RESONANCES FOR A SEMI-CLASSICAL SCHRÖDINGER OPERATOR NEAR A NON TRAPPING ENERGY LEVEL

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Abstract

We give an example of a short range potential $V$ on the real line that is dilation analytic at infinity, non trapping at energy $E > 0$, but oscillating in the neighborhood of some points, so rapidly that the Schrödinger operator $P = -\hbar^2 \Delta + V$ shows a string of resonances near $E$ in the lower half plane as $\hbar > 0$ is small enough. The extended states behave as standing waves partially reflected off the bumps of $V$. Such a potential is the analogue of the Wigner-Von Neumann potential in the case of embedded eigenvalues.

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0. INTRODUCTION

Let \( V \) be a potential on \( \mathbb{R}^n \) such that \( V(\Delta + 1)^{-1} \) is compact, and consider, for \( h > 0 \), the Schrödinger operator \( P = -h^2 \Delta + V \) on \( L^2(\mathbb{R}^n) \). Then \( \sigma_{sc}(P) = [0, +\infty) \). To fix the ideas, assume \( V(x) \) goes to zero as \( |x| \to \infty \). One would expect the discrete spectrum of \( V \) to be negative, for a quantum particle with strictly positive energy would eventually escape to infinity through tunneling; the celebrated Wigner-Von Neumann example shows that this guess is wrong. Namely there exists a potential \( V \) on the real line as above such that, for \( h = 1 \), \( P \) has an embedded eigenvalue \( E = 1 \). The asymptotic behaviour of \( V \) reads (see [ReSi]):

\[
V(x) = -\frac{\sin 2x}{x} + O(|x|^{-2}), \quad |x| \to \infty
\]

and the eigenfunction associated to \( E = 1 \)

\[
\psi(x) = \frac{\sin x}{x} (1 + O(|x|^{-2}))
\]

behaves as a standing wave reflected coherently off the bumps of \( V \).

(Of course, it should be pointed out that a coherence condition in the frequencies is not sufficient to get an eigenvalue, as shows the case of periodic potentials; the decay of \( V \) at infinity also plays a role).

Conversely, Froese and Herbst (see [CFKS]) showed that if \( V \) behaves at infinity like \( \frac{\sin 2x}{x} \), then, under some other suitable conditions, \( P \) verifies a Mourre estimate at all positive energies but 0 and 1, which implies that \( P \) has no (strictly) positive eigenvalue except possibly at \( E = 1 \). In a recent paper [Ku], Klaus gives asymptotics of the analytic continuation of the \( S \)-matrix near such a critical energy (see also [Ku]).

Assume further that \( V \) is dilation analytic at infinity, i.e. \( V \) is smooth and extends analytically outside a compact set \( K \) of \( \mathbb{R}^n \) in a domain:

\[
\Gamma = \{ x \in \mathbb{C}^n ||\text{Im}z| \leq C(1 + |\text{Re}z|), \quad \text{Re}z \in \mathbb{R}^n \backslash K \}
\]

for some \( C > 0 \), where:

\[
\lim_{x \in \Gamma, |x| \to \infty} V(x) = 0
\]

Then \( P \) has only continuous spectrum above 0 and we may define the resonances of \( P \) near the energy level \( E > 0 \) by the method of analytic distortions of Hunziker [Hu]. In the semi-classical limit \( (h \to 0) \) the existence of resonances relies very much on the underlying dynamical system. Namely, let \( \mathcal{H}(x, \xi) = \xi^2 + V(x) \) be the classical hamiltonian. For \( I = [E - \varepsilon, E + \varepsilon] \) define the outgoing and incoming tails:

\[
\Gamma_\pm = \{ (x, \xi) \in \mathcal{H}^{-1}(I) | \exp tH_\rho(x, \xi) \to \infty \quad \text{as} \quad t \to \pm \infty \}
\]

(Here \( H_\rho \) denotes the hamiltonian vector field) and \( K(I) = \Gamma_+ \cap \Gamma_- \) as the set of trapped trajectories (see [Ge Sj]). We say that \( V \) is non-trapping at energy \( E \) if \( K(I) = \emptyset \) for \( \varepsilon > 0 \) small enough. Generically, if \( V \) is non trapping at energy \( E \), there are no resonances too close to the real axis below \( E \); resonances free domains were extensively studied by many authors. Thus, if \( V \) is everywhere analytic, there are no resonances in a \( h \)-fixed neighborhood of \( E \) in the lower half plane ; if \( V \) is only of Gevrey class \( G^s (s > 1) \) is some compact set of \( \mathbb{R}^n \), there are no resonances in a box like \( |E - \lambda, E + \lambda| - i|0, \hbar^{1-s}/\hbar| \)

for some \( \delta > 0 \) (see [Ro] and references therein). Note that these results have a natural counterpart in the scattering by an obstacle for the wave equation.

On the other hand we know many examples of resonances created by trapped rays for the classical hamiltonian flow. Among most typical cases there are the shape resonances ([Co Du Kle Sej], [He Sj], [Hi Si], [Na 1], [Br Co Du 2]...), the barrier top ([Si], [Br Co Du 1] [Na 2]... and the closed trajectory of hyperbolic type [Ge Sj]. The last example is a quantum analogue of wave scattering by convex bodies which has also received considerable attention (see [Lax Ph] and [Pet Stel] for recent surveys).

One can then address the following problem : is there any potential which would give rise to resonances below a non trapping energy (necessarily far enough from the real axis) ? Of course, the Wigner-Von Neumann potential would not do, nor any of its natural semi-classical extensions, for it is not dilation analytic and is certainly trapping at the relevant energies. But we keep in mind the salient feature of this example, namely the creation of a standing wave pattern. This can also be achieved for resonances if part of the extended state is reflected coherently between some bumps of the potential, which nevertheless would not be sensitive to the classical hamiltonian flow ; indeed, the virial...
condition holds near the resonance. Our example relies on a certain (local) non analyticity of $V$, due to fast oscillations.

Let $s > 3$, $a > 0$ ($a$ has to be slightly adjusted in function of $h$), and

$$V(x, h) = \sqrt{2h/\pi} e^{-h^{-1/4}} \left( e^{-x^2/2h} \cos(x/h) + e^{-(x-a)^2/2h} \cos(x-a)/h \right)$$

We look for resonances of $P = -h^2 \Delta + V(x, h)$ near $E = \frac{1}{4} \lambda$ (we could also introduce a coupling constant $\lambda \in \mathbb{R}$; our results for $P_\lambda = -h^2 \Delta + \lambda V(x, h)$ hold with a good uniformity with respect to $\lambda$). Note that $V$ is analytic, but if we are also interested in the dependence in $h$, as $h \to 0$ we only have:

$$|\partial_x^k V(x, h)| \leq C^{|k|}(j)^s, \quad 0 < h \leq 1, \quad x \text{ near } 0 \text{ and } a$$

for some $C > 0$, i.e. $V$ is uniformly in $C^s$ as $h \to 0$, near $0$ and $a$, while $V$ verifies uniform Cauchy estimates away from these points. Then the result of [Ro] shows that we have to search for resonances with an imaginary part less than $\sim -h^{1-1/s}$.

This is precisely the order of magnitude we shall obtain. Let $P = -h^2 \Delta + V$ be as above. For $\xi = \frac{1}{4} + o(1) \in C$ we define $P(\xi) = P - \xi^2$ so that $\xi^2 \in \sigma(P) \cap \sigma(P(\xi))$. For $0 < a$ small enough (but independent of $h$), let $\Gamma_a$ be a family of standard distortions of $\mathbb{R}$ in the sense of [Hu], so that $P$ extends as an unbounded operator on $L^2(\Gamma_a)$ with domain $H^2(\Gamma_a)$. $\Gamma_a$ is parametrised by $x \mapsto x_a = x e^{\theta(x)}$ so that $\theta(x) = 0$ for $-b < x < a + b$, and $\theta(x) = \theta$ for $x < -2b$ or $x > a + 2b$. Here $b > 0$ will be also chosen small enough but independent of $h$ (the choice of $\Gamma_a$ is specified in Appendix A). We still denote by $P(\xi)$ the distorted operator.

**Theorem 0.1**: Let $C_0 > 0$ and $W_a = \{ \eta \in C, \eta = 2\xi - 1, \text{Im} \eta \leq 0, \eta h^{1-1/s}, \text{al} \eta + 2h^{1-1/s} > -\varepsilon h^{1-1/s}, \varepsilon > 0 \text{ small enough} \}$.

For $s > 3$, we have the following: if $a = (2m + 1)h$, $m = m(h) \in \mathbb{N}$ in such a way that $|\varepsilon - a_0| < \text{Const. } h(a_0) > 0$, there exists a discrete subset $\Omega_a \subset W_a$ such that, for $h > 0$ small enough:

1° If $\eta \in W_a \setminus \Omega_a$, then $P(\xi) : H^s(\Gamma_a) \to L^2(\Gamma_a)$ is bijective with bounded inverse.

2° If $\eta \in \Omega_a$, then $0 \in \sigma(P(\xi))$, $P(\xi)$ is Fredholm of index $0$ and splits into a direct sum $P(\xi) : F(\xi) \oplus (F(\xi) \cap H^2(\Gamma_a)) \to F(\xi) \oplus F(\xi)$, where $F(\xi)$ is finite dimensional, $F(\xi)$ is closed and $P(\xi) : F(\xi) \oplus H^2(\Gamma_a) \to F(\xi)$ is bijective with bounded inverse.

3° If $\eta \in \Omega_a$, then either $\dim F(\xi) = 2$ and $P(\xi)|_{H(\xi)}$ is nilpotent of order 2 (i.e. $P(\xi)|_{H(\xi)} \neq 0$, $P(\xi)|_{H(\xi)} = 0$) or there exists $\eta \in \Omega_a$ with $\eta = \frac{1}{2} \eta + \frac{1}{2}$ such that $0 < |\eta - \eta| = O\left( e^{-c h^{1/s}} \right), c > 0$, and $\dim F(\xi) = \dim \tilde{F}(\xi) = 1, F(\xi) \cap \tilde{F}(\xi) = 0$ (we say then that we have splitting of resonances).

4° There exists a surjection $f$ from $\Omega_a$ onto the set of roots of the equation:

$$2\pi h e^{-\eta^2/4h} e^{\eta h^p/4h} \gamma(\xi) = 1$$

in $W_a$, such that $f(\eta) - \eta = O\left( e^{-C h^{1/s}} \right), C > 0$, where $\gamma(\xi) = -\left( 8\pi h^{p/2} \right)^{-1} e^{-2h^{1/s}}$. Moreover, if $\xi_a$ is such a root, then

$$\eta_a = h^{1-1/s} \left( \frac{2\pi^2}{a} h^{1-1/s} + \frac{2h^{1-1/s}}{a} \right) + h^{1/s},$$

where $\xi_a \sim \sum_{j=1}^{\infty} h^{1/s} \xi_a(j)$ in the sense of classical analytic symbols in $h = h^{1-2/s}$.

Let us make a few remarks.

1° The range of the eigenprojector $\Pi(\xi_a)$ associated to some $\eta \in \Omega_a$ ("double" or "splitting" resonance) is spanned by functions exponentially close to $F_a(h)$ and $F_a(x)$ (see (2.4)). The frequency set of $F_a(x)$ (as defined in [Gu St]), is concentrated near $[0, \infty] \times \left\{ \frac{1}{2} \right\}$ (finite set); this of $F_a(x)$ near $[-\infty, a] \times \left\{ \frac{1}{2} \right\}$ (finite set). So the union of these (disconnected) frequency sets looks like a closed trajectory for the classical flow in phase space (although it is not):
4° It could be possible and also more natural to choose a potential independent of \( h \), i.e. replace for instance \( V(x, h) \) by:

\[
V(x) = \int f(\lambda)(e^{-\lambda x^2/2} \cos \lambda x + e^{-\lambda(x-a)^2/2} \cos \lambda(x-a))d\lambda
\]

with \( f(\lambda) \) decaying as \( e^{-\lambda^{1/4}} \) as \( \lambda \to \infty \). Presumably by stationary phase arguments, the main contribution of this potential to resonances would come from \( \lambda = h^{-1} \).

