Temperature Dependence of Extended and Fractional $SU(3)$ Monopole Currents

Ken Yee
Dept. of Physics and Astronomy, L.S.U.
Baton Rouge, Louisiana 70803-4001

email: kyee@rouge.phys.lsu.edu

Abstract

We examine in pure $SU(3)$ the dependence of extended monopole current $k$ and cross-species extended monopole current $k^{\text{cross}}$ on temperature $\tau$, monopole size $L$, and fractional monopole charge $\sim 1/q$. We find that features of both $k$ and $k^{\text{cross}}$ are sensitive to $\tau$ for a range of $L$ and $q$. In particular, the spatial-temporal asymmetry ratios of both $k$ and $k^{\text{cross}}$ are sensitive over a range of $L$ and $q$ to the $SU(3)$ deconfinement transition. The motivation for studying cross, extended, and fractionally charged monopoles in $SU(3)$ is given.
1 APQCD

The abelian projection of lattice $SU(3)$ gauge configurations [1] yields a $U(1) \times U(1)$ invariant lattice gauge theory which we call abelian projected QCD (APQCD). A working hypothesis [2], supported by recent numerical results in lattice $SU(2)$ [3] and $SU(3)$ [4], is that APQCD captures essential features of QCD confinement. Taking advantage of the equivalence between the two $U(1)$ APQCD fields [5], one may imagine that one of the $U(1)$ species is integrated out leaving a single representative $U(1)$ copy to be studied. While nothing is wrong with this approach, it is also important to understand APQCD as a full $U(1) \times U(1)$ model because the two $U(1)$ gauge fields are correlated, that is, they interact. Obviously integrating out a $U(1)$ copy surrenders dynamical information about the interspecies interaction, which may, for example, play a role in avoiding the Kosterlitz-Thouless deconfinement transition in the continuum limit of APQCD.

A general $U(1) \times U(1)$ action consistent with APQCD symmetries is

\[-S_{\text{APQCD}} = \sum_{L=1}^{\infty} \sum_{P(L)} \left\{ F(\Theta_{P(L)}^1) + F(\Theta_{P(L)}^2) + G(\Theta_{P(L)}^1, \Theta_{P(L)}^2) \right\} + \cdots \]  

where $P(L)$ is a square $L \times L$ plaquette and superscripts 1 and 2 refer to the two $U(1)$ gauge fields. "\cdots" refers to allowed operators we are neglecting, such as nonsquare Wilson loops, Polyakov loops, and interactions between Wilson loops of different shapes and sizes. Gauge, hermitian conjugation of the $SU(3)$ links ($U_{x,\mu} \rightarrow U_{x,\mu}^{-1}$), and species permutation [5] symmetry requires that

- $F(x + 2\pi) = F(x)$ and $G(x + 2\pi, y) = G(x, y);$
\begin{itemize}
  \item $F(-x) = F(x)$ and $G(-x, -y) = G(x, y)$;
  \item $G(y, x) = G(x, y)$.
\end{itemize}

Therefore, up to a constant $F$ and $G$ must be of the form

\begin{equation}
F(x) = \sum_{q=1}^{\infty} \beta_q \cos(qx),
\end{equation}

\begin{equation}
G(x, y) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \left\{ A_{p, q} \cos(px + qy) + B_{p, q} \cos(px - qy) \right\}
\end{equation}

where $p, q$ are integers, $A_{q, p} = A_{p, q}$, and $B_{q, p} = B_{p, q}$.

As previously mentioned, there are two ways to view APQCD. Firstly, one can integrate out the species 2 links from $S_{APQCD}$ leaving an effective $U(1)$ action

\begin{equation}
-S_{eff}^{APQCD} = \sum_{L=1}^{\infty} \sum_{p(L)}^{\sum_{q=1}^{\infty}} \beta_{q}^{eff} (L) \cos(q\Theta_{p(L)}^{1}).
\end{equation}

$S_{APQCD}^{eff}$ includes $L > 1$ “fenetre” and $q > 1$ “mixed [6]” plaquette operators. Extended monopoles [7] are fundamental dynamical variables in fenetre models whereas fractionally charged $\sim 1/q$ monopoles [8] are fundamental variables of mixed models. Previous studies of APQCD monopoles have been restricted to $q = 1$ and, in $SU(3)$, also $L = 1$.

