HHH

LFTB 93.23 rev

I Introduction

see 8404

THE SUPERALGEBRA EMBEDDING OF

$OSP(1|2)$ IN $SL(1|2)$ AND

THE GRADE STAR REPRESENTATIONS OF $SL(1|2)$

Michel DAUMENS

Laboratoire de Physique Théorique*, Université de Bordeaux I, France**

Abstract

First, it is shown that the superalgebra embedding of $osp(1|2)$ in $sl(1|2)$ allows to characterize the irreducible representation states of $sl(1|2)$ without any missing label. Through this embedding the general non positive definite scalar product in the $osp(1|2)$ representation space can be generalized to the $sl(1|2)$ representation space. Then grade star representations of $sl(1|2)$ are derived by considering this non positive scalar product.

PACS.02.20 - Group Theory.
PACS.11.30P - Supersymmetry.

* Unité associée au CNRS 764
** Postal address: Rue du Solarium, 33175 Gradignan CEDEX
LFTB-93-23 Revised Version

December 1993

The concept of grade star representation for superalgebras was introduced by Scheunert, Nahm and Rittenberg [1]. As an example [2], by means of the sub-algebra $gl(1) \oplus sl(2)$, these authors derived the irreducible representations of $sl(1|2)$ algebra (also denoted by $spi(2,1)$, and by $A(0,1)$ in the Kac's notations). They are characterized by two numbers (p, q) , $p \in \mathbb{R}$, $2q \in \mathbb{N}$, and by a parity λ taking the values 0 or 1. The general representations occur for $|p| \neq q$ and $q > 1/2$, and their dimension is $8q$; they contain four multiplets of states $(-1)^\lambda$ and $(b = p \pm 1/2, l = q - 1/2)$ with Grassman degree $(-1)^{q+l}$. The particular cases $(q = 1/2)$ and $|p| = q$ are recalled in Section III. Besides, by using always *positive definite* scalar product in the representation space, Scheunert et al. derived the star representations of $sl(1|2)$ and the grade star ones; they found two classes $S^\pm$ for $\pm p > q$ of star representations and one class of grade star ones for $q = 1/2$ and $|p| < 1/2$ only. They concluded that considering grade star representations of $sl(1|2)$ is almost useless.

However, in order to develop Racah-Wigner calculus for the super-rotation algebra (or $osp(1|2)$, also denoted by $gsu(2)$) [3, 4], Minnaert and Mozrymas [5] showed that the states must be only pseudo-orthonormalized, in order to the tensor product of representations can be reduced by pseudo-orthogonal super-rotation Clebsch-Gordan coefficients (for a review of Racah-Wigner calculus see Ref.[6]). Let us recall that an irreducible representation of $osp(1|2)$, characterized by the superspin j , $(2j \in \mathbb{N})$ and the parity λ , contains two rotation multiplets with spins $l = j$ and $l = j - 1/2$. Then the general form of the scalar product can be written: $[5] < (j, \lambda) | m_l(j, \lambda) | l' m' > = (-1)^{k+l+\psi} \delta_{ll'} \delta_{mm'}$, where $k = 2(j - l) \pmod{2}$ and the two binary variables ϕ, ψ only depend on the representation.

In this article, first the embedding of $osp(1|2)$ in $sl(1|2)$ is presented. Next it is shown that this embedding allows to characterize the representation states of $sl(1|2)$ through a two variable problem, without any missing label. The corresponding basis states are denoted by: $(|p, q, \lambda)(j), l, m >$ and they have the Grassman degree $\lambda + 2(q - l) \pmod{2}$. A general representation (p, q) of $sl(1|2)$ contains two multiplets of $osp(1|2)$:

$$(j = q, l = q \text{ or } q - 1/2) \text{ and } (j = q - 1/2, l = q - 1/2 \text{ or } q - 1).$$

Then, the above scalar product in $osp(1|2)$ representation space, is generalized to the $sl(1|2)$ representation space, as:

$$< (p, q, \lambda)(j) | m_l(p, q, \lambda)(j') | l' m' > = (-1)^{k(\phi+\psi)+\psi} \delta_{jj'} \delta_{ll'} \delta_{mm'},$$

where $h = 2(q - j)$, $k = 2(j - l)$, $(\text{mod } 2)$ and as previously, the two binary variables ϕ, ψ only depend on the representation, i.e. on the numbers p, q and λ .

Finally, the grade star representations of $sl(1|2)$ are derived by using this *non positive definite* scalar product. The result is the following: in the general case $(\pm p \neq q, q > 1/2)$, there are three classes of grade star representations

$$G^\pm, \text{ for } \pm p > q; \quad G^0, \text{ for } |p| < q.$$

The classes $G^\pm$ are equivalent to $S^\pm$ for star representations defined in [2] while G^0 generalizes the grade star representation class defined in [2] for positive definite scalar product.

The organization of this article is as follows. In Section II, the definition of the superalgebra $sl(1|2)$ is given in a form which explicits both sub-algebras $gl(1) \oplus sl(2)$ and $osp(1|2)$. The characterization of the representation states by means of $gl(1) \oplus sl(2)$ [2] is recalled in Section III. Section IV is devoted to the characterization of the states by means of $osp(1|2)$. Finally, the grade star representations of $sl(1|2)$ are derived in Section V.