5° We conjecture that there are no other string of resonances near \( E = \frac{1}{9} \) when dimension is 1 ; namely if we keep in mind the analogy with a closed trajectory as described above, we can infer from Bohr-Sommerfeld quantization rule that \( 2\alpha \xi \) is the only relevant action that is responsible for creation of resonances (and no multiple of \( 2\alpha \xi \)). In other terms there will be no new resonance revealed by winding around the loop several times. Note that we have found the first string of resonances (if several).

6° The proof we present here is straightforward but very tedious. It is quite frustrating to note that the quantization condition can be obtained (heuristically) as follows: let \( u_0(x) \) be the solution of \( (P - \xi^2)u_0 = 0 \) in \( L^2(\Gamma_0) \) with \( u_0 \sim e^{ix\xi L/h}, x \to \pm \infty \); then \( \xi^2 \) is a resonance iff \( W(u_+, u_-) = 0 \) where \( W(u_+, u_-) \) denotes the Wronskian of \( u_+ \) and \( u_- \). To compute \( W(u_+, u_-) \) it suffices to know the asymptotics of \( u_+ \) and \( u_- \) near a point between 0 and \( a \). Let \( G_+(\xi) = i(2\xi h)^{-1}x e^{i(x-y)\xi L/h} \) and \( G_-(-\xi) = i(2\xi h)^{-1}y e^{-i(x-y)\xi L/h} \) be the half Green kernels for \( Q(\xi) = -h^2\Delta - \xi^2 \). Put:

\[
u_- = e^{-i\xi h} - G_+(\xi)Ve^{-i\xi h}(x) \quad \text{and} \quad \nu_+(x) = e^{i\xi h} - G_-(\xi)Ve^{-i\xi h}(x).
\]

For \( \delta < x < a - \delta \) we have:

\[
v_-(x) = e^{i\xi h} - c(h)e^{-i\xi h}e^{i\xi h}(a - x, \eta) + O(e^{-1/CN})
\]

and \( \nu_+(x) = e^{-i\xi h} - c(h)e^{i\xi h}(a, \eta) + O(e^{-1/CN}) \). (see the definition of \( \alpha(x, \eta) \) in the beginning of Sect. 1 ; here \( \delta = h^{1/4}, \xi = h^{1/2}, c(h) = i(2\xi h)^{-1}h \). Then \( W(\nu_+, \nu_-) = \frac{2\xi h}{\pi} \left( 1 - \frac{\pi^2}{4} \right) e^{-\eta \xi h} + O(e^{-1/CN}) \) and the equation \( W(\nu_+, \nu_-) = 0 \) gives precisely the quantization condition in the theorem. Unfortunately we have not been able to give this idea a rigourous form.

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I. FREDHOLM ALTERNATIVE FOR $I - (G(\xi) V)^2$

1. The kernel

Let $\xi = \frac{1}{2} + o(1) \in \mathbb{C}$ and $Q(\xi) = -\hbar^2 \Delta - \xi^2$ be the free hamiltonian. The Green kernel for $Q(\xi)$ is:

$$G(x,y,\xi) = i/(2\hbar)^{-1} \left( \chi(y < x) e^{i(x-y)\xi/\hbar} + \chi(x < y) e^{-i(x-y)\xi/\hbar} \right), x, y \in \mathbb{R}$$

where, here and in the following, $\chi$ denotes the (sharp) characteristic function. (Among all arguments of the various functions we shall consider, we omit the variable $\hbar$ which appears everywhere.) All quantities will depend holomorphically on $\xi$ in a neighborhood of $\frac{1}{2}$. Let $x \rightarrow x_\theta$ as above be the parametrization of $\Gamma_\theta$, defined for $\theta > 0$ small enough, and $A$ a subset of entire functions on $\mathbb{C}$ such that:

i) for any $\theta$ in a complex neighborhood of $[0, \theta_0]$, $(x_\theta)^*(A)$ is a dense subset of $L^2(\mathbb{R})$.

ii) for any $\psi \in A$, $\theta \rightarrow (x_\theta)^* \psi = \psi \circ x_\theta$ is an analytic family for $\theta$ in a complex neighborhood of $[0, \theta_0]$. Such a choice is given by (see [Hu]):

$$A = \{ \psi \text{ entire }, \psi(x) = O_N((1 + |\text{Re}| x)^{-N}) \text{ on } \mathbb{R} \},$$

if $\text{Im} x < C(1 + |\text{Re} x|)$ for some $C > 0$.

We shall extend $G(x,y,\xi)$ to complex variables $x, y \in \Gamma_\theta$. For this purpose we can consider the action of $G(\xi)$ on $u \in A$. Namely, for real $x$ and $y$, we have:

$$G(\xi)u(y) = i/(2\hbar)^{-1} \left( \int_{-\infty}^{x} e^{i(x-y)\xi/\hbar} u(y)dy + \int_{x}^{+\infty} e^{-i(x-y)\xi/\hbar} u(y)dy \right), u \in A$$

and this equality still makes sense for $x, y \in \Gamma_\theta$, $u \in A$ provided $\text{Im}(e^{i\theta}) \geq 0$. It is also clear that for fixed $\theta > 0$, $\xi \rightarrow G(\xi)$ is an analytic family of bounded operators from $L^2(\Gamma_\theta)$ to $H^2(\Gamma_\theta)$ (the usual Scholows space) when $\xi$ varies in a compact neighborhood of $\frac{1}{2}$. $G(\xi)$ is symmetric in the sense:

$$\langle G(\xi) u, v \rangle = \langle u, G(\xi) v \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the bilinear symmetric pairing:

$$\langle u, v \rangle = \int_{\mathbb{R}} (x_\theta)^* u(x)v(x)dx$$

that we simply denote by:

$$\langle u, v \rangle = \int (u(x)v(x)dx)$$

Let now $P(\xi) = Q(\xi) + V$. It is known that when $\theta > 0$, $\xi \neq 0$, $P(\xi)$ is an unbounded operator on $L^2(\Gamma_\theta)$ with domain $H^2(\Gamma_\theta)$. We are interested in the values of $\xi$ such that $0 \in \sigma(P(\xi))$. When $0 \notin \sigma(P(\xi))$, the left inverse of $P(\xi)$ is:

$$L(\xi) = (I + G(\xi) V)^{-1} G(\xi)$$

Let us write instead:

$$L(\xi) = (I - (G(\xi) V)^2)^{-1} G(\xi)(I - V G(\xi))$$

This standard substitution provides a better approximation to the spectral radius of $P$; further it codifies or "quantizes" the loop we have mentionned above. We shall apply classical Fredholm theory to $I - (G(\xi) V)^2$. Following [RN] in a very concrete way, we approximate $(G(\xi) V)^2$ by a finite rank operator. We begin with computing $(G(\xi) V)^2$ in the real domain. Set $\eta = \frac{2}{\pi} - 1$, $\eta' = \frac{2}{\pi} + 1$ and define the functions for real $x, y$:

$$\alpha(x, y) = \int_{-\infty}^{+\infty} e^{-t^2/4\hbar} e^{-i(x-y)t/\hbar} dt , \tau = \eta \text{ or } \eta'$$

$$\beta(x, y) = \chi(x < y) \int_x^{+\infty} \tilde{V}(t) dt$$

where:

$$\tilde{V}(t) = 2e^{-t^2/2\hbar} \cos(t/\hbar) + 2e^{-(t-a)^2/2\hbar} \cos(t-a)/\hbar$$

$$\varphi_1(y) = e^{-y^2/2\hbar} + e^{ia/\hbar} e^{-i(a-y)^2/2\hbar}$$

$$\varphi_2(y) = e^{-ia/\hbar} \varphi_1(a-y)$$

$$\varphi_3(y) = \alpha(y, \eta') + \alpha(y, \eta) + e^{-2ia/\hbar}(\alpha(y-a, \eta') + \alpha(y-a, \eta))$$

$$\varphi_4(y) = e^{2ia/\hbar} \varphi_3(a-y).$$

The frequency set of $\alpha(x, y)$ is concentrated near $[0, +\infty] \times [0]$; this function can be considered as a smoothed characteristic function of $[0, +\infty]$ (up to a normalization
constant.) On the other hand, \( \alpha(x, y') \) is oscillating and its frequency set is concentrated near \( \{(0, -2)\} \).

We set also:
\[
\psi_+(y) = e^{iy\gamma} \psi_1(y) + e^{iy\gamma} \psi_2(y)
\]
\[
\psi_-(y) = e^{-iy\gamma} \psi_2(y) + e^{-iy\gamma} \psi_1(y)
\]
that is,
\[
\psi_-(y) = e^{-2iy\xi h} \psi_+(a - y)
\]

Let \( \gamma(\xi) \) be defined in Theorem 0.1 - 4^a, and put as usual for real \( x \) and \( y \) : \( x \land y = \min(x, y) \), \( x \lor y = \max(x, y) \). After some straightforward computation, we find:
\[
(G(\xi)V)^2 = \gamma(\xi)K(\xi) + \gamma(\xi)K'(\xi)
\]
with:
\[
K(\xi) = e^{i(x+y)\xi h} \psi_+(y) \psi_1(x \land y) + e^{i(x+y)\xi h} \psi_-(y) \psi_2(x \lor y)
\]
\[
K'(\xi) = e^{i(x+y)\xi h} \psi_-(y) \psi_1(y, x) + e^{i(x+y)\xi h} \psi_+(y) \psi_2(y, x, y)
\]
(we have identified an orthonormal operator with its kernel.) Let us make a few comments on the kernel \( K(\xi) \) and \( K'(\xi) \) which might help to understand the interface pattern, at least on a heuristic level. It is easy to see, by stationary phase arguments (at least after reading the Appendix) that \( K(\xi) \) roughly takes a function outgoing at \( +\infty \) and whose frequency set is concentrated on \( [0, +\infty) \times \{ \frac{1}{4} \} \) (we can check this for \( e^{i\xi h} \alpha(y, \eta) \)) into a function which is also outgoing at \( +\infty \) and whose frequency set is concentrated near \( [0, +\infty) \times \{ \frac{1}{4} \} \cup \) (finite set). In the same way, \( K(\xi) \) roughly takes a function outgoing at \( -\infty \) and whose frequency set is concentrated on \( ] -\infty, a [ \times \{ \frac{1}{4} \} \) into a function whose frequency set is concentrated near \( ] -\infty, a [ \times \{ \frac{1}{4} \} \cup \) (finite set). So we can say, somewhat indefinitely, that \( K(\xi) \) essentially preserves each of the frequency sets \( [0, +\infty) \times \{ \frac{1}{4} \} \) and \( ] -\infty, a [ \times \{ \frac{1}{4} \} \) (modulo a finite set in phase space). So \( K(\xi) \) sets up constructive interference at frequencies \( \pm \frac{1}{4} \). The action of the operator \( K'(\xi) \) is somewhat more difficult to describe but it will be treated mainly as a perturbation.

Again, the kernels \( K(\xi) \) and \( K'(\xi) \) can be extended to \( x, y \in \Gamma_x \). For instance \( \beta(x, y) \) can be viewed as a kernel:
\[
\beta(x, y) u(x) = \int_{-\infty}^{+\infty} dy \int_{a}^{b} U(t) dt, \ u \in A
\]
where the integration is performed along \( \Gamma_x \), with \( x \in \Gamma_x \). Similarly, \( \alpha(x \land y, \tau) \) denotes the kernel
\[
\alpha(x \land y, \tau) u(x) = \int_{-\infty}^{+\infty} e^{-it/\sqrt{2}h} e^{-it/\sqrt{2}h} dt \int_{-\infty}^{+\infty} u(y) dy, \ u \in A
\]
which can be written also in the more convenient form:
\[
\alpha(x \land y, \tau) u(x) = \int_{-\infty}^{+\infty} e^{-it/\sqrt{2}h} e^{-it/\sqrt{2}h} u(y) dy \land dy, \ u \in A
\]
where, for \( x \in \Gamma_x \), \( \gamma_x(x) = \{(t, y) \in \Gamma_x \mid t < y \text{ and } t < x\} \) and \( \land \) means : put on increasing order on \( \Gamma_x \). Here we have ordered \( \Gamma_x \) with the natural order on \( R \) induced by the injective map \( x \rightarrow x \). In the sequel we will often identify \( \Gamma_x \) wit \( R \), and carry out the computations exactly as if the variables were real, dropping the indices \( \theta \).