Section 2 presents our numerical results for $L \geq 1$ and $q \geq 1$ monopoles corresponding to $S_{APQCD}^{eff}$ and discusses their nonabelian gauge and temperature $\tau$ dependence. We find that the monopole density $\rho$ in APQCD has its minimum at $L = q = 1$. There is no a priori reason why $\rho$ should be smaller at $L = q = 1$ than, say, at $L = 2$, $q = 3$. As argued below, this result suggests $\beta_{1}^{eff}(1)$ is the dominant coupling in $S_{APQCD}^{eff}$. We also find that our generalized monopole current $k$ has subtle but detectable temperature variations between and within the confining and finite temperature phases.
These variations are such that the ratio of spatial to temporal monopole densities is, as long as \( L < 1/\tau \), an approximately \( L \)-independent nontrivial order parameter for the \( SU(3) \) finite temperature deconfinement transition. In this sense, \( L > 1 \) monopole currents are as sensitive to the deconfinement transition as the \( L = 1 \) current is.

Secondly, one can think in terms of the \( U(1) \times U(1) \) action, Eq. (1). As illustrated in Ref. [8], such an action has “cross-species” monopoles in addition to the single species “diagonal” monopoles. Section 3 presents some exploratory results for cross-species monopoles as a function of \( L \), \( q \), and \( \tau \). In general, we find in APQCD that cross-species monopoles have the same qualitative dependence on \( L \) and \( \tau \) as diagonal monopoles. For a range of \( L \), the ratios of spatial to temporal cross-species monopole densities are nontrivial order parameters for the \( SU(3) \) deconfinement transition. Such parallel behavior between diagonal and cross-species monopoles is consistent with what is expected from the analysis in Ref. [8].

## 2 Diagonal Monopoles

In Toussaint-Degrard notation [9], define the generalized monopole current

\[
k_\mu(L, q) \equiv \frac{1}{2\pi} \sum_{P(L) \in C(L, \mu)} \left\{ q \Theta_{\partial P(L)} \right\}_{\text{mod} 2\pi}, \quad q = 1, 2, \cdots
\]

where \( C(L, \mu) \) refers to the \( L^3 \) cube oriented in direction \( \mu \) assuming \( D = 3+1 \) dimensions. When \( q = 1 \) and \( L > 1 \), \( k \) reduces to the Type I extended monopoles of Ref. [7]. Integer current \( k \) is topologically conserved for all \( L \) and \( q \) provided one uses an extended derivative when \( L > 1 \). Because \( k_4 \) carries magnetic charge \( \geq 1/qe \) if the fundamental representation Wilson line carries electric charge \( Q = e/2 \), we say \( q > 1 \) monopoles are fractionally charged or “fractional.” By Dirac’s quantisation condition, fractional
monopoles of charge $1/q_e$ can only interact with electric charge $Q \geq q_e/2$ Wilson loops.

Our monopole density is calculated as

$$\rho(L, q) \equiv |\langle |k_4(L, q)| \rangle| = \frac{\sum_{\text{config}} |k_4(L, q)|}{\sum_{\text{config}} 1}. \quad (6)$$

