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Gravitational waves from a point particle in circular orbit around
a black hole : logarithmic terms in the post-Newtonian expansion

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ABSTRACT

We found logarithmic terms in a post-Newtonian expansion of gravitational radiation induced by a particle traveling circular orbit of radius r_0 around a Schwarzschild black hole of mass M . We calculated gravitational wave luminosity using Teukolsky equation in high accuracy (~ 20 figures) and determined the coefficients of the post-Newtonian expansion by means of least square fitting. We found that there are terms proportional to $x^6 \log x$ and $x^8 \log x$ where $x = (M/r_0)^{1/2}$. We also examine the accumulated phase of coalescing compact star binaries by means of the post-Newtonian expansion as it sweeps through the bandwidth at which the future laser interferometric detectors have good sensitivity.

One of promising sources of gravitational waves for the planned laser interferometer detectors^[1] is the last stage of inspiral of compact binaries. Accurate theoretical models of wave forms are needed to extract physical information from gravitational waves^[2]. Many theoretical efforts have already been done for this purpose^[3-11]. In the case when one of the stars in a binary is much more massive than the other, a linear perturbation theory is available. To investigate the effectiveness of a post-Newtonian method, Cutler et al.^[4] numerically integrated the Teukolsky equation in high accuracy and fitted their numerical data for the gravitational wave luminosity to the post-Newtonian expansion form by means of least square fitting up to the sixth order.

In this paper, we try to extend the post-Newtonian expansion to the eighth orders by using fitting method. To do this it is essential to calculate the gravitational wave luminosity in higher accuracy than Cutler et al.'s work in which numerical integrations were used to obtain a solution of the Teukolsky equation. In this paper we adopt a method which was originally proposed by Leaver^[12] to obtain higher accuracy. In this method, solutions of the Teukolsky equations are represented in analytic manner.

We consider the case when a test particle of mass μ travels circular orbit around a Schwarzschild black hole of mass $M \gg \mu$. Then the specific energy $\tilde{E}$ and the angular momentum $\tilde{L}$ of the particle is given by

$$\tilde{E} = (\tau_0 - 2M) / \sqrt{\tau_0(\tau_0 - 3M)},$$

and

$$\tilde{L} = \tau_0 / \sqrt{\tau_0 - 3},$$

where τ_0 is orbital radius. The angular frequency is given by $\Omega = (M/\tau_0^3)^{1/2}$. To calculate the gravitational wave luminosity we use the Teukolsky equation^{[13][14]}

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$$\left[\Delta^2 \frac{d}{dr} \left(\frac{1}{\Delta} \frac{d}{dr} \right) - U(r) \right] R_{lm\omega} = T_{lm\omega}(r), \quad (1)$$

where

$$U(r) = \frac{r}{r-2} [\omega^2 r^2 - 4i\omega(r-3)] - (\ell-1)(\ell+2), \quad (2)$$

and $\Delta = r(r-2M)$. $T_{lm\omega}$ is given by

$$\begin{aligned} T_{lm\omega}/\pi/\mu = & -2b_{0lm} \delta(r-r_0) \\ & - {}_{-1}b_{lm} 2i\tau_0 [(\tau_0-2)^2 \delta'(r-r_0) - (\tau_0-2)(2-i\omega\tau_0) \delta(r-r_0)] \\ & + {}_{-2}b_{lm} \left[\tau_0^2 (\tau_0-2)^2 \delta''(r-r_0) + \{2i\omega\tau^3(\tau_0-2) - 2\tau_0(3\tau_0^2-8\tau_0+4)\} \delta'(r-r_0) \right. \\ & \left. + \{4\tau_0^2-8-\omega^2\tau_0^4-6i\omega\tau_0^2(\tau_0-1)\} \delta(r-r_0) \right], \end{aligned} \quad (3)$$

where $\delta(r)$ is the usual delta function and $\omega = m\Omega$. ${}_sb_{lm}$ are given by

$$\begin{aligned} {}_0b_{lm} &= \frac{1}{2} [(\ell-1)\ell(\ell+1)(\ell+2)]^{1/2} {}_0Y_{lm} \left(\frac{\pi}{2}, 0 \right) \bar{E}\tau_0/(\tau_0-2), \\ {}_{-1}b_{lm} &= [(\ell-1)(\ell+2)]^{1/2} {}_{-1}Y_{lm} \left(\frac{\pi}{2}, 0 \right) \bar{L}/\tau_0, \\ {}_{-2}b_{lm} &= {}_{-1}Y_{lm} \left(\frac{\pi}{2}, 0 \right) \bar{L}\Omega, \end{aligned}$$

and

where ${}_sY_{lm}$ are the spin weighted spherical harmonics^[14].