We approximate the kernel \( (G(\xi)V)^2 \) by the finite rank operator \( \gamma(\xi)T(\xi) \), where (again identifying an operator with its kernel):
\[
T(\xi) = \sum_{j=1}^{12} f_j(x) g_j(y)
\]
With Dirac's notations, \( f_j(x) \) will range among:
\[
\{< 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, | \pm 1 \pm 2 \pm 3 \}
\]
and \( g_j(y) \):
\[
\{|1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, | \pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9 \pm 10 \pm 11 \pm 12 \}
\]
respectively (precise labelling will be given in (1.2)) where:
\[
< 1 + 1 = \chi(x < \delta) e^{ix \xi h} \psi_1(x)
\]
\[
< 1 - 1 = \chi(x < \delta) e^{-ix \xi h}
\]
\[
< 2 + 1 = \chi(x < \delta) e^{ix \xi h}
\]
\[
< 5 + 1 = \chi(x < \delta < a - \delta) e^{ixa \xi h}
\]
\[
< 5 - 1 = \chi(\delta < x < a - \delta) e^{-ixa \xi h}
\]
\[
< 6 - 1 = \chi(\delta < x < \delta) e^{-ix \xi h} e^{-ixa \xi h} e^{-a \xi h} e^{-ixa \xi h}
\]
\[ H''(\xi) = H_3(\xi) + H_4(\xi) + H_5(\xi) + H_6(\xi) + H_{10}(\xi) + H_{11}(\xi) \]

where:

\[ H_3(\xi) = \chi(x < y < -\delta) \left( e^{i(x-y)\xi/h} \psi_+^*(y) \psi_1(x) + e^{-i(x-y)\xi/h} \psi_-^*(y) \psi_2(y) \right) \]

\[ H_4(\xi) = \chi(y > x < -\delta) \left( e^{i(y-x)\xi/h} \psi_+^*(y) \psi_1(y) + e^{-i(y-x)\xi/h} \psi_-^*(y) \psi_2(x) \right) \]

\[ H_5(\xi) = \chi(-\delta < x < y < \delta) \left( e^{i(x-y)\xi/h} \psi_+^*(y) \psi_1(x) + e^{-i(x-y)\xi/h} \psi_-^*(y) \psi_2(y) \right) \]

\[ H_6(\xi) = \chi(x < a < -\delta) \left( e^{i(x-a)\xi/h} \psi_+^*(y) \psi_+^*(y) + e^{-i(x-a)\xi/h} \psi_-^*(y) \psi_-^*(y) \right) \]

\[ H_{10}(\xi) = \chi(a + \delta < x < y) \left( e^{i(x-y)\xi/h} \psi_+^*(y) \psi_1(x) + e^{-i(x-y)\xi/h} \psi_-^*(y) \psi_2(y) \right) \]

\[ H_{11}(\xi) = \chi(a + \delta < y < x) \left( e^{i(x-y)\xi/h} \psi_+^*(y) \psi_1(y) + e^{-i(x-y)\xi/h} \psi_-^*(y) \psi_2(x) \right) \]

Here we have put \( \delta = \hbar^{1/4} \), and omitted the subscript \( \theta \) in the formulae, according to our abuse of notation. Actually the resonant function will be built (in a good approximation) from a linear combination of the \( f_j \)'s. Now we define the remainder term \( \sum_{j=13}^{16} f_j(x) g_j(y) \) where \( f_j(x) \) ranges among \( \{ 2+, 6+, 8-, 12- \} \) and \( g_j(y) = \{ 2+, 6+, 8-, 12- \} \) respectively, with:

\[ H'(\xi) = \sum_{j=13}^{16} f_j(x) g_j(y) \]
so that, by construction:

\[ K(\xi) = T(\xi) + H'(\xi) + H''(\xi) \]

and:

\[ (G(\xi)V)^2 = \gamma(\xi)T(\xi) + \gamma(\xi)H(\xi). \]

Our labelling from 1 to 13 corresponds to a tiling of the (x, y) plane: we did not try to make the most astute partition, nor to keep as few relevant terms as possible; they should be estimated anyway. Our first task is to estimate the remainder term \( H(\xi) \).

Let \( h' = h^{1-2/\alpha} \). We shall work modulo exponentially small errors, whose order of magnitude is either \( \exp(-1/Ch^{1/\alpha}) \), or \( \exp(-1/Ch') \) \( (C > 0) \). Since \( s > 3 \), the \( \exp(-1/Ch') \) errors are very small; namely for any quantity \( f(h) \) of temperate growth in the scales \( \exp(c/h^{1/\alpha}) \), \( c > 0 \), i.e. \( f(h) = \mathcal{O}(\exp(c/h^{1/\alpha})) \), \( h \to 0 \), we have \( \exp(-1/Ch') f(h) = \mathcal{O}(\exp(-1/C'h')) \) for some \( C' > 0 \) when \( h > 0 \) is small enough.

In Appendix, we prove the following lemmas.

**Lemma 1.1:** Let \( s > 3 \), \( C_0 > 0 \) and \( \eta \in \mathbb{C} \) as above such that \( |\eta| \leq C_0 h^{1-1/\alpha}, \Im \eta \leq 0 \). Then for all \( b > 0 \) small enough, there exists \( \delta_0 > 0 \) such that if \( 0 < \theta < \delta_0 \) we have:

1° \[ H_7(\xi) = \mathcal{O}(e^{-1/CK'}) \]

2° \[ H_8(\xi), H_9(\xi) = \mathcal{O}(1) \]

3° \[ H_{10}(\xi), H_{10}(\xi), H_{11}(\xi), H_{11}(\xi) = \mathcal{O}(e^{-1/CK'}) \]

4° \[ |6 > < 6|, | -8 > < 8|, | -12 > < 12| = \mathcal{O}(e^{-1/CK'}) \]

5° \[ |2 > < 2| = \mathcal{O}(1) \]

in operator norm \( L^2(\Gamma_\delta) \to L^2(\Gamma_\delta) \), when \( h > 0 \) is sufficiently small.

From Lemma 1.1, we have:

\[ H'(\xi) + H''(\xi) = \mathcal{O}(1) \colon L^2(\Gamma_\delta) \to L^2(\Gamma_\delta) \]

The estimates on \( K'(\xi) \) are not so good; this is why we need shrink from below the domain where we are looking for resonances.

**Lemma 1.2:** As in Lemma 1.1, we have

\[ K'(\xi) = \mathcal{O}(e^{-\epsilon \frac{\theta}{h^{1-1/\alpha}}} \Im \eta / 2h) \colon L^2(\Gamma_\delta) \to L^2(\Gamma_\delta) \]

for \( h > 0 \) small enough, and any fixed \( b' > b \).

In the sequel, we shall keep the notation \( b \) instead of \( b' \). It follows from Lemmas 1.1 and 1.2 that for given \( b \) and \( \theta \) as above, and any \( \epsilon > 0 \), we have \( ||\gamma(\xi)H(\xi)|| < 1 \) when \( -\frac{4\pi}{\epsilon} h^{1-1/\alpha} < \Im \eta \), and \( h > 0 \) is small enough. Next we estimate the \( L^2 \)-norm of the \( b_j \)'s and \( q_j \)'s; using estimates of Appendix, we easily show:

**Lemma 1.3:** As in Lemma 1.1, we have:

\[
\begin{align*}
1 + 1 & = \mathcal{O}(1) \\
1 - 1 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
2 - 1 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
5 + 5 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
5 - 5 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
6 - 6 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
8 + 8 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
9 + 9 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
9 - 9 & = \mathcal{O}(1) \\
12 + 12 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
13 + 13 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1) \\
13 - 13 & = e^{-\epsilon \Im \eta / 2h} \mathcal{O}(1)
\end{align*}
\]

where all the estimates are understood as \( L^2(\Gamma_\delta) \) norms.
Note that we have disregarded possible powers of $h$ as prefactors for they are irrelevant.

Now we are ready to compute the inverse for $I - (\gamma(\xi) V)^2$. Let $f_j = (I - \gamma(\xi) H(\xi))^{-1} f_j$ and $T'(\xi)$ the operator with kernel $T'(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^*} f_j(x) g_j(y) \, dy$.

\[
I - (\gamma(\xi) V)^2 = (I - \gamma(\xi) H(\xi))(I - \gamma(\xi) T'(\xi)).
\]

When $-\text{Im} \eta$ is not too large as above, the operator $(I - \gamma(\xi) H(\xi))^{-1}$ makes sense as a Neuman series. The inversion of $I - \gamma(\xi) T'(\xi)$ is standard, namely, to solve the equation $(I - \gamma(\xi) T'(\xi)) u = v$, we try $u = v + \gamma(\xi) \sum_{j=1}^{12} \xi_j f_j$, and set $\eta_j = < v, g_j >$. Using that the $f_j$'s are linearly independent, we are led to the system:

\[
\frac{\xi_i}{\gamma(\xi) f_j} \xi_j = \eta_i, \quad i = 1, 2, \ldots, 12 \tag{1.1}
\]

where $\gamma(\xi) f_j = \gamma(\xi) < f_j, g_j >$.

2. The interaction matrix

Before we proceed we need to recall some definitions; in the sequel we shall compute many wave packet interferences, that is, gaussian integrals. We recall from [Le] that a formal analytic symbol in $h'$ is an expression

\[
a(x, h, h') = \sum_{k \geq 0} a_k(x, h) h'^k
\]

where $a_k(x, h)$ is a sequence of holomorphic functions with respect to $x$, in a neighborhood $V$ of $x_0 \in C^n$ such that $\forall K \subset V$, $3 \epsilon > 0$, $\forall x \in K$ : $|a_k(x, h)| \leq C^{k+1} k!$ uniformly for $0 < h < h_0$.

We call a realization of the formal symbol $a(x, h, h')$ any holomorphic function $\tilde{a}$ in $V$ such that $\forall K \subset V$

\[
\tilde{a} - \sum_{k=0}^{N} a_k(x, h) h'^k \leq \tilde{C}^{N+1}(N+1)! h'^{(N+1)} \quad (\tilde{C} > 0)
\]

uniformly for $x \in K$, $0 < h < h_0$. In particular

\[
\tilde{a}(x, h, h') = \sum_{0 \leq k \leq \tilde{C} \epsilon V} a_k(x, h) h'^k
\]

is a realization of $a$. We shall denote by the same letter a symbol and one of its realizations. Classical analytic symbols in the usual sense, i.e. like $\sum_{k \geq 0} a_k(x) h'^k$ are analytic symbols in $h'$, for $h^k = (h'^k/2^k/\epsilon) h'^k$. These will be denoted simply by $a(x, h)$; actually all symbols we shall encounter here are of this type, except this giving the expansion of the resonances. Now we are ready to compute the scalar products $< f_j, g_i >$, and start with the leading term $< f_j, g_i >$. In Appendix, we prove the following:

**Lemma 1.4**: Within the same conditions as in the previous Lemmas, we have:

1° $< 1 + | + 5 > = < 1 + | + 13 > = O(h^{1/2})$

$< 1 - | - 2 > = 2\pi \epsilon e^{\gamma(\epsilon) h^2} = O(h)$

$< 1 - | - 5 > = < 1 - | - 13 > = \sqrt{2\pi \epsilon} e^{\gamma(\epsilon) h^2} = O(e^{-1/C_\epsilon})$

$< 6 - | - 2 > = \sqrt{2\pi \epsilon} e^{\gamma(\epsilon) h^2} + O(h)$

$< 6 - | - 5 > = < 6 - | - 13 > = 2\pi \epsilon e^{\gamma(\epsilon) h^2} + O(e^{-1/C_\epsilon})$

$< 6 + | + 1 > = < 6 + | + 8 > = < 6 + | + 9 > = 2\pi \epsilon e^{\gamma(\epsilon) h^2} + O(e^{-1/C_\epsilon})$

$< 6 + | + 12 > = \sqrt{2\pi \epsilon} e^{\gamma(\epsilon) h^2} + O(h)$

$< 13 + | + 1 > = < 13 + | + 8 > = < 13 + | + 9 > = \sqrt{2\pi \epsilon} e^{\gamma(\epsilon) h^2} + O(e^{-1/C_\epsilon})$

$< 13 + | + 12 > = 2\pi \epsilon e^{\gamma(\epsilon) h^2} + O(h)$

$< 13 - | - 1 > = < 13 - | - 9 > = O(h^{1/2})$

while all other brackets entering the definition of $T'(\xi)$ are either 0 (for reason of support), or $O(e^{-1/C_\epsilon})$.

