2\Delta}{Md^2} \left[ \dot{d}^k \left( \mathcal{R}^a(A) \mathcal{R}^a(A) + 3(\dot{\mathcal{R}}^a(A))^2 \right) - 2\mathcal{R}^k(A) \dot{\mathcal{R}}^a(A) \right] &= 0 \\
\frac{d}{dt} \mathcal{L}^k(A) &= 0 \\
\frac{d}{dt} \mathcal{R}^k(A) &= -\frac{\Delta}{2Md} \mathcal{R}^a(A) \epsilon^{kbc} \dot{d}^b \mathcal{R}^c(A)
\end{align*}
\]

(27)

We see that if \( A \) is chosen at \( t = 0 \) such that \( \dot{d} \) is parallel or perpendicular to \( \mathcal{R}^k(A) \) then the last two equations become \( \dot{\mathcal{L}} = 0 \) and \( \dot{\mathcal{R}} = 0 \) and are satisfied if \( \mathcal{L}(A) \) and \( \mathcal{R}(A) \) are constants. This means that \( A \) corresponds to an iso-rotation about a fixed axis with constant angular velocity.

Let us begin by choosing \( \mathcal{R}^k(A) \) parallel to \( \dot{d} \). This does not actually lead to a scattering because \( \dot{d} \) does not have the liberty to change direction with time (it has to stay parallel to \( \mathcal{R}^k(A) \) which is a constant). Substituting the condition \( \mathcal{R}^k = \alpha \dot{d}^k = \) constant in the equation for \( \ddot{d} \) we find an equation for a particle constrained on a line with an attractive Coulomb potential. The trajectory will necessarily lead to \( d \) small and hence to regions where we can no longer trust our equations or their predictions.

The case where \( \mathcal{R}^k(A) \) is perpendicular to \( \dot{d} \) is more interesting. Again the last two equations are satisfied with \( A \) corresponding to a steady iso-rotation, however \( \dot{d} \) now has the freedom to move in a plane according to the equation:

\[
\ddot{d}^k + \frac{2\Delta}{Md^2} \mathcal{R}^a(A) \mathcal{R}^a(A) \dot{d}^k = 0.
\]

(28)

This is the equation of a Kepler system: the two Skyrmions scatter in a plane while keeping their isospins and spins constant and perpendicular to the plane of the orbit.

These geodesics are the simplest to compute algebraically but are almost all that we can compute by hand. Indeed if we take \( \mathcal{R}^a \) at \( t = 0 \) not exactly
parallel or perpendicular to $d^u$, complicated precessions appear. It seems that the typical motions of the Skyrmions described by equations (28) are very complex and allow for large nutations, precessions and spin flips. Quantum mechanically these would correspond to neutral and charged pion exchange which we expect from a dynamical model representing nucleon-nucleon interactions.

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