We solve Eq. (1) under the outgoing boundary condition at infinity and the ingoing boundary condition at the horizon. Then gravitational wave luminosity is given by

$$\frac{dE}{dt} = \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} |Z_{lm}^{\text{out}}|^2 / 2\pi\omega^2, \quad (4)$$

$$Z_{lm}^{\text{out}} = \frac{1}{2i\omega B_{lm}} \int_{-\infty}^{\infty} dr R_{lm}(r) T_{lm\omega}(r) / \Delta^2, \quad (5)$$

where ℓ and m are the spherical harmonic degree and order. $R_{lm}(r)$ is a solution of the homogeneous Teukolsky equation and has a boundary condition

$$R_{lm}(r) = \begin{cases} \Delta^2 e^{-i\omega r^*} & , r^* \rightarrow -\infty \\ r^3 B_{lm}^{\text{out}} e^{i\omega r^*} + r^{-1} B_{lm}^{\text{in}} e^{-i\omega r^*} & , r^* \rightarrow +\infty, \end{cases} \quad (6)$$

where $r^* = r + 2M \ln(\tau/2M - 1)$. In the case of the circular orbit, $T_{lm\omega}$ contains only the terms which are proportional to a delta function and its derivatives. So Z_{lm}^{out} can be evaluated using $R_{lm}(\tau_0)$ and its derivatives.

It is often convenient to use a Regge-Wheeler (RW) equation^[15] instead of the Teukolsky equation. If we transform $R_{lm\omega}$ as^{[16] [17][18]}

$$R_{lm\omega}(r) = \Delta \left(\frac{d}{dr^*} + i\omega \right) \frac{\tau^2}{\Delta} \left(\frac{d}{dr^*} + i\omega \right) \tau X_{lm}, \quad (7)$$

the homogeneous Teukolsky equation becomes

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V(r) \right] X_{lm\omega}(r) = 0, \quad (8)$$

and

$$V(r) = \left(1 - \frac{2}{r} \right) \left(\frac{\ell(\ell+1)}{r^2} - \frac{6}{r^3} \right).$$

The asymptotic form of X_{lm} , after normalized appropriately, is given by

$$X_{lm}(r) = \begin{cases} e^{-i\omega r^*} & , r^* \rightarrow -\infty \\ A_{lm}^{\text{out}} e^{i\omega r^*} + A_{lm}^{\text{in}} e^{-i\omega r^*} & , r^* \rightarrow +\infty. \end{cases} \quad (9)$$

To solve X_{lm} we put Leaver's procedure into practice. The details of the method are presented in Leaver^[12] so that we present here only an outline. The Teukolsky equation is classified into generalized spheroidal wave equations which can be computed by means of power series expansion around the horizon or a series of Coulomb wave functions.

First, X_{in} can be represented as

$$X_{\text{in}} = \exp(-i\omega r^*) \sum_{n=0}^{\infty} a_n \left(1 - \frac{2}{r}\right)^n. \quad (10)$$

This expansion is a power series expansion in variable $u = 1 - 2/r$ around regular singular point $u = 0$. a_n 's are determined by a three term recurrence relation

$$(n+1-4i\omega)a_{n+1} = \left(2n + \frac{\ell(\ell+1)-3}{n+1}\right)a_n - \frac{n^2-4}{n+1}a_{n-1}, \quad (n \geq 1) \quad (11)$$

$$a_1 = [\ell(\ell+1) - 3]/(1 - 4i\omega) \times a_0.$$

From the boundary condition (9), we must set $a_0 = 1$.

For convergence of (10), we need summation only up to $n \sim 10^5$ for $r_0 < 10^4 M$. As R_{in} is expressed by X_{in} and their derivatives, it also converges for $r_0 < 10^4 M$. Thus $Z_{\ell m}^{\text{out}}$ can be calculated almost analytically for $r_0 \lesssim 10^4 M$ except for the term B_{in} . To obtain B_{in} , we need to determine A_{in} . However since $r = \infty$ is an irregular singular point, a power series expansion with variable r^{-1} converges only asymptotically. So it is hard to obtain an accurate value of A_{in} using (10). To overcome this, we expand X_{in} using a series of Coulomb wave functions discussed by Leaver [12]. That is

$$X_{\text{in}} = \left(A_{\text{out}} \sum_{L=-\infty}^{\infty} b_L h_{L+\nu}^{(+)}(z) + A_{\text{in}} \sum_{L=-\infty}^{\infty} b_L h_{L+\nu}^{(-)}(z) \right) \left(\frac{r}{r-2} \right)^{2i\omega} \quad (12)$$

where $h(\pm)$ are a certain kind of Coulomb wave functions. In Eq. (12), ν is real and is determined so that a series of Coulomb wave functions converges. We will show the details elsewhere [19].