2° We also have:

$< 1 + | + 2 > = O(h)$

$< 6 + | + 2 > = 2\pi \epsilon e^{\gamma(\epsilon) h^2} + O(e^{-1/C_\epsilon})$

while $< j \pm | + 2 > = O(e^{-1/C_\epsilon})$ for the other $j$'s.

We now estimate the remainder terms in $< f_j, g_i >$ i.e.

$< (\gamma(\xi) H(\xi) + \gamma^2(\xi) H^2(\xi) + \ldots) f_j, g_i >$. It is clear that the rough $L^2$ estimate from
Lemmas 1.1 to 1.3 are not sufficient, since some \( f_j \) or \( g_i \) are of temperate growth in the scales \( \exp c/\sqrt{h} \). Nevertheless, the first correction in \( H(\xi) \) will be sufficient, for the higher order terms \( <(\gamma^i(\xi)H^j(\xi) + \ldots) f_j| g_i > \) can be controlled only by the \( L^2 \) norms. This amounts to say that \( H(\xi) \) acts only once in perturbation. In Appendix, we compute \( \langle K^i(\xi)|j \rangle \pm \). To simplify the notations we shall denote without labelling, by \( a(x, h) \) the realization of any analytic symbol. Further we adopt the following convention: let \( \nu_j \in \{0, \pm 1\} \), \( k_j \in \{1, 2, 3, \ldots\} \), \( l_j \in \{0, \pm 1, \pm 2, \ldots\} \) be some finite sequences of integers. We denote by \( L_\nu \) (resp. \( L_\nu \)) some finite subset of \( N \) such that \( l_j \) is even (resp. odd) for \( j \in I_\nu \) (resp. \( j \in L_\nu \)), and by \( S_{\nu}(x) = \sum_{j \in I_\nu} e^{-k_j x^2/2h} e^{i\xi(j+\nu_j)/h} a_0(\pm x, \eta) \) a sum of such terms, where \( a_0(\pm x, \eta) \) means that the factor \( a(\pm x, \eta) \) may occur at the \( j \) th place or not. \( S_{\nu}(x) \) holds for a sum of the same type, but indexed over some subset \( L_\nu \subset N \).

**Lemma 1.5**: Under the same conditions as above, and omitting the subscript \( \theta \), we have:

1° \[ < K^\prime(\xi)|2 \rangle - = O \left( e^{-1/CH} \right) \]

2° \[ < K^\prime(\xi)|1 \rangle + = \chi(x < \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi x^2/2h} e^{-ix\eta/h} a(y, h) dy + \int_{-\delta}^{\delta} e^{-\xi y^2/2h} e^{-iy\eta/h} a(y, h) dy + S_\nu(x) \]
\[ + \chi(x > \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi x^2/2h} e^{-ix\eta/h} a(x, h) \int_{-\delta}^{\delta} e^{-\xi y^2/2h} e^{-iy\eta/h} a(y, h) dy + S_{\nu}(x) \]

3° \[ < K^\prime(\xi)|1 \rangle - = \chi(x < \delta) e^{\xi/h} \]
\[ - e^{\xi x^2/2h} e^{-ix\eta/h} a(x, h) \int_{-\delta}^{\delta} e^{-\xi y^2/2h} e^{-iy\eta/h} a(y, h) dy + S_\nu(x) \]

4° \[ < K^\prime(\xi)|6 \rangle - = \int_{-\delta}^{\delta} e^{-\xi x^2/2h} e^{-ix\eta/h} a(y, h) dy + S_\nu(x) \]
\[ + \chi(x > \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi y^2/2h} e^{-iy\eta/h} a(y, h) dy + S_{\nu}(x) \]

5° \[ < K^\prime(\xi)|8 \rangle + = \chi(x < a - \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi x^2/2h} e^{-ix\eta/h} a(y, h) dy + S_\nu(x) \]
\[ + \chi(x > a + \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi y^2/2h} e^{-iy\eta/h} a(y, h) dy + S_{\nu}(x) \]

6° \[ < K^\prime(\xi)|13 \rangle - = \int_{-\delta}^{\delta} e^{-\xi x^2/2h} e^{-ix\eta/h} a(y, h) dy + S_\nu(x) \]
\[ + \chi(x > a + \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi y^2/2h} e^{-iy\eta/h} a(y, h) dy + S_{\nu}(x) \]

7° \[ < K^\prime(\xi)|13 \rangle + = \chi(x < a - \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi x^2/2h} e^{-ix\eta/h} a(y, h) dy + S_\nu(x) \]
\[ + \chi(x > a + \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi y^2/2h} e^{-iy\eta/h} a(y, h) dy + S_{\nu}(x) \]

8° \[ < K^\prime(\xi)|13 \rangle + = \chi(x < a - \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi x^2/2h} e^{-ix\eta/h} a(y, h) dy + S_\nu(x) \]
\[ + \chi(x > a + \delta) e^{\xi/h} \]
\[ + \int_{-\delta}^{\delta} e^{-\xi y^2/2h} e^{-iy\eta/h} a(y, h) dy + S_{\nu}(x) \]
\[\begin{align*}
&\chi(a - \delta < x < a + \delta) e^{-i\xi / h} + 2\pi e^{i\xi / h} \chi(a - \delta < x < a + \delta) e^{-i\xi / h} \\
&\left( \int_{-a}^{a} e^{-\pi y^2 / 2h} \alpha(-y, \eta) a(y, h) dy + \int_{-a}^{a} e^{-\pi y^2 / 2h} e^{\pi y^2 / h} \alpha(y, h) dy + \right) \\
&+ \mathcal{O}\left( e^{-1/c_k} \right) \right) \\
&\text{being uniform in } L^2(\Gamma_h) \text{ norm; the } \mathcal{O}(1) \text{ terms are constant,} \\
&\text{just depending on } h \text{ and } \xi.
\end{align*}\]

It remains to evaluate the other components of \( H(\xi) \). Most significant terms are
\(< H_0(\xi)|j| ± 1 > < H_0(\xi)|j| ± 1 \). Their computation is analogous to this of
\(< K(\xi)|j| ± 1 > < K(\xi)|j| ± 1 \), although it is complicated by the fact we need compute integrals like \( \int_{-a}^{a} e^{-\pi y^2 / 2h} e^{\pi y^2 / h} \alpha(y, \eta) a(y, h) dy \) but this can be done as in Lemma 1.5. We shall just mention the following important fact:
\(< H_0(\xi)|2 - | > = \mathcal{O}(e^{-1/c_k}) < H_0(\xi)|2 + | > = \mathcal{O}(e^{-1/c_k}) < H_0(\xi)|9 - | > = \mathcal{O}(e^{-1/c_k}) < H_0(\xi)|9 + | > = \mathcal{O}(e^{-1/c_k}) < H_0(\xi)|12 - | > = \mathcal{O}(e^{-1/c_k}) < H_0(\xi)|12 + | > = \mathcal{O}(e^{-1/c_k})\)

From this, Lemma 1.5, Lemma 1.4 2°, we have:
\(< H(\xi)|2 - | = \mathcal{O}(e^{-1/c_k}) < H(\xi)|2 + | = \mathcal{O}(e^{-1/c_k}) < H(\xi)|9 - | = \mathcal{O}(e^{-1/c_k}) < H(\xi)|9 + | = \mathcal{O}(e^{-1/c_k}) < H(\xi)|12 - | = \mathcal{O}(e^{-1/c_k}) < H(\xi)|12 + | = \mathcal{O}(e^{-1/c_k})\)

all these estimates being valid uniformly in \( L^2(\Gamma_h) \) and \( \eta \in C[|\eta| < C h^{1-1/\epsilon}, \text{Im} \eta < 0, \text{if } h > 0 \text{ is small enough.} \)

Now we can compute the matrix elements \( < f_i|g_j >, 1 \leq i, j \leq 12. \)

If all quantities \( < \gamma(\xi) H(\xi)f_i|g_j > \) are evaluated as in Lemma 1.4 with the same accuracy, then the remainder term \(< \gamma(\xi) H(\xi) + \ldots|f_i|g_j > \) can be estimated with a crude \( L^2 \)-inequality. So the accuracy to which matrix elements are computed in the following Lemma depends on the terms. This remarkable fact hinges on the structure of the matrix which will be discussed below. Now we precise (with a comfortable margin) the domain allowed for \( \eta \). We put:
\[ W_h = \{ \eta \in C[|\eta| < C h^{1-1/\epsilon}, \text{Im} \eta + 2h^{1-1/\epsilon} > -\frac{a}{h} \} \}

We set also \( j'± | = (I-\gamma(\xi)H(\xi))^{-1}|j± |. \) We have as in Lemma 1.4

Lemma 1.6: For \( \eta \in W_h \), the following estimates hold when \( h > 0 \) is small enough:

1° \(< 1' + |1 + > = e^{-h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' + |1 - > = e^{-h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' + |1 - 2 > = e^{h^2/4h^2} \mathcal{O}(1) < 1' + |1 + 5 > = \mathcal{O}(1) < 1' + |1 - 5 > = < 1' + |1 - 6 > = e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' + |1 + 8 > = < 1' + |1 + 9 > = e^{-h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' + |1 - 9 > = e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' + |1 + 12 > = \gamma(\xi) e^{2h^2/4h^2} \mathcal{O}(1) < 1' + |1 - 13 > = \mathcal{O}(1) < 1' + |1 - 13 > = e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1)\)

2° \(< 1' - |1 - 1 > = e^{-h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' - |1 - 1 > = e^{-h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' - |1 - 2 > = 2\pi h e^{-\pi y^2 / 2h} e^{2\pi y^2 / h} (1 + e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1)) < 1' - |1 - 5 > = e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' - |1 - 6 > = \sqrt{2\pi h e^{-\pi y^2 / 2h}} (1 + e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1)) < 1' - |1 + 8 > = < 1' + |1 + 9 > = e^{-h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' - |1 - 9 > = e^{-h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' - |1 + 12 > = e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' - |1 + 13 > = e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1) < 1' - |1 - 13 > = \sqrt{2\pi h e^{-\pi y^2 / 2h}} (1 + e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1))\)

3° \(< 6' - |1 + 1 > = e^{-2\text{Im} \xi / 2h} e^{h^2/4h^2} \mathcal{O}(1) < 6' - |1 - 1 > = \gamma(\xi) e^{2h^2/4h^2} \mathcal{O}(1) < 6' - |1 - 2 > = (2\pi h)^2 e^{-\pi y^2 / 2h} e^{2\pi y^2 / h} (1 + e^{-2h^{1-1/\epsilon}} e^{h^2/4h^2} \mathcal{O}(1))\)

