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A Constraint on the Pressure in Static Stellar Models

TETSUYA SHIROMIZU

*Department of Physics, Kyoto University
Kyoto 606-01, Japan*

ABSTRACT

We show that there exists at least a point such that pressure is positive without using any assumption in general relativistic and static stellar models. This supports an intuitive consideration which one needs positive pressure in order to balance with gravitational force.

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In order to get equilibrium state of stars, one may think intuitively that positive pressure is necessary, because gravitational force is always attractive. If one consider usual matter as components of stars, in Newton gravity positivity of pressure will be guaranteed by asymptotic vacuum condition. However, in general relativity(Einstein gravity) positivity of pressure is not obvious in general. For example, one may consider boson stars^[1] whose components are scalar fields (e.g. axion...). Boson stars are candidate for dark matter and lens objects for microlensing^[2]. Thus, the sign of pressure is worth discussing without assuming symmetry and energy conditions, because one has not shown yet that the shape of static stars is spherical^[3] and scalar fields can have negative pressure in general. In this letter, we treat two cases — perfect fluid and complex scalar fields coupled with electro-magnetic fields—.

In Newton gravity, there are restriction on the state of stars. For example, polytrope index n has the upper limit $n = 5$ from regularity condition and $n < 3$ from stability condition^[4]. On the other hand, people was believing that also in general relativistic frame there exists such restriction, but it was examined only by numerical analysis in asymptotically flat space-times. Recently, Lindblom and Masood-ul-Alam showed it analytically^[5]. Their method is mathematical and positive energy theorem^[6] is used as a powerful tool. We concentrate on asymptotically flat space-times and investigate on the line of their method.

For notation and conventions, we follow Ref.[7].

(1)Perfect fluid.

The metric of static space-times is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -V^2 dt^2 + q_{IJ}dx^I dx^J, \quad (1)$$

where $V^2 := -g(\xi, \xi)$, $I, J = 1, 2, 3$ and $\xi := \partial_t$ is timelike Killing vector such that $(L_\xi q)_{ab} = 0$. q_{ab} is metric of time-symmetric surfaces Σ . Now, we examine cases which do not have Killing horizons (event horizons) and satisfy global

hyperbolicity^[7]. Einstein equations are

$$D^a D_a V = 4\pi(\rho + 3P)V, \quad (2)$$

$$({}^{(3)}R_{ab} = \frac{1}{V} D_a D_b V + 4\pi(\rho - P)q_{ab}, \quad (3)$$

where ρ and P are density and pressure, respectively. D_a is covariant derivative on Σ . Note that one cannot use V as one coordinate of space, because eq. (2) does not guarantee that V is monotonic function without energy condition. Bianchi identity yields Euler equation;

$$D_a P = -\frac{1}{V}(\rho + P)D_a V. \quad (4)$$

Next, we perform conformal transformation for time-symmetric surface Σ ;

$$\tilde{q}_{ab} = \Omega^4 q_{ab}, \quad (5)$$

where

$$\Omega = V^{\frac{1}{4}} \quad (6)$$

Under this transformation, three-manifold $(\tilde{\Sigma}, \tilde{q}_{ab})$ is asymptotically massless space. In fact, as the behavior of three metric q_{IJ} and V near infinity are

$$q_{IJ} = (1 + \frac{2M}{r})\delta_{IJ} + O(\frac{1}{r^2}), \quad (7)$$

$$V = 1 - \frac{M}{r} + O(\frac{1}{r^2}), \quad (8)$$

one gets

$$\tilde{q}_{IJ} = V^2 q_{IJ}$$

$$\begin{aligned} &= (1 - \frac{M}{r})^2 (1 + \frac{2M}{r})\delta_{IJ} + O(\frac{1}{r^2}) \\ &= \delta_{IJ} + O(\frac{1}{r^2}). \end{aligned} \quad (9)$$

Furthermore, three Ricci scalar after the transformation becomes

$$\begin{aligned} ({}^{(3)}\tilde{R}) &= 16\pi\Omega^{-4} \left[\rho - \frac{2}{\Omega} \frac{d\Omega}{dV} (\rho + 3P)V \right] - 8\Omega^{-5} \frac{d^2\Omega}{dV^2} D^a V D_a V \\ &= -48\pi \frac{P}{V^2} + \frac{2}{V^2} D^a V D_a V \end{aligned} \quad (10)$$

If one assume $P \leq 0$ everywhere, inequality $({}^{(3)}\tilde{R}) \geq 0$. Recall corollary of positive energy theorem^[6];

Asymptotically massless hypersurface with non-negative Ricci scalar is three-Euclid space.

This yields $({}^{(3)}\tilde{R}) = 0$ and then

$$P = D_a V = 0. \quad (11)$$

Eq. (2) and (11) give $\rho = 0$. So, from eq. (3) one get $({}^{(3)}R_{ab}) = 0$ and then space-time is Minkowski space. Thus, $P > 0$ must holds at *some point* when stars exist.