A BKT transformation [11] of the action $S_{QED} = \sum_{P_{(1)}} \beta \cos \Theta_{P_{(1)}}$ reveals that only $L = q = 1$ monopoles are elementary dynamical variables in compact QED. In this model $k_{\mu}(L \neq 1, q \neq 1)$ is a composite operator whose dynamical relevance is indirect. Figure 1A depicts a plot of the monopole density in compact QED as a function of $q$ above and below the critical point. As shown, $\rho$ is suppressed only near $L = q = 1$. As $q$ becomes greater than 1, $\rho$ converges to

$$\rho_R = \frac{7}{15} = \frac{A}{B}, \quad (7)$$

the monopole density when links are completely random [12, 13]. Similarly, as $L$ becomes larger the extended links making up $k(L, q)$—which are superpositions of $L = 1$ links—become more and more disordered. Hence

$$\rho(L >> 1, q) \rightarrow \rho_R \text{ for all } q.$$

In a model with $L > 1$ and/or $q > 1$ plaquettes in its action, monopoles of corresponding $L$ and $q$ become fundamental dynamical variables. Figure 1B depicts $\rho$ as a function of $q$ for the action

$$-S_{\text{mixed}} = \sum_{P_{(1)}} \beta_2 \cos(2\Theta_{P_{(1)}}) + \beta_5 \cos(5\Theta_{P_{(1)}}) \quad (8)$$

at $\beta_5 = .7$ and for a range of $\beta_2$. When $\beta_2 \leq 1.0$, $\rho(1,2), \rho(1,5)$ and to a lesser extent $\rho(1,3), \rho(1,4),$ and $\rho(1,7)$ are suppressed relative to $\rho_R$. When

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1The sum over $x$ ranges over all dual lattice sites; the cubes of $k$ are permitted to overlap. Thusly defined, $\langle |k_4(L, q)| \rangle$ vanishes identically on our periodic lattices. Our normalization of $\rho$ disagrees with Ref. [10] by a factor of $L^3$. Also, since our $SU(3)$ results presented below are fixed at $\beta = 6.0$ we do not need to convert $\rho$ to physical units for interpretation.
Figure 1: 1A depicts the $q$-dependence of the $U^3$ monopole density in compact QED; 1B, in a mixed $U(1)$ model with action comprised of $q = 2$ and $q = 5$, $L = 1$ plaquettes. The solid horizontal lines refer to random-link monopole density $\rho_R = \frac{7}{15}$. The dotted and dashed lines are guides-to-eye. As illustrated, $\rho(L, q)$ equals $\rho_R$ unless it is disrupted by an operator in the action of related $L$ and $q$. $\rho$ is in lattice units.
\( q >> 5 \), \( \rho \) uniformly approaches \( \rho_R \). On the other hand, at \( \beta_2 = 1.2 \) the model crosses over to a weak coupling frozen phase where \( \rho \) is greatly suppressed for a range of \( q \).

In the \( \beta_2 \leq 1.0 \) phase, much of the \( q \)-dependence of \( \rho \) depicted in Figure 1 can be understood in terms of strong coupling arguments. If the action is zero, the \( U(1) \) links are random and \( \rho = \rho_R \). If we turn on a representation \( q \), \( L \times L \) plaquette in the action, then such a plaquette can dress \( k(L, q) \) and drive \( \rho(L, q) \) below \( \rho_R \). If a second plaquette operator is turned on, then \( k \) can be dress by the two plaquettes separately and in combination. In Figure 1 \( \rho(1, 3), \rho(1, 4) \), and \( \rho(1, 7) \) are suppressed because representations \( q = 3, 4, 7 \) are products of \( q = 2 \) and \( q = 5 \). \( \rho(1, 1) \) and \( \rho(1, 6) \) are much less suppressed because \( q = 1 \) and \( q = 6 \) cannot be made from \( 2 \) and \( 5 \). As \( \beta_2 \) increases to 1.2, the strong coupling picture breaks down.

The strong coupling picture predicts the following behavior for the \( L \)-dependence of \( \rho \). If the action contains only plaquettes of size \( L_0 \) then \( \rho(L < L_0, q) \sim \rho_R \) since it is geometrically very difficult for larger plaquettes to dress smaller monopole operators. Similarly, \( \rho(L > L_0, q) \) would differ from \( \rho_R \) only due to high order strong coupling graphs. Hence only \( \rho \) at \( L_0 \) and \( q_0 \) associated with corresponding operators in the action (or simple products thereof) would be appreciably suppressed.