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\[ <6' - | + 5 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <6' - | - 5 > = <6' - | - 6 > = 2\pi he^{-\nu/2b} e^{2i\xi/\alpha}(1 + e^{-2\nu - 1/\alpha} e^{6b/\alpha b^{1/2}} O(1)) \]
\[ <6' - | + 8 > = <6' - | + 9 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ <6' - | - 9 > = \gamma(\xi)e^{2i\xi/\alpha} O(1) \]
\[ <6' - | + 12 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ <6' - | - 13 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <6' - | + 13 > = 2\pi he^{-\nu/2b} e^{2i\xi/\alpha}(1 + e^{-2\nu - 1/\alpha} e^{6b/\alpha b^{1/2}} O(1)) \]

\[ 4^* \]
\[ <8' + | + 1 > = 2\pi he^{-\nu/2b} e^{2i\xi/\alpha}(1 + e^{-2\nu - 1/\alpha} e^{6b/\alpha b^{1/2}} O(1)) \]
\[ <8' + | - 1 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <8' + | - 2 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ <8' + | + 5 > = \gamma(\xi)e^{2i\xi/\alpha} O(1) \]
\[ <8' + | - 5 > = <8' + | - 6 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ <8' + | + 8 > = <8' + | + 9 > = 2\pi he^{-\nu/2b} e^{2i\xi/\alpha}(1 + e^{-2\nu - 1/\alpha} e^{6b/\alpha b^{1/2}} O(1)) \]
\[ <8' + | - 9 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <8' + | + 12 > = (2\pi h/2b) e^{-\nu/2b} e^{2i\xi/\alpha}(1 + e^{-2\nu - 1/\alpha} e^{6b/\alpha b^{1/2}} O(1)) \]
\[ <8' + | + 13 > = \gamma(\xi)e^{2i\xi/\alpha} O(1) \]
\[ <8' + | - 13 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]

\[ 5^* \]
\[ <13' + | + 1 > = \sqrt{2\pi h} e^{-\nu/2b} e^{2i\xi/\alpha}(1 + e^{-2\nu - 1/\alpha} e^{6b/\alpha b^{1/2}} O(1)) \]
\[ <13' + | - 1 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' + | - 2 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' + | + 5 > = \gamma(\xi)e^{2i\xi/\alpha} O(1) \]
\[ <13' + | - 5 > = <13' + | - 6 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' + | + 8 > = <13' + | + 9 > = \sqrt{2\pi h} e^{-\nu/2b} e^{2i\xi/\alpha}(1 + e^{-2\nu - 1/\alpha} e^{6b/\alpha b^{1/2}} O(1)) \]
\[ <13' + | - 9 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' + | + 12 > = 2\pi he^{-\nu/2b} e^{2i\xi/\alpha}(1 + e^{-2\nu - 1/\alpha} e^{6b/\alpha b^{1/2}} O(1)) \]
\[ <13' + | + 13 > = \gamma(\xi)e^{2i\xi/\alpha} O(1) \]
\[ <13' + | - 13 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ 6^* \]
\[ <13' - | + 1 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' - | - 1 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' - | - 2 > = \gamma(\xi)e^{2i\xi/\alpha} O(1) \]
\[ <13' - | + 5 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' - | - 5 > = <13' - | - 6 > = \gamma(\xi)e^{2i\xi/\alpha} O(1) \]
\[ <13' - | + 8 > = <13' - | + 9 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' - | - 9 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' - | + 12 > = e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' - | + 13 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]
\[ <13' - | - 13 > = e^{-6im/2b} e^{6b/\alpha b^{1/2}} O(1) \]

\[ 7^* \]
\[ <2' - | \pm i > = O(e^{-1/\alpha}) \forall i \]
\[ <2' + | \pm i > = O(e^{-1/\alpha}) \forall i \]
\[ <2' - | \pm i > = O(e^{-1/\alpha}) \forall i \]
\[ <2' + | \pm i > = O(e^{-1/\alpha}) \forall i \]
\[ <2' - | \pm i > = O(e^{-1/\alpha}) \forall i \]
\[ <2' + | \pm i > = O(e^{-1/\alpha}) \forall i \]

Remark: Of course we believe that, e.g., \( H(\xi) | + 1 > = e^{-6im/2b} O(1) \) rather than \( e^{-6im/2b} O(1) \), etc... but we don’t know how to prove it; a way would be to dilate e.g., from \( a \geq b \) rather than \( a \geq b \), but we have not been able then to control simply the remainder \( H(\xi) \). However, this uncertainty has no consequence in the sequel.

3. The system
We label as follows:

\[
\begin{align*}
&f'_1 = \xi' - 1, & f'_2 = \xi' + 1, & f'_3 = \xi'' - 1, & f'_4 = \xi'' + 1, \\
&f'_5 = \xi''' - 1, & f'_6 = \xi''' + 1, & f'_7 = \xi'''' - 1, & f'_8 = \xi'''' + 1,
\end{align*}
\]

and correspondingly for the \(g_i\)'s. We solve explicitly (1.1) modulo errors of order \(e^{-C/\sqrt{\varepsilon}}\).

Introduce

\[
\xi^* = (\xi_1, \ldots, \xi_6), \quad \xi^8 = (\xi_7, \ldots, \xi_{12})
\]

\[
\eta^* = (\eta_1, \ldots, \eta_6), \quad \eta^8 = (\eta_7, \ldots, \eta_{12})
\]

Relations of almost orthogonality in Lemma 1.6, 7° show:

\[
\xi_i = \eta_i + \sum_{j=1}^{6} c_{ij}(\xi^8) \eta_j + O(e^{-1/\sqrt{\varepsilon}}) \xi^8, \quad i = 1, \ldots, 12
\]

(1.3)

where \(O(e^{-1/\sqrt{\varepsilon}})\) denotes a linear form in \(\xi^8\) with \(O(e^{-1/\sqrt{\varepsilon}})\) coefficients if \(\eta \in W_h\).

Now we solve for the first 6 equations (1.1). We have:

\[
\sum_{j=1}^{6} (-\delta_{ij} + c_{ij}(\xi^8)) \xi_j = -\eta_i + O(e^{-1/\sqrt{\varepsilon}}) \xi^8, \quad i = 1, \ldots, 6
\]

(1.4)

The determinant of the system (1.1) vanishes of nearly second order for some discrete values of \(\xi\). It will be computed in the next section. However, since its roots (say \(\xi_k(h)\)) have been computed, we need not solve (1.1) for \(\xi\) arbitrary close to the \(\xi_k(h)\)'s. For instance when \(|\xi - \xi_k(h)| \geq \varepsilon h\), \(\varepsilon > 0\), the system (1.1) can be decoupled in such a way that we do not "feel" the existence of nearly double roots. So we proceed to discuss the resolution of (1.1).

First we solve for \((\xi_1, \xi_4)\) vs. \((\eta_1, \eta_4, \xi_1, \xi_4, \xi_5, \xi_6)\). For \(\eta \in W_h\), the determinant of this \(2 \times 2\) system is \(1 + \gamma(\xi) O(1)\), so is very close to 1. Then we plug \((\xi_1, \xi_4)\) in the last of equations (1.4), i.e.

\[
\sum_{j=1}^{6} (-\delta_{ij} + c_{ij}(\xi^8)) \xi_j = -\eta_i + O(e^{-1/\sqrt{\varepsilon}}) \xi^8
\]

This gives a linear equation for \(\xi_5\) vs. \((\eta_1, \eta_4, \xi_1, \xi_4, \xi_5, \xi_6, \xi^8)\). The coefficient of \(\xi_5\) writes:

\[
d_5(\xi) = 1 - \gamma(\xi) 2 e^{-4h} h e^{24h} e^{26h} + O(1)
\]

(1.5)

The roots of \(d_5(\xi)\) will be shown to be exponentially close (in the scales \(e^{-C/\sqrt{\varepsilon}}\)) to the resonances of our operator. Assume for the moment that \(\xi\) is chosen in such a way that:

\[
|d_5(\xi)| \geq \text{Const.} > 0
\]

(1.6)

(this condition could be substantially weakened, e.g. \(|d_5(\xi)| \geq e^{-C/\sqrt{\varepsilon}}\) for small \(C > 0\), but it is sufficient to our purpose.) If we report \(\xi_5\) vs. \((\eta_1, \eta_4, \xi_1, \xi_4, \xi_5, \xi_6, \xi^8)\) into \((\xi_1, \xi_4)\) when (1.6) holds, we get \((\xi_1, \xi_4)\) vs. \((\eta_1, \eta_4, \xi_1, \xi_4, \xi_5, \xi_6, \xi^8)\). Next, using 3rd and 5th of equations (1.4), we solve for \((\xi_3, \xi_7)\) vs. \((\eta_3, \eta_7, \xi_3, \xi_7, \xi_8, \xi^8)\). Again, we get a \(2 \times 2\) system whose determinant is \(1 + \gamma(\xi) e^{-C/\sqrt{\varepsilon}} e^{6h} e^{8h} O(1)\). Then we report these values for \((\xi_3, \xi_7)\) in the expressions giving \((\xi_1, \xi_4, \xi_5)\) : so whenever (1.6) holds, we get \((\xi_1, \xi_4, \xi_5, \xi_6)\) vs. \((\eta_1, \eta_4, \xi_1, \xi_4, \xi_5, \xi_6, \xi^8)\), which turn gives \((\xi_2, \xi_5, \xi_6)\) vs. \((\eta_2, \eta_5, \xi_2, \xi_5, \xi_6, \xi^8)\). Substituting into the first of equations (1.4) we get a linear equation for \(\xi_4\) vs. \((\eta^*, \xi^8)\), and the coefficient of \(\xi_4\) is found to be:

\[
d_4(\xi) = 1 - \gamma(\xi) 2 e^{-4h} h e^{24h} e^{26h} + \gamma^2(\xi) e^{-4h} e^{24h} O(1)
\]

(1.7)

where \(O(1)\) is holomorphic in \(W_h\) ; so \(d_4(\xi)\) is very close to \(d_4(\xi)\). We still assume:

\[
|d_4(\xi)| \geq \text{Const.} > 0
\]

(1.8)

Having determined \(\xi_1\) vs. \((\eta^*, \xi^8)\) we can substitute into the former expression giving \(\xi^*.\)

Using also (1.3) to get rid of the \(\xi^*\) dependence, we eventually obtain the following Lemma.

**Lemma 1.7:** Assume \(\eta \in W_h\) and (1.6), (1.8) hold. Then the system (1.1) has a unique solution given by:

\[
\begin{align*}
\xi_1 &= \frac{1}{d_1(\xi)} \eta + e^{-4h} e^{24h} O(1) \eta_1 + \\
&+ \frac{1}{d_1(\xi)} \sqrt{2\pi h} e^{-\xi^2 / 2h} \gamma(\xi) \left(1 + e^{-4h} e^{24h} O(1)\right) \eta_1
\end{align*}
\]

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\[\xi_2 = e^{-2\theta h} \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_1(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_1 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_1(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_2 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_1(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_3 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_1(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_4 + \mathcal{O} \left(e^{-1/\sqrt{\hbar}}\right) (\eta_5).\]

\[\xi_3 = \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_3(\xi) \eta_1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1) \eta_2 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_3(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_3 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1) \eta_4 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1) \eta_5 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1) \eta_6 + \mathcal{O} \left(e^{-1/\sqrt{\hbar}}\right) (\eta_7).\]

\[\xi_4 = e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1) \eta_1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1) \eta_2 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_4(\xi) \eta_3 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_4(\xi) \eta_4 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_4(\xi) \eta_5 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_4(\xi) \eta_6 + \mathcal{O} \left(e^{-1/\sqrt{\hbar}}\right) (\eta_7).\]

\[\xi_5 = \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_5(\xi) \gamma(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1) \eta_2 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1) \eta_3 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_5(\xi) \gamma(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_4 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_5(\xi) \gamma(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_5 + \frac{1}{d_1(\xi)} e^{2\pi i/\hbar} \kappa_5(\xi) \gamma(\xi) \left(1 + e^{-2\theta h} e^{128/\hbar^{1/4}} \mathcal{O}(1)\right) \eta_6 + \mathcal{O} \left(e^{-1/\sqrt{\hbar}}\right) (\eta_7).\]

Here \(\kappa_1(\xi), \kappa_3(\xi), \kappa_4(\xi) = \mathcal{O}(1)\) are holomorphic functions in \(W_\hbar\) (\(\xi_1, \ldots, \xi_6\) are then determined by (1.3)).