If one consider the cases in which strong energy condition^[7] holds, one can show $P > 0$ *everywhere*. From eq. (2) one gets $D^a D_a V \geq 0$ and then maximum principle implies that V does not have maximum points. Furthermore, note that for all points p on Σ there always exist curves which start from p and end at infinity and V is monotonically increasing function along the curve. Then one can show that P is monotonically decreasing function from eq. (4). If there exists a point such that P is negative, P has a non-negative value at infinity. However, this is contradiction with asymptotically flat condition. Thus, P is positive everywhere.

This situation that one could not show positivity of pressure *everywhere* unless one imposes strong energy condition is similar to that one could not explain focusing effects unless one impose strong energy condition or generic condition on Raychaudhuri equation^[7].

(II) Complex scalar fields.

Lagrangian density for matter and energy-momentum tensor are

$$\mathcal{L} := -\frac{1}{2}(\mathcal{D}_a\psi)^*(\mathcal{D}^a\psi) - U(\psi) - \frac{1}{16\pi}F^{ab}F_{ab}. \quad (12)$$

and

$$\begin{aligned} T_{ab} = & \frac{1}{2} \left[(\mathcal{D}_a\psi)^*(\mathcal{D}_b\psi) + (\mathcal{D}_a\psi)(\mathcal{D}_b\psi)^* \right] - \frac{1}{2}g_{ab} \left[(\mathcal{D}^c\psi)^*(\mathcal{D}_c\psi) + 2U(\psi) \right] \\ & + \frac{1}{4\pi} (F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}), \end{aligned} \quad (13)$$

respectively, where $U(\psi)$ is potential, $F_{ab} := \nabla_a A_b - \nabla_b A_a$, $D_a = \nabla_a - ieA_a$, and $A_a = \phi(dt)_a$. Let us assume that $\psi = e^{-i\omega t}\Phi(x^1, x^2, x^3)$ for simplicity. Our results are independent of this assumption. The equation which they satisfy are

$$D^a D_a V = 4\pi(\rho_{eff} + 3P_{eff})V, \quad (14)$$

$$\begin{aligned} {}^{(3)}R_{ab} - \frac{1}{V}D_a D_b V = & 4\pi(D_a\Phi^*D_b\Phi + D_a\Phi D_b\Phi^* + 2U(\Phi)q_{ab}) \\ & - \frac{2}{V^2}D_a\phi D_b\phi + \frac{1}{V^2}q_{ab}D_c\phi D^c\phi, \end{aligned} \quad (15)$$

where ρ_{eff} and P_{eff} are effective density and pressure, respectively, and have

expression

$$\rho_{eff} := T_{ab}q^{ab} = \frac{\xi^a\xi^b}{3} = \frac{(e\phi + \omega)^2}{2V^2}|\Phi|^2 + \frac{1}{2}D^a\Phi^*D_a\Phi + U(\Phi) + \frac{1}{8\pi V^2}D_a\phi D^a\phi, \quad (16)$$

and

$$P_{eff} := \frac{1}{3}T_{ab}q^{ab} = \frac{(e\phi + \omega)^2}{2V^2}|\Phi|^2 - \frac{1}{6}D^a\Phi^*D_a\Phi - U(\Phi) + \frac{1}{24\pi V^2}D_a\phi D^a\phi. \quad (17)$$

Equations for matter are

$$\frac{1}{V}D_a(VD^a\Phi) + \frac{(e\phi + \omega)^2}{V^2}\Phi = 2\frac{\partial U}{\partial\Phi^*}, \quad (18)$$

and

$$D_a\left(\frac{1}{V}D^a\phi\right) = 4\pi e(e\phi + \omega)|\Phi|^2V. \quad (19)$$

Three Ricci scalar after the same transformation as the case (I) becomes

$${}^{(3)}\tilde{R} = -48\pi\frac{P_{eff}}{V^2} + \frac{2}{V^2}D^aV D_aV. \quad (20)$$

Here, note that this expression is independent of the assumption for scalar fields. If one assume $P_{eff} \leq 0$ everywhere, one get ${}^{(3)}\tilde{R} = 0$ by the same discussion of (I) and then

$$P_{eff} = D_aV = 0, \quad (21)$$

that is, $\Phi = 0$. Thus, $P_{eff} > 0$ must holds at *some points*. For scalar fields, $\frac{(e\phi + \omega)^2}{2V^2}|\Phi|^2 + \frac{1}{24\pi V^2}D_a\phi D^a\phi > \frac{1}{6}D^a\Phi^*D_a\Phi + U(\Phi)$. This fact shows that star composed by real scalar fields must not exist. Furthermore, note that one cannot get the same results as the latter half of (I) even if one imposes strong energy condition.

Let us summarize the above discussions. We drew the flow chart (Fig. 1) so that one can understand easily our results. We have examined static stellar asymptotically flat models. We have shown non-existence of stars such that pressure is negative everywhere. There exists at least a point where pressure is positive. As one can see easily, these statements can be apply to all “pressure” defined by $P := \frac{1}{3}T_{ab}q^{ab}$. We have obtained these results without any energy conditions. This means that energy conditions are not very important, although in the ordinary discussion one such as the latter half of (I) one use it. Such nature may appear also in another aspect.

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Figure captions.

The flow chart of our main results, where $P = \frac{1}{3}T_{ab}q^{ab}$.

Fig. 1