Numerical results of dE/dt are shown in Table 1. These data contain multipoles from $\ell = 2$ to $\ell = 6$. All numerical calculations are done with precision

of 32 figures. Checking the convergence, we confirmed that the accuracy of the summation in Eqs. (10) (12) and (5) has at least 20 figures. We compared our numerical data with those by Cutler et al. [4] and found that their data agree with ours up to 8 $\sim$ 9 figures which seems to be consistent with their error estimates.

Using these data, we perform the post-Newtonian expansion. First we assume that dE/dt can be expanded in powers of x ($= (M/r_0)^{1/2}$), that is, we assume that

$$\frac{dE}{dt} = \left(\frac{dE}{dt} \right)_N \sum_{k=0}^{\infty} a_k x^k, \quad (13)$$

where $(dE/dt)_N$ is derived from a quadrupole formula,

$$\left(\frac{dE}{dt} \right)_N = \frac{32 \mu^2 M^3}{5 r_0^5}. \quad (14)$$

First four terms are known analytically [3] [21],

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -1247/336, \quad a_3 = 4\pi. \quad (15)$$

The next three terms are known approximately [4],

$$a_4 \simeq 4.89(2\%), \quad a_5 \simeq -38(10\%), \quad a_6 \simeq 135(50\%), \quad (16)$$

where numbers in the parentheses denote the error. We calculated multipoles from $\ell = 2$ to $\ell = 6$. Then we can determine the coefficients up to the order x^9 because $\ell = 7$ multipole contributes from order x^{10} .

We determine the coefficients a_n by means of least square fitting implemented with the method of singular value decomposition [20]. We used 120 data of dE/dt in the fitting. The results are shown in Table 2(a). Using these results we plot

in Fig.1 the residual ζ_n of dE/dt which is defined by

$$\frac{dE}{dt} \times \zeta_n = \frac{dE}{dt} - \left(\frac{dE}{dt} \right)_N \sum_{k=0}^n a_k x^k. \quad (17)$$

We see that, at $n = 6$, $|\zeta_n|$ shows a strange behavior. In general, it is argued that the post-Newtonian expansion may contain terms proportional to $x^k (\log x)^p$ (k, p : non-negative integer) [3][4][22]. To take into account this, we add $b_6 x^6 \log x$ term to the model functions. That is

$$\frac{dE}{dt} = \left(\frac{dE}{dt} \right)_N \left(\sum_{k=0}^n a_k x^k + b_6 x^6 \log x \right). \quad (18)$$

The result is shown in Table 2(b). It is clear that accuracy of the determination of first four values are improved about three orders. The residual of dE/dt including $x^6 \log x$ term also becomes better. From the value in Table 2(b), we can understand the behavior of $|\zeta_6|$ in Fig.1. Since $b_6 x^6 \log x + a_7 x^7$ becomes zero near $r \sim 20$, ζ_6 change sign near this point. All of these facts are consistent with the existence of the term proportional to $x^6 \log x$ in the post-Newtonian expansion. From the analysis of the data dE/dt of each modes (ℓ, m) , we find that this term comes from $\ell = m = 2$ mode. We also tried model functions with $x^6 (\log x)^n$ ($n \geq 2$) and found that the coefficients of these terms are less than 10^{-2} . This fact suggests that the terms proportional to $x^6 (\log x)^n$ ($n \geq 2$) do not exist.

In a similar way, we examine the coefficients at the order x^7 and x^8 . We find that there seems to exist no $x^7 (\log x)^n$ ($n \geq 1$) terms but there is $b_8 x^8 \log x$ term. This term comes from both $\ell = m = 2$ and $\ell = m = 3$ modes. Fitting results are show in Table 2(c). We see that accuracy of the determination of first four values are improved about two orders compared with Table 2(b).

We also performed the fitting to the data in which first four terms are sub-

tracted analytically and found that the post-Newtonian expansion has the form

$$\begin{aligned} \frac{dE}{dt} = & \left(\frac{dE}{dt} \right)_N \left(1 - \frac{1247}{336} x^2 + 4\pi x^3 - 4.9284611990 x^4 - 38.292835 x^5 \right. \\ & \left. + 115.730 x^6 - 16.3049 x^6 \log x - 101.450 x^7 - 93 x^8 + 57 x^8 \log x \right). \end{aligned} \quad (19)$$

To investigate the accuracy of these coefficients, we compare these results to the value in Table 2(c) and count the number of figures at which the value are not changed. We adopt these numbers as significant figures of the coefficients and the accuracy of these values are estimated as follows; a_4 : $10^{-5}\%$, a_5 : $10^{-3}\%$, a_6 : 0.1% , b_6 : 0.1% , a_7 : 1% , a_8 : 30% , b_8 : 20% .