4. The determinants

A straightforward computation shows that the determinant of the whole system (1.1) is given by:

\[\det(\xi) = \left(1 - 2\theta h e^{-2\pi i/\hbar} e^{2\pi i/\hbar} \gamma(\xi) + \mathcal{O}(1) \right) \left(1 - 2\theta h e^{-2\pi i/\hbar} e^{2\pi i/\hbar} \gamma(\xi) + \mathcal{O}(1) \right) \left(1 - 2\theta h e^{-2\pi i/\hbar} e^{2\pi i/\hbar} \gamma(\xi) + \mathcal{O}(1) \right) + \mathcal{O}(1) \left(e^{-1/\sqrt{\hbar}}\right) - C > 0.\]

all \(C\) being holomorphic in \(W_\hbar\). Now we investigate the roots of \(d_\xi(\xi), d_\xi(\xi)\) and \(d_\xi(\xi)\).

First we rescale \(\eta\) by setting \(\eta = \eta h^{-1/4}\), so that \(\tilde{\eta} = \mathcal{O}(1)\). For \(k \in \mathbb{Z}\) such that:

\[\frac{2\pi a}{\hbar} |k| h < \frac{1}{2} C_\theta h^{-1/4} (1.10)\]

we set \(\tilde{\eta}_k = 2\pi a h^{-1/4} a\), and \(\tilde{\eta}_k = \tilde{\eta}_k - \frac{a}{2} + \tilde{\xi}_k\), where \(|\text{Re}\tilde{\xi}| < \frac{x}{2} h^{1/8}, |\text{Im}\tilde{\xi}| < \frac{y}{2} h^{1/8}\). Let : \(B_k(h) = \{\eta = h^{-1/4} (\tilde{\eta}_k - \frac{a}{2} + \tilde{\xi}_k), |\text{Re}\tilde{\xi}| < \frac{x}{2} h^{1/8}, |\text{Im}\tilde{\xi}| < \frac{y}{2} h^{1/8}\}\), so that the union over \(k \in \mathbb{Z}\) satisfying (1.10) of small neighborhoods of the \(B_k(h)\) cover the "bottom" of \(W_\hbar\). It is clear that \(d_\xi(k), d_\xi(\xi)\) and \(d_\xi(\xi)\) do not vanish in \(W_\hbar \setminus \bigcup B_k(h)\). Next we fix \(a = a(h) = 26\).
\[(2n + 1)xh \quad \text{where} \quad m = m(h) \in \mathbb{N} \quad \text{is so large that} \quad |a - a_0| < \text{Const} \cdot h, \quad \text{for some constant} \quad a_0.\]

This is a coherence condition to get a standing wave pattern. We set:

\[d(\xi) = 1 - 2\pi h \, e^{-\eta^2/4h} e^{ia\xi/h} \gamma(\xi) \quad (1.11)\]

We have:

\[1 - d(\xi) = \left(1 + h^{-1/2} \left(\frac{\eta - 2i}{a} + C\right)\right)^{2} \exp \left[ia\xi/h \right] \exp \left(-\frac{\eta - 2i}{a} + C\right)^{2} h^2 \]

We rescale again \(\zeta\) by setting \(\zeta = h^{1/4} \bar{\zeta}\), \(\bar{\zeta} = O(1)\), which amounts to shrink \(\zeta\) in \(B_h(1)\); again it is clear that none of \(d_k(\xi)\), \(d_1(\xi)\), \(d_0(\xi)\), or \(d(\xi)\) vanish in \(B_h(1)\). \(B_h(1)\), where:

\[B_h(1) = \left\{ \eta = h^{1/4} \bar{\zeta} \left(\frac{\eta - 2i}{a} + h\bar{\zeta}\right), \left|\text{Re} \bar{\zeta}\right| < \frac{\pi}{a}, \left|\text{Im} \bar{\zeta}\right| < C_1 \right\}\]

where \(C_1 > 0\) is so far arbitrary. So:

\[1 - d(\xi) = \left(1 + \frac{2\pi k h}{a} - \frac{2i}{a} h^{-1/2} + h\bar{\zeta}\right)^{2} e^{ia\bar{\zeta}} \exp \left[-\left(\frac{\eta - 2i}{a} + h\bar{\zeta}\right)^{2} h^2 \right] \]

By a Taylor expansion, we see easily that:

\[1 - d(\xi) - e^{ia\bar{\zeta}} = O(h') \quad (1.12)\]

uniformly for \(\eta \in B_h(1)\). Now, if \(\text{Re} \bar{\zeta} < \frac{\pi}{a}\) and \(\text{Im} \bar{\zeta} < C_1\), the equation \(e^{ia\bar{\zeta}} = 1\) has the unique solution \(\bar{\zeta} = 0\) and this is a simple root. Moreover, it is easy to see that \(1 - e^{ia\bar{\zeta}}\) is bounded away from 0 on the boundary of \(\text{Re} \bar{\zeta} < \frac{\pi}{a}\), \(\text{Im} \bar{\zeta} < C_1\). So by Rouche's theorem, one can assert from (1.12) that equation \(d(\xi) = 0\) has a simple root in \(B_h(1)\). The same holds for \(d_k(\xi)\) and \(d_1(\xi)\), and \(d_0(\xi)\) has exactly 2 roots in \(B_h(1)\). Further, all these roots depend smoothly on \(h > 0\), and admit clearly asymptotic expansions. Namely, when:

\[\eta_k = h^{1/4} \left(\frac{\eta - 2i}{a} + h\bar{\zeta}_k\right) \quad (1.13)\]

denotes such a root (the roots of \(d_1(\xi)\), \(d_0(\xi)\) or \(d_0(\xi)\) differ only by \(O\left(e^{-C/h^{1/2}}\right)\) from these of \(d(\xi)\), then we have:

\[\bar{\zeta}_k \sim \sum_{j=1}^{\infty} h^{-j/4} \zeta_{k,j}(h) \quad (1.14)\]
II. END OF THE PROOF

We show that \( \Omega_\alpha \) is actually a set of resonances for \( P \). Let \( \xi_0 \) be one of the roots of \( d(\xi) = 0 \) in \( W_0 \). We know that there exist two roots (counted with possible multiplicity) of \( d(\xi) = 0 \) in \( [\xi - \xi_0] < c \epsilon \) \((\epsilon > 0)\) and that they are exponentially close to \( \xi_0 \) in the scale \( e^{-C/|\lambda|} \). Consider the spectral projector for \( P \) in \( [\xi - \xi_0] \leq c \epsilon \):

\[
\Pi(\xi_0) = \frac{1}{2\pi i} \oint (I - (G(\xi)V)^{-1})(I - G(\xi)V)G(\xi) d\xi
\]

where \( \oint \) denotes integral along the closed loop \( [\xi - \xi_0] = c \epsilon \). It is easy to check that (1.6) and (1.8) hold on this loop. So we have:

\[
(I - (G(\xi)V)^{-1})(I - \gamma(\xi)S(\xi)(I - \gamma(\xi)H(\xi))^{-1})
\]

where:

\[
S(\xi) = \sum_{i,j=1}^{12} d_{ij}(\xi) f_i(x) g_j(y)
\]

and \( d_{ij}(\xi) \) is given by \( \xi_0 = \sum_{j=1}^{12} d_{ij}(\xi) \eta_j \). By Cauchy's formula:

\[
\Pi(\xi_0) = \frac{1}{2\pi i} \oint d_{ij}(\xi) f_i(x) A(\xi) y_j(y) (\xi - \xi) d\xi
\]

where \( A(\xi) = G(\xi)(I - VG(\xi))I - \gamma(\xi)^t H(\xi))^{-1} \) and \( ^tH(\xi) \) denotes transpose of \( H(\xi) \).

Because of (1.3), Lemma 1.7 and Cauchy's formula, we get:

\[
\Pi(\xi_0) = \frac{1}{2\pi i} \sum_{k=1}^{6} \oint_{\gamma_k} F_k(\xi) A(\xi) G_k(y) (\xi - \xi_0)^{-1} + O(e^{-1/Cx}) \quad [2.1]
\]

where:

\[
F_k(x) = F_k^*(x) + \sum_{i=1}^{12} \gamma_i(\xi) < f_i'(x) g_i(y), \quad k = 1, \ldots, 6
\]

\[
G_k(y) = \sum_{j=1}^{6} d_{ij}(\xi) y_j(y).
\]

We shall compute \( F_k(x) \) and \( G_k(y) \) in the following lemmas.

Lemma 2.1: For \( \eta \in W_0 \), we have:

1. \( F_1(x) = \sqrt{2\pi \hbar \epsilon} e^{-\frac{i\eta}{2\hbar}} e^{i\epsilon \xi / \hbar} \left[ \chi(-\delta < x < \delta) + (1 - d(\xi))\chi(x < -\delta) + (1 - d(\xi))\chi(x < -\delta) + e^{-2k_{x}} e^{i\xi / \hbar} \right] + e^{i\xi / \hbar} O(1)

2. \( F_2(x) = (\chi < \delta) e^{i\xi / \hbar} \varphi_1(x) + e^{-2k_{x}} e^{i\xi / \hbar} O(1)

3. \( F_3(x) = e^{-i\xi / \hbar} [(1 - d(\xi))\chi(x < -\delta) + (1 - d(\xi))\chi(x < -\delta) + e^{-2k_{x}} e^{i\xi / \hbar} O(1)

4. \( F_4(x) = e^{i\xi / \hbar} \left[ \chi(x > a - \delta) + (1 - d(\xi))\chi(x > a + \delta) + (1 - d(\xi))\chi(x > a + \delta) + e^{-2k_{x}} e^{i\xi / \hbar} O(1)

5. \( F_5(x) = (\chi < a - \delta) e^{-i\xi / \hbar} \varphi_2(x) + (1 - d(\xi))\chi(x > a - \delta) + e^{-2k_{x}} e^{i\xi / \hbar} O(1)

6. \( F_6(x) = \sqrt{2\pi \hbar \epsilon} e^{-\frac{i\eta}{2\hbar}} e^{i\epsilon \xi / \hbar} \left[ (1 - d(\xi))\chi(x < a - \delta) + (1 - d(\xi))\chi(x > a + \delta) + e^{-2k_{x}} e^{i\xi / \hbar} O(1) \right] + e^{-2k_{x}} e^{i\xi / \hbar} O(1)

We see that:

\[
F_1(x)\big|_{\xi = \xi_0} = \sqrt{2\pi \hbar \epsilon} e^{-\frac{i\eta}{2\hbar}} e^{i\epsilon \xi / \hbar} \chi(x < a - \delta) + e^{i\xi / \hbar} O(1) \quad [2.2]
\]

and similarly:

\[
F_6(x)\big|_{\xi = \xi_0} = \sqrt{2\pi \hbar \epsilon} e^{-\frac{i\eta}{2\hbar}} e^{i\epsilon \xi / \hbar} \chi(x > a - \delta) + e^{-2k_{x}} e^{i\xi / \hbar} O(1) \quad [2.3]
\]
After gluing \( F_5(x) \) and \( F_2(x) \) to these functions we will get the approximate resonant functions. We put:

\[
\tilde{\psi}_1(x) = \left( \sqrt{2\pi} e^{-\nu^2/2h} \right)^{-1} \psi_1(x) \\
\tilde{\psi}_2(x) = \left( \sqrt{2\pi} e^{-\nu^2/2h} e^{2\nu x/\hbar} \right)^{-1} \psi_2(x) = \tilde{\psi}_2(a-x)
\]

(\( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) are “normalized” in the sense (see (a.3), (a.4)) : \( \tilde{\psi}_1(x) = 1 + O(e^{-1/\hbar^4}) \), \( \delta < x < a - \delta \)).

**Lemma 2.2**: For \( |\xi - \xi_0| = \hbar, \) we have:

1. \( G_1(y) = \frac{1}{d(\xi)} e^{i \xi y/\hbar} \psi_-(y) \left[ (1 - d(\xi))^2 \chi(-\delta < y < \delta) \tilde{\psi}_1(y) + (1 - d(\xi))^2 \chi(\delta < y < a - \delta) \tilde{\psi}_2(y) \right] + e^{-2\hbar^{-1/4} e^{123/\hbar^{1/4}} O(1)}.