All APQCD results presented in this Note are on \( 24^3 \times T, \beta = 6.0 \) lattices with periodic boundary conditions. Temperature \( \tau \) is given in lattice units as \( \tau \equiv 1/T \). On our lattices the APQCD Polyakov loop \( \langle P \rangle \) vanishes when \( T = 40 \) and \( T = 8 \), and is nonzero when \( T = 6 \) and \( T = 4 \). Accordingly, the \( T = 40 \) and \( T = 8 \) lattices are below and \( T = 6 \) and \( T = 4 \) are above the deconfinement temperature \( \tau_c \). In the Figures, all quantities are plotted in lattice units with (sometimes unresolvable) jackknife error bars. The dotted lines are guide-to-eye lines; the solid horizontal line indicates the value of \( \rho_R \).
Figure 2: 2A and 2B depict the $q$ and $L$ dependence of the APQCD monopole density $\rho$ in different gauges. $\rho$ is given in dimensionless lattice units. In the three gauges examined, the $q = L = 1$ monopoles are disrupted from the random $\rho_R = 7/15$ value more than $q \neq 1$ or $L \neq 1$ monopoles. Jackknife error bars are, in principle, drawn for all data points in this Note, although sometimes they are too small to be visible.
Since APQCD is in a confining rather than a weak coupling phase [1, 3, 4], one can hope to obtain an indication of the dominant operators in $S_{APQCD}^{\text{eff}}$ from $\rho(L, q)$ in APQCD. Figures 2A and 2B are plots of $\rho$ at $T = 40$ in maximal abelian (MA) gauge and, for comparison purposes, Landau and axial gauge. In addition to what is shown, we have also evaluated $\rho$ at other $L$ and $q$ combinations and, furthermore, produced similar results on $16^3 \times 24$, $\beta = 5.7$ lattices. In all cases, we find $\rho$ assumes its minimum value\(^2\) at $L = q = 1$, and that $\rho$ is a monotonically increasing function of $L$ for fixed $q$ and $q$ for fixed $L$. This suggests that between $\beta = 5.7 - 6.0$ the biggest coupling in $S_{APQCD}^{\text{eff}}$ is $\beta_1(1)$ although we certainly do not rule out the existence or significance of other nonzero couplings. This conclusion is fully consistent with an earlier analysis of $S_{APQCD}^{\text{eff}}$ based on plaquette spectral densities [4].

Figures 3 and 4 depict the temperature $\tau$ dependence of $q = 1$ monopoles in MA gauge for a range of $L$. $q > 1$ monopoles (not shown) behave similarly although, since $\rho(L, q > 1)$ is closer than $\rho(L, 1)$ to $\rho_R$, they are less sensitive order parameters than $q = 1$ monopoles. Figures 3A and 3B depict the behavior of the spatial and temporal monopole densities for $L = 1 - 4$. The temporal density is $\rho$ of Eq. (6); the spatial densities are analogously defined using the spatial $k_\mu$ components. Firstly, the temporal and spatial densities are only mildly $\tau$-dependent with the $L = 1$ densities being the most sensitive. None of the densities decrease dramatically across the deconfinement transition as occurs in compact QED. Secondly, while both densities rise with $\tau$ within the confinement phase, the spatial density falls while the temporal density mildly rises with $\tau$ in the high temperature phase. Since spatial monopoles share time-oriented links with Polyakov loops $P$, the falloff

\(^2\)If we let integer $q$ take on fractional values, the minimum nonzero value of $\rho$ occurs at $q = \frac{1}{2}$, corresponding to magnetic charges $\pm 2$. Since lattice monopole charges don’t exceed magnitude 2, $\rho(L, q < \frac{1}{2}) = 0$. 