Using these data and log terms, we plot in Fig. 2 the residual ζ'_n which is defined by

$$\frac{dE}{dt} \times \zeta'_n = \frac{dE}{dt} - \left(\frac{dE}{dt} \right)_N \sum_{k=0}^n (a_k x^k + b_k x^k \log x), \quad (20)$$

where $b_k \neq 0$ for $k = 6$ and 8 . We see all ζ'_n except ζ'_7 show power law behavior. ζ'_7 changes sign around $r_0 \sim 15M$. This may suggest that the coefficients of x^9 may be very large (about 650). However for $n \geq 9$, more detailed analysis is needed.

Finally, we estimate the accuracy of the post-Newtonian expansion to construct the template wave form from an inspiraling compact binary. Consider that a binary starts to inspiral from orbital separation r_0 and that there are two template waveforms. One is constructed using correct $\dot{E}$ and the other is constructed using an approximate form for $\dot{E}$ (dot denote a time derivative). Then at the end of the inspiraling, accumulated phase difference $\Delta\Phi$ between two template waveforms is given by [4]

$$\Delta\Phi/\Phi \sim \Delta\dot{E}(r_0)/\dot{E}(r_0), \quad (21)$$

where Φ is the total phase and right hand side refers to the relative error in $\dot{E}$.

The planned ground based laser interferometer detectors^[1] will observe inspiring compact binaries in wave frequency $f \sim 10\text{Hz}$ to 1000Hz . In the case of a $1.4M_{\odot}$ neutron star binary, when the wave frequency is 10 Hz , r_0 is about 723km ($175M$) and the total accumulated phase from 10 Hz to 10^3 Hz is $\Phi \simeq 10^5$. We assume that the expansions for non-negligible mass μ/M will be subject to the same convergence problems as for negligible μ/M . Then from figure 2, we can estimate the accumulated phase error $(\Delta\Phi)_n$ when we include the post-Newtonian expansion of order x^n , $(\Delta\Phi)_3 \sim 20$, $(\Delta\Phi)_4 \sim 7$, $(\Delta\Phi)_5 \sim 3$, $(\Delta\Phi)_6 \sim 0.2$, $(\Delta\Phi)_7 \sim 0.02$. This suggests that if we construct the template wave form which includes the post-Newtonian effect up to $n = 7$, we will obtain accuracy $\sim 3 \times 10^{-3}$ cycle after $\sim 1.5 \times 10^4$ cycles. Note that $b_6 x^6 \log x$ term contributes to $\Delta\dot{E}(r_0)/\dot{E}(r_0)$ one third of $a_6 x^6$ term at $r_0 \simeq 175M$. This means the importance of $b_6 x^6 \log x$ term in constructing the template wave form.

The details of present rapid communication will be shown elsewhere^[19].

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REFERENCES

1. R. E. Vogt, in *Proceeding of the Sixth Marcel Grossmann Meeting on General Relativity*, eds. H. Sato and T. Nakamura (World Scientific, Singapore, 1992), p244. A. Abramovici *et al.*, Science **256**, 325 (1992).
2. C. Cutler *et al.*, Phys. Rev. Lett. **70**, 2984 (1993a).
3. E. Poisson, Phys. Rev. D**47**, 1497 (1993).
4. C. Cutler, L.S. Finn, E. Poisson and G.J. Sussman, Phys. Rev. D**47**, 1511 (1993b).
5. C.W. Lincoln and C.M. Will, Phys. Rev. D**42** 1123 (1990).
6. A.G. Wiseman, Phys. Rev. D**46** 1517 (1992).
7. L.E. Kidder, C.M. Will and A.G. Wiseman, Class. Quantum. Grav. **9** L125 (1992).
8. L. E. Kidder, C. M. Will and A. G. Wiseman, Phys. Rev. D**47**, R4183 (1993).
9. T. Tanaka, M. Shibata, M. Sasaki, H. Tagoshi and T. Nakamura, Prog. Theor. Phys. **90**, 65 (1993).
10. M. Shibata, Phys. Rev. D**48**, 663 (1993).
11. M. Shibata, Prog. Theor. Phys. **90**, 595 (1993).
12. E. W. Leaver, J. Math. Phys. **27**, 1238 (1986).
13. S. A. Teukolsky, Astrophys. J. **185**, 635 (1973).
14. R. A. Breuer, *Gravitational Perturbation Theory and Synchrotron Radiation*, Lecture Notes in Physics **44**, (1975), (Springer Verlag).
15. T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).
16. S. Chandrasekhar, Proc. R. Soc. London **A343**, 289 (1975).
17. M. Sasaki and T. Nakamura, Phys. Lett. **87A**, 85 (1981).