2. \( G_2(y) = \frac{1}{d(\xi)} e^{i \xi y/\hbar} \psi_+(y) \left[ (1 - d(\xi))^2 \chi(-\delta < y < \delta) \tilde{\psi}_1(y) + (1 - d(\xi))^2 \chi(\delta < y < a - \delta) \tilde{\psi}_2(y) \right] + e^{-2\hbar^{-1/4} e^{123/\hbar^{1/4}} O(1)}.

3. \( \tilde{G}_1(y) = \frac{1}{d(\xi)} \left[ y^2 \xi e^{i \xi y/\hbar} \chi(y < \delta) e^{i \xi y/\hbar} \psi_-(y) \right] + \text{hol} + e^{-k^{-1/4} e^{123/\hbar^{1/4}} O(1)}.

4. \( \tilde{G}_2(y) = \frac{1}{d(\xi)} \left[ y^2 \xi e^{i \xi y/\hbar} \chi(y < \delta) e^{i \xi y/\hbar} \psi_+(y) \right] + \text{hol} + e^{-k^{-1/4} e^{123/\hbar^{1/4}} O(1)}.

5. \( \tilde{G}_3(y) = \frac{1}{d(\xi)} e^{i \xi y/\hbar} \psi_-(y) \left[ (1 - d(\xi))^2 \chi(-\delta < y < \delta) \tilde{\psi}_1(y) + (1 - d(\xi))^2 \chi(\delta < y < a - \delta) \tilde{\psi}_2(y) \right] + e^{-2\hbar^{-1/4} e^{123/\hbar^{1/4}} O(1)}.

6. \( \tilde{G}_4(y) = \frac{1}{d(\xi)} e^{i \xi y/\hbar} \psi_+(y) \left[ (1 - d(\xi))^2 \chi(-\delta < y < \delta) \tilde{\psi}_1(y) + (1 - d(\xi))^2 \chi(\delta < y < a - \delta) \tilde{\psi}_2(y) \right] + e^{-2\hbar^{-1/4} e^{123/\hbar^{1/4}} O(1)}.

where all estimates are uniform on a neighborhood of \( |\xi - \xi_0| = \hbar \), and “hol” denotes an holomorphic function in \( W_\hbar \) (with respect to the \( \xi \) or \( y \) variable).

Denote by \( \tilde{G}_1(y) \) the meromorphic part of \( G_1(y) \) in the formulae above. Modulo an error \( O\left(e^{-C/\hbar^{1/4}}\right) \), \( C > 0 \), we have:

\[
\tilde{G}_1(y)|_{\xi = \xi_0} = \tilde{G}_0(y)|_{\xi = \xi_0} \\
= e^{i \xi_0 y/\hbar} \psi_-(y) \left[ \chi(-\delta < y < \delta) + \chi(\delta < y < a + \delta) \tilde{\psi}_2(y) \right]_{|_{\xi = \xi_0}}
\]

\[
\tilde{G}_2(y)|_{\xi = \xi_0} = \tilde{G}_0(y)|_{\xi = \xi_0} \\
= e^{i \xi_0 y/\hbar} \psi_+(y) \left[ \chi(-\delta < y < \delta) \tilde{\psi}_1(y) + \chi(\delta < y < a + \delta) \right]_{|_{\xi = \xi_0}}
\]

So we define the outgoing functions:

\[
F_+(x) = F_2(x) + F_3(x)|_{\xi = \xi_0} \\
= \sqrt{2\pi e^{123/\hbar^{1/4}}} \left[ \chi(-\delta < x < \delta) \tilde{G}_1(x) + \chi(x > \delta) \right]_{|_{\xi = \xi_0}} + e^{-k^{-1/4} e^{123/\hbar^{1/4}} O(1)}
\]

\[
F_-(x) = F_1(x) + F_2(x)|_{\xi = \xi_0} \\
= \sqrt{2\pi e^{123/\hbar^{1/4}}} \left[ \chi(-\delta < x < \delta) \tilde{G}_1(x) + \chi(x > \delta) \right]_{|_{\xi = \xi_0}} + e^{-k^{-1/4} e^{123/\hbar^{1/4}} O(1)}
\]

(the last term in \( F_1(x) \), and the last two terms in \( F_-(x) \) can be neglected modulo relative errors of order \( O\left(e^{-C/\hbar^{1/4}}\right) \), \( C > 0 \).

Now we apply the formula of residues to (2.1):

\[
\Pi(\xi_0) = \frac{2}{d(\xi_0)} \gamma(\xi_0) \left[ F_1(x)A(\xi)\tilde{G}_1(x)|_{\xi = \xi_0} + F_-(x)A(\xi)\tilde{G}_2(x)|_{\xi = \xi_0} + F_3(x)A(\xi)\tilde{G}_3(x)|_{\xi = \xi_0} + F_2(x)A(\xi)\tilde{G}_4(x)|_{\xi = \xi_0} + e^{-k^{-1/4} e^{123/\hbar^{1/4}} O(1)} \right] + O\left(e^{-1/\hbar^{4}}\right)
\]

where the dot denotes derivative with respect to \( \xi \). Actually, \( F_1(x)A(\xi)\tilde{G}_1(x)|_{\xi = \xi_0} = 0 \), and \( F_2(x)A(\xi)\tilde{G}_2(x)|_{\xi = \xi_0} \) can be neglected because it is only \( O(1) \). Moreover it is easy to check that \( A(\xi)\tilde{G}_1(x)|_{\xi = \xi_0} \) and \( A(\xi)\tilde{G}_2(x)|_{\xi = \xi_0} \) are linearly independent and give the main
contribution to the kernel of $\Pi(\xi)$, modulo errors of order $e^{-C/(h)^{1/4}}$. So we have, putting $G_+(y) = \lambda(\xi) \bar{G}_1(y)\|_{L^2}$ and $G_-(y) = \lambda(\xi) \bar{G}_2(y)\|_{L^2}$ (we check that $G_+(y)$ [resp. $G_-(y)$] is essentially outgoing at $+\infty$ [resp. at $-\infty$]):

$$\Pi(\xi) = \frac{2}{d(\xi)} \gamma(\xi) \xi_5 \left[ F_+ (\xi) G_+(y) + F_- (\xi) G_-(y) \right] + O \left( e^{-C/(h)^{1/4}} \right).$$

(2.4)

The projector $\Pi(\xi)$ is of rank at least 2. Our purpose was to solve the eigenvalue problem $Pu = -h^2 u'' + V(x, h)u = \xi^2 u$ on $L^2(\mathbb{R}_x)$ for $\xi$ close to $\frac{x}{2}$. We know from the general theory of ordinary differential equations that the eigenspace corresponding to a resonance $\xi^2$ is one dimensional. Namely, for given $\xi$, $Re^{i} > 0$, let $u_+(x)$ the solution of $(P - \xi^2)u = 0$ in $L^2(\mathbb{R}_x)$, with $u_+ \sim e^{i\xi x/h}$, $x \to -\infty$ (i.e. $u_+$ is outgoing at $+\infty$, $u_-$ is outgoing at $-\infty$); then $\xi^2$ is a resonance iff $u_+$ and $u_-$ are collinear. But as $P$ is not self adjoint, it cannot be excluded that $\xi^2$ is a multiple pole for the resolvent $(P - \xi^2)^{-1}$; in that case, stability arguments (see [Ra]) and also [Si], [KaRo] for related results) show that this multiplicity is at most 2. So we can conclude that the eigenprojector $\Pi(\xi)$ is of rank exactly 2 and that the alternative of Theorem 0.1, 3° holds. The other statements easily follow from Proposition 1.8. So we have proved the Theorem.

Appendix:

It will be useful to estimate (when $t$ is close to 0) $a(t, \tau)$ by the following lemma which is a standard exercise on contour integrals:

**Lemma a.1**: Let $F(t) = \frac{1}{2} \left( 1 + (2\pi h)^{-1/2} \int_{t}^{t+1/2} e^{-y^2/2h} dy \right)$. Then for all $t, \tau \in \mathbb{C}$:

$$e^{t\tau/h} a(t, \tau) = \sqrt{2\pi h} F(t - \tau) + ie^{t(1-\tau)/h} \int_{0}^{Re} e^{y^2/h} e^{-iy(1-\tau)/h} dy$$

(a.1)

In particular for real $t$ we have:

$$|a(t, \tau)| \leq \frac{1}{\sqrt{2\pi h}} \exp \left\{ (t \tau)^2 - (t \tau)^2 \right\} / 2h$$

$$+ |\tau| \left( (t - \tau)^2 - (t - \tau)^2 \right) / 2h$$

(a.2)

Note that we don’t really use this lemma; simple non stationary phase arguments would suffice (see below).

1° Sketch of the proof of Lemma 1.1

We shall content ourselves to give the proof of estimates for $H_1(\xi), H_2(\xi)$ for the other cases can be treated in the same spirit.

i) $H_1(\xi)$. This will be an easy consequence of Lemma a.1. There is no distortion for $\delta < x, y < a - \delta$. First we notice that when $x, y \in \mathbb{R}, |x| > \delta$ and $\eta \in \mathbb{C}, |\eta| \leq C_h^{-1/2}$ we have $\langle x - \tau \rangle, \delta > \delta/2$ for $h > 0$ small enough, since $s > 2$. It follows easily from (a.1) and (a.2) that:

$$\alpha(x, \eta) = \sqrt{2\pi h} e^{-y^2/2h} + O \left( e^{-1/C} \right), \quad x > \delta$$

(a.3)

$$\alpha(x, \eta) = O \left( e^{-1/C} \right), \quad |z| > \delta$$

(a.4)

while clearly:

$$\alpha(x, \eta) = O \left( e^{-1/C} \right), \quad x < -\delta$$

(a.5)

all these estimates being uniform in the intervals we consider. Then:

$$\varphi_1(x \wedge y) = \sqrt{2\pi h} e^{-y^2/2h} + O \left( e^{-1/C} \right), \quad \delta < x, y < a - \delta$$

$$\varphi_2(x \vee y) = \sqrt{2\pi h} e^{-y^2/2h} e^{2\pi i h} + O \left( e^{-1/C} \right), \quad \delta < x, y < a - \delta.$$
On the other hand:

\[ |e^{i(x-y)\ell/k}\psi_+(y)| \leq 4e^{-1/2\delta}e^{-\alpha y} \Im \psi_+, \quad \delta < x, y < a - \delta \]
\[ |e^{i(x-y)\ell/k}\psi_-(y)| \leq 4e^{-1/2\delta}e^{\alpha y} \Im \psi_-, \quad \delta < x, y < a - \delta \]

The estimate for \( H_1^{(c)}(\xi) \) follows easily.

ii) \( H_3^{(c)}(\xi) \). First we specify the (standard) distortion. Let \( b > 0 \), and \( t \) a smooth function on \( \mathbb{R} \) such that \( t \equiv 1 \) on \( (-\infty, -2b) \cup (a + 2b, +\infty), \ t \equiv 0 \) on \( [-a, -b], \) and \( t \) is strictly decreasing on \( [-2b, -b] \), strictly increasing on \( [a + b, a + 2b] \). We put \( x_\theta = xe^{\theta t(x)} \), which we shall denote \( x_\theta = xe^{\theta t(x)} \) for simplicity. We have:

\[ H_3^{(c)}(\xi) = \chi(x < y < \delta) \left( e^{i(x-y)\ell/k}\psi_+(y)\varphi_1(x_\theta) + e^{-i(x-y)\ell/k}\psi_-(y)\varphi_2(x_\theta) \right) \]

By Schur's lemma, we need estimate:

\[ \sup_{x < \delta \lambda} \int_{-\delta}^{\delta} |H_3^{(c)}(\xi)| dy \tag{a.6} \]
\[ \sup_{y < \delta \lambda} \int_{0}^{\delta} |H_3^{(c)}(\xi)| dx \tag{a.7} \]

For \( -b < x < \delta \), we may replace (a.6) by

\[ \sup_{-b < y < \delta} |H_3^{(c)}(\xi)| = O \left( e^{-1/C \delta} \right). \]

For \( x < -b \), we apply distortion. Write \( H_3^{(c)}(\xi) = H_3^{(c)}(\xi) + H_3^{(c)}(\xi) \). We analyse \( H_3^{(c)}(\xi) \) first. We have:

\[ \alpha(-y_\theta, \tau) = \alpha(-\Re y_\theta, \tau) \]
\[ -i \Im y_\theta e^{-\tau^2/2b} \int_{0}^{\tau} \exp \left[ -(\Re y_\theta + i\Im y_\theta - i\tau)^2/2b \right] dt \tag{a.8} \]

For \( \gamma = \eta \) we have:

\[ \alpha(-y_\theta, \eta), \alpha(a - y_\theta, \eta) = \sqrt{2\pi} e^{-\eta^2/2b} + O \left( e^{-1/C \eta} \right), \quad y < \delta \]

while for \( \gamma = \eta' \), it is readily seen that for \( \gamma > 0 \) small enough (depending on \( b \) only)

\[ \alpha(-y_\theta, \eta'), \alpha(a - y_\theta, \eta') = O \left( e^{-1/C \eta'} \right), \quad y < \delta \tag{a.9} \]

all the estimates being uniform on the given intervals.