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Figure 3: 3A and 3B show that the spatial and temporal monopole densities both decrease across the deconfinement transition and both increase within the confined phase with increasing $\tau$. Within the deconfined phase the spatial density decreases whereas the temporal density rises with increasing $\tau$. Therefore, in 3C the spatial-temporal asymmetry ratio $R$ has enhanced sensitivity to $\tau$ in the deconfined phase for all $L$. The $4^3$ monopoles suffers boundary effects at $\tau = 1/4$. 
Figure 4: 4A and 4B present the $q$ and $L$ dependence of the ratio $R$. While $R$ converges to the uninteresting value of unity at $q > 4$ at all temperatures, it is only very mildly $L$-dependent. At this time we do not understand the significance of such near $L$-independence, which indicates that certain features of $L > 1$ monopoles correlate to temperature as positively as $L = 1$ ones.
of spatial densities and the rise of $\langle P \rangle$ with $\tau$ probably share the same origin: freezing of the time-oriented links with increasing $\tau$. $k_4$ does not contain time-oriented links and so it is less $\tau$-sensitive. These $L = 1 - 4$ results in $SU(3)$ are not inconsistent with previous $L = 1$ $SU(2)$ results [14]. Figure 3C depicts the ratio $\mathcal{R}$ of spatial to temporal monopole densities. As shown, $\mathcal{R}$ is a sensitive finite temperature order parameter not just at $L = 1$ [1] but for all $L = 1 - 4$.

Figures 4A and 4B reveal the $q$ and $L$ dependence of $\mathcal{R}$ at different temperatures. While $\mathcal{R}$ becomes less and less informative as $q$ increases, what is striking is that $\mathcal{R}$ is approximately $L$-invariant as long as $L$ is smaller than the smallest lattice width $T$.

### 3 Cross Species Monopoles

If one does not integrate out the second $U(1)$ species, then the APQCD action is of the form (1). Then, in addition to $k(L, q)$ there are cross-species monopole currents

$$k^{\text{cross}}(L, p, q, \pm) \equiv \frac{1}{2\pi} \sum_{P(L) \in \mathcal{C}(L, p)} \left\{ (p\Theta_{P(L)}^1 \pm q\Theta_{P(L)}^2)_{\text{mod} 2\pi} \right\}$$

for integers $p$ and $q$. Cross species monopoles are elementary dynamical variables arising from the interspecies interaction operator $G$ of Eq. (3). For a special case of (1) we showed in Ref. [8] that both $k$ and $k^{\text{cross}}$ occur as elementary dynamical variables. In such $U(1) \times U(1)$ models the dual Meissner effect depends on the combined status of $k$ and $k^{\text{cross}}$, which may in principle condense or freeze out independently for each value of $L$ and $q$.

In APQCD $k^{\text{cross}}(L, q, q, +)$ is equivalent to $k(L, q)$ by species permutation symmetry. Henceforth we focus on $k^{\text{cross}}(L, q, q, -)$, for which some MA gauge results are presented in Figure 5. $\rho^{\text{cross}}$ and $\mathcal{R}^{\text{cross}}$ are defined in
Figure 5: While cross-species monopoles are generally denser than the diagonal ones, they have the same general $q$, $L$, and temperature dependence as the latter. As depicted in 5A the cross-species monopole density is smallest at $L = 1$ and rises quickly to $\rho_R$ at bigger $L$ for all $\tau$. 5B shows that the asymmetry ratio $R^{\text{cross}}$ is a nontrivial order parameter. Note the boundary effect on the $L = 4$ current at $\tau = 1/4$. 
parallel to their diagonal counterparts. Our exploratory calculations indicate that generally $k^{cross}$, while denser, has qualitative features reminiscent of $k$. As demonstrated in Figure 5B, $R^{cross}$ is also a nontrivial order parameter for the $SU(3)$ finite temperature transition. However, it is noticeably more $L$-dependent than its diagonal counterpart.

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References