18. T. Nakamura, K. Oohara and Y. Kojima, Prog. Theor. Phys. Supple. **90** (1987).
19. H. Tagoshi and T. Nakamura, to be submitted to Prog. Theor. Phys.
20. W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, *Numerical Recipes*, (Cambridge University Press).
21. R. V. Wagoner and C. M. Will, Astrophys. J. **210**, 764 (1976).
22. L. Blanchet and T. Damour, Phil. Trans. R. Soc. London **A320** 379 (1986), J. L. Anderson, R. E. Kates, L. S. Kegeles and R. G. Madonna, Phys. Rev. **D25**, 2038 (1982), T. Futamase, Phys. Rev. **D28**, 2373 (1983).

Table 1 Numerical value of dE/dt including multipoles up to $\ell = 6$. This table is not intended to indicate the accuracy of these value are over 30 figures. However accuracy is at least 20 figures.

Table 2 The results of the least square fitting. The model functions of (a) are power in x . (b) contains $x^6 \log x$ term and (c) contains both $x^6 \log x$ and $x^8 \log x$ terms.

Figure 1 The plot of the residual $|\zeta_n|$ of dE/dt using the coefficients in Table 2(a).

Figure 2 The plot of the residual $|\zeta'_n|$ of dE/dt using the coefficients in Eq.(19).

Table 1

r_0/M	$(M/\mu)^2 dE/dt$
100.0000	6.23820398978509412124176192704334E-10
300.0000	2.607333426816527765774956086335520E-12
400.0000	6.2015746132938382603167325567270E-13
500.0000	2.035048609173431022305459014285948E-13
600.0000	8.186438196960486600132340003588864E-14
700.0000	3.790284023911351535919547135042982E-14
800.0000	1.9451303639855005289216165030152E-14
900.0000	1.07987258129729460059199934934023E-14
1000.0000	6.37875266047998631545677531270354E-15
1200.0000	2.564828833240345107029159654956498E-15
1400.0000	1.187107733351135240943577415669260E-15
1600.0000	6.090542750440027869703840948937458E-16
1800.0000	3.380585421529206602594378236237482E-16
2000.0000	1.996566834529030273039578847096921E-16
2200.0000	1.239897486733535981491638714136889E-16
2400.0000	8.02597352148203988440516887789591E-17
2600.0000	5.379398644465723875822126316487929E-17
2800.0000	3.714072389983239585415708907420717E-17
3000.0000	2.630886432040147798074361413233360E-17
3200.0000	1.905267894555203290888243178670E-17
3400.0000	1.407143600435670282337691474289615E-17
3600.0000	1.057412945896035827263146787254939E-17
3800.0000	8.069762995913967648059791767590368E-18
4000.0000	6.244509390803978517372128153303494E-18
4200.0000	4.892935859814708021886556563356111E-18
4400.0000	3.877650922832691958929296005872690E-18

Table 2 (a)

n	a_n
0	0.99999999999956737032
1	2.5540107268919450071E-10
2	-3.7113095967507652120
3	12.566384350926089206
4	-4.9204816403237679929
5	-37.999982610332799195
6	179.273234265923125902
7	-817.51080139077674299

Table 2 (b)

n	a_n
0	0.9999999999999971856
1	1.926112188029453128E-13
2	-3.7113095238748910921
3	12.5663706290971842005
4	-4.9284639028225238633
5	-38.292319775967105056
6	115.913907630116282263
$x^6 \ln x$	-16.282941934716851792
	-104.97250734572709616
	-259.25092983916886042
8	

Table 2 (c)

n	a_n
0	1.000000000000000000053030
1	-4.115960526368818532705E-15
2	-3.71130952380792330949
3	12.566370613929664016796
4	-4.928461103278450962933
5	-38.29285880105704321767
6	115.7194712671279087861
$x^6 \ln x$	-16.30735495755162544055
	-101.1791705803646726845
	-72.60876280963051767049
8	64.2185303965965494097
$x^8 \ln x$	

Fig. 1

Fig. 2