Next we have:

\[ \psi_\theta(y_\theta) = e^{-y_\theta^2/2b} \left( e^{\delta(y_\theta + e^{i\theta t(x)}\psi_+(y_\theta))} + O \left( e^{-1/C \delta} \right), \quad y < \delta \]

so:

\[ |\psi_\theta(y_\theta)| \leq \exp \left( \frac{\delta^2}{2} - \frac{\omega^2}{2} \right) \] \( \tag{a.11} \)

From these estimates, it follows easily that if \( \delta > 0 \) is small enough, we have:

\[ \sup_{-b < y < \delta} \int_{-\delta}^{\delta} |H_3^{(c)}(\xi)| dy = O \left( e^{-1/C \delta} \right). \]

For the integral with respect to \( x \), we use also that

\[ |e^{-i\ell\xi/k}| \text{ is integrable when } x \to -\infty. \]

So:

\[ \sup_{y < \delta} \int_{-\delta}^{\delta} |H_3^{(c)}(\xi)| dx = O \left( e^{-1/C \delta} \right). \]

Now consider \( H_3^{(c)}(\xi) \). The result follows this time from the exponential decrease of \( \varphi_1(x_\theta) \).

Namely, by (a.8) we have:

\[ \alpha(x_\theta, \tau) = \int_{-\infty}^{\Re x_\theta} e^{-t^2/2b} e^{it\tau/2b} dt + i \Im x_\theta e^{-\tau^2/2b} \]
\[ \times \int_{0}^{\tau} \exp \left[ -(\Re x_\theta + i\Im x_\theta + i\tau)^2/2b \right] dt \]

so when \( \delta > 0 \) is small enough:

\[ \alpha(x_\theta, \tau) = e^{-\tau^2/12b} O(1), \quad x < \delta \]

with uniform estimate on \( x < \delta \), with error integrability. The exponential decrease, again is given by \( \psi_+(y_\theta) \). Thus we have estimated \( H_3^{(c)}(\xi) \). \( \square \)

2° Proof of Lemma 1.2

We have:

\[ K'(\xi) = e^{i(x-y)\ell/k} \psi_-(y) \beta(y_\theta, x_\theta) + e^{-i(x-y)\ell/k} \psi_+(y_\theta) \beta(x_\theta, y_\theta) \]

We need first estimate \( \beta(y_\theta, x_\theta) \); as \( V(t) \) is a sum of terms like \( e^{-1/2b - i(x+y)\theta /2b} \) and \( e^{-1/2b - i(x+y)\theta /2b} \) we can compute \( \beta(x_\theta, y_\theta) \) by integrating the function \( e^{-i\ell \xi /k} \) along
suitable polygons. For instance:

\[
\int_{\gamma} e^{-i(xz)/\sqrt{2h}} dt = \int_{\Gamma_0} e^{-i(z^2/2h)} dt
\]
\[
- i(x \sin \theta(x) \pm 1) \int_{\gamma} \exp \left[ - (x \cos \theta(x) \pm i(y \sin \theta(x) \pm 1))^{2}/2h \right] dt
\]
\[
+ i(y \sin \theta(y) \pm 1) \int_{\gamma} \exp \left[ - (y \cos \theta(y) + i(y \sin \theta(y) \pm 1))^{2}/2h \right] dt.
\]

So if \( \theta > 0 \) is small enough, we have, uniformly along \( \Gamma_0 \):

\[
|\beta(x, x_0)| \leq C_\theta (y < x) \left( e^{-x^2/2h} + e^{-y^2/2h} + e^{-y^2/2h} \sqrt{2\pi h} \right) \]  \hspace{1cm} (a.12)

\[
K^{(i)}(x) = O \left( \exp \frac{a + b}{2} \right), \quad -b < x, y < a + b. \]  \hspace{1cm} (a.13)

Now we apply distortion outside \(-b < x, y < a + b\). As above, Schur's lemma, together with the above estimates, show easily that (a.12) holds in the sense of operator norm in \( L^2(\Gamma_0) \), when the constant \( \delta \) on the right hand side, is replaced by any \( \delta' > \delta \). \( \square \)

3\textsuperscript{a} Proof of Lemma 1.4

Again, we shall content ourselves to compute a few terms which suffice to indicate the main ingredient of the proof. We start with:

Lemma a.2: Let \( \gamma = \pm y' \). Then there exists a classical analytic symbol \( a(x, h) \sim \sum_{k \in \mathbb{Z}} a_k h^k \) defined in a neighborhood of \( 0 \) such that:

\[
a(x, \gamma) = h e^{-x^2/2h} e^{-ix\gamma/\sqrt{2h}} a(x, h) + O \left( e^{-1/Ch} \right), \quad C > 0.
\]

Proof: This follows from repeated integrations by parts since the phase is non-stationary, and from the fact that \( \operatorname{Im} = O \left( h^{-1/2} \right) \).

i) \( |1 + z| = \int_{-\delta}^{\delta} e^{2i\pi \xi/h} \psi_1(x) \psi_2(x) dx \)....

Expanding \( \psi_1(x) \psi_2(x) \) with Lemma a.2, we get:

\[
\psi_1(x) \psi_2(x) = a(x, \gamma) a(x, \gamma) + a(x, \gamma) e^{-ix\gamma/\sqrt{2h}} a(x, h) + \]
\[
e^{ix\gamma/\sqrt{2h}} a(x, h) + e^{ix\gamma/\sqrt{2h}} \Delta a(x, h) + \Delta a(x, h) + O \left( e^{-1/Ch} \right), \quad -\delta < x < \delta
\]

where we sum over the \( \pm \) signs, and do not label the different analytic symbols for simplicity. Consider for instance:

\[
\int_{-\delta}^{\delta} e^{ix\gamma/\sqrt{2h}} \psi_1(x) \psi_2(x) dx = \int_{-\delta}^{\delta} e^{-x^2/2h} e^{-ix\gamma/\sqrt{2h}} \psi_1(x) \psi_2(x) dx + O \left( e^{-1/Ch} \right)
\]

As the phase is not stationary we shift the integration contour in the complex domain \( x \to x \pm \gamma \). We have \( a(x, \gamma) \) so that, choosing \( \epsilon > 0 \) small enough, the integral is \( O \left( e^{-1/Ch} \right) \). It is clear that all other terms can be evaluated in the same way, so \( <1 \to -2 > = O \left( e^{-1/Ch} \right) \).

ii) \( 1 + |+ 5 | = \int_{-\delta}^{\delta} (\psi_1(x) \psi_2(x) dx) \)....

It is clear, from (a.8) to (a.11), that the integral over \( |1 + \delta| \) can be replaced by the corresponding one over \( |1 - \delta| \). Again by Lemma a.2:

\[
\psi_1(x) \psi_2(x) = a(x, \gamma) a(x, h) + a(x, h) e^{-ix\gamma/\sqrt{2h}} a(x, h) + \]
\[
e^{ix\gamma/\sqrt{2h}} a(x, h) + O \left( e^{-1/Ch} \right), \quad -\delta < x < \delta
\]

and:

\[
\psi_1(x) \psi_2(x) = e^{-x^2/2h} e^{-ix\gamma/\sqrt{2h}} a(x, h) + \]
\[
+ 2e^{-x^2/2h} a(x, h) + e^{-x^2/2h} e^{-ix\gamma/\sqrt{2h}} a(x, h) + \]
\[
+ 2e^{-x^2/2h} a(x, h) + e^{-x^2/2h} e^{-ix\gamma/\sqrt{2h}} a(x, h) + \]
\[
+ e^{ix\gamma/\sqrt{2h}} a(x, h) e^{-ix\gamma/\sqrt{2h}} a(x, h) + \]
\[
+ O \left( e^{-1/Ch} \right), \quad -\delta < x < \delta
\]

All non stationary terms are treated as above and give a contribution \( O \left( e^{-1/Ch} \right) \). We are left with:

\[
\int_{-\delta}^{\delta} e^{-x^2/2h} e^{-ix\gamma/\sqrt{2h}} a(x, h) dx + 2\delta \int_{-\delta}^{\delta} e^{-x^2/2h} a(x, h) a(x, h) dx
\]

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which can be roughly estimated by $O(h^{1/2})$. This would actually suffice for our purpose, but to get $O(h^{3/2})$ we can proceed as follows: by a contour integral:

$$I = \int_0^\infty e^{-t^{1/2}h} e^{iry/h} a^2(x, \eta) dx$$

$$= e^{-y^{1/2}h} \int e^{-t^{1/2}h} a^2(x + \text{i}y, \eta) dx + O(\varepsilon^{-1/C'})$$

Applying analytic stationary phase one can easily show:

$$(2\pi)^{-1/2} \int e^{-x^{1/2}h} a^2(x + \text{i}y, \eta) dx = a(\eta, \eta) + o(\eta, \eta) + O \left( h^{-1/4} \right)$$

On the other hand, we have

$$o(\eta, \eta) = (2\pi h)^{1/2} \left( 1 + o(1) \right) / 2$$

which gives $I = O(h^{3/2})$. The other integral $II = 2h \int_0^\infty e^{-t^{1/2}h} a(x, \eta) a(x, h) dx$ can be estimated in the same way.

4' Proof of Lemma 1.5

It makes no difficulty, once we have established the following elementary Lemmas. The first one gives an expression for $b(x, y)$ in terms of analytic symbols.

Lemma a.3: There exist classical analytic symbols $a^k(x, h) \sim \sum_{k \geq 0} a_k(x) h^k$ defined in a neighborhood of 0 such that:

$$\int e^{-t^{1/2}h} \cos \frac{t}{h} dt = h e^{-t^{1/2}h} e^{\pm \text{i}r/h} a^2(x, h)$$

$$+ h e^{-y^{1/2}h} e^{\pm \text{i}y/h} a^2(y, h) + O(\varepsilon^{-1/C'})$$

(again the same letter denotes several symbols)

Proof: By Cauchy's formula we have, for instance:

$$\int e^{-t^{1/2}h} e^{-i(t^{1/2}h)} dt = -e^{-1/2h} \int e^{-t^{1/2}h} dt - i e^{-1/2h} \int_0^1 e^{-(y+i1)^{1/2}h} dy$$

$$+ i e^{-1/2h} \int e^{-(y+i1)^{1/2}h} dy.$$
where \( L_1(x, h) \) is of the same form as \( L_0(x, h) \) for \( a_1(y, h) = \partial_y \left( \frac{a(y, h)}{\sqrt{1 + y^2}} \right) \). Iterating this procedure \( const./h' \) times (we could do it \( const./h \) times, which would give a smaller remainder, but we do not need it) and using simple induction arguments based on Cauchy inequalities, we are led to (a.14).

\[ \Box \]

**Remark a.5:** Simple variants of this Lemma are also needed, e.g. replacing the integral over \([-\delta, x]\) by an integral over \([x, \delta]\), \( \alpha(y, \eta) \) by \( \alpha(y, \eta) \), etc...