II The superalgebra $sl(1|2)$ and its representations

From the $sl(1|2)$ generators $\{\tilde{Q}, B, V_\alpha, W_\alpha\}$, ($\alpha = \pm 1/2$), used by Scheunert et al. [2], one defines the new generators $\{\tilde{J}, J_\alpha, B_0, B_\alpha\}$ according as

$$\tilde{J} = \tilde{Q}, \quad J_\alpha = \frac{1}{2}(V_\alpha + W_\alpha), \quad (2.1)$$

$$B_0 = B, \quad B_\alpha = -\frac{1}{2}(V_\alpha - W_\alpha). \quad (2.2)$$

These generators satisfy the (anti)commutation rules

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [\tilde{J}, J_\alpha] = \frac{1}{2}J_\beta(\tilde{\sigma})^\beta_\alpha, \quad \{J_\alpha, J_\beta\} = \frac{1}{2}(\Gamma\tilde{\sigma})_{\alpha\beta} \cdot \tilde{J}, \quad (2.3)$$

$$[\tilde{J}, B_0] = 0, \quad [J_\alpha, B_0] = \frac{1}{2}B_\alpha, \quad [\tilde{J}, B_\alpha] = \frac{1}{2}B_\beta(\tilde{\sigma})^\beta_\alpha, \quad \{J_\alpha, B_\beta\} = \frac{1}{2}\Gamma_{\alpha\beta}B_0, \quad (2.4)$$

$$[B_0, B_\alpha] = -\frac{1}{2}J_\alpha, \quad \{B_\alpha, B_\beta\} = -\frac{1}{2}(\Gamma\tilde{\sigma})_{\alpha\beta} \cdot \tilde{J}, \quad (2.5)$$

where $i, j, k = 1, 2, 3$; $\alpha, \beta = \pm \frac{1}{2}$; $\tilde{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are Pauli matrices, and $\Gamma = i\sigma^2$.

Equations (2.3) gather the (anti)commutation relations of the $osp(1|2)$ algebra (super-rotation), and therefore express the embedding of $osp(1|2)$ in $sl(1|2)$. The five generators $(\tilde{J}, J_\alpha)$ constitute a vector for super-rotations, i.e. a representation of super-spin 1, with parity 0; the J generators are called the super-spin operators and the square of J is the Casimir of $osp(1|2)$

$$J^2 = \tilde{J}^2 + \Gamma_{\alpha\beta}J_\alpha J_\beta. \quad (2.6)$$

Equations (2.4) mean that the three generators (B_0, B_α) form a spinor for super-rotation, i.e. a representation of super-spin 1/2, with parity 0. The square of this spinor is defined as

$$B^2 = B_0^2 + \Gamma_{\alpha\beta}B_\alpha B_\beta. \quad (2.7)$$

The set $(B_0, \tilde{J})$ generates the sub-algebra $gl(1) \oplus sl(2)$, used in Ref [2] to build the irreducible representations of $sl(1|2)$. The Casimirs of $sl(1|2)$ are

$$C_2 = B^2 - J^2, \quad (2.8)$$

$$C_3 = B_0(2C_2 + \tilde{J}^2 - B_0^2 - 1) - \tilde{J} \cdot (\Gamma\tilde{\sigma})_{\alpha\beta}J_\alpha B_\beta, \quad (2.9)$$

they have opposite sign to those defined in [2].

In a grade star representation, the generators satisfy the relations

$$\tilde{J}^\dagger = \tilde{J}, \quad B_0^\dagger = B_0, \quad (2.10)$$

$$J_\alpha^\dagger = (-1)^\epsilon \Gamma_{\alpha\beta}J_\beta, \quad B_\alpha^\dagger = -(-1)^\epsilon \Gamma_{\alpha\beta}B_\beta, \quad (2.11)$$

where the number ϵ , that takes the values 0 and 1, characterizes two classes of such representations. In the following, the generators $(B_0, \tilde{J})$ will always be Hermitean.

The irreducible representations of $sl(1|2)$ are characterized by two numbers (p, q) . $p \in \mathbb{R}$, $2q \in \mathbb{N}$ and a parity λ taking the values 0 or 1. Let us denote by V_0 and V_1 the even and odd parts of the representation space. Then, there are two families of representations [2]

i) *the typical representations* for ($\pm p \neq q$). They can be characterized by the eigenvalues of the Casimirs

$$C_2 = p^2 - q^2, \quad C_3 = p(p^2 - q^2), \quad (2.12)$$

and their dimension is $8q$ with $\dim V_0 = \dim V_1 = 4q$.

ii) *the atypical representations* for ($\pm p = q$). They cannot be characterized by the Casimirs which vanish and their dimension is $4q+1$ with $\dim V_\lambda = 2q+1$, $\dim V_{\lambda+1} = 2q$, i.e. they are such that $\dim V_0 \neq \dim V_1$.

A complete description of these representations and of their basis states is given in the following section.

III Characterization of the states by means of

$$gl(1) \oplus sl(2)$$

In this section, the results of Ref.[2] are expressed in terms of the (J, B) generators. The irreducible representation states, denoted by $|(p, q, \lambda)bm\rangle$, are eigenstates of the commuting operator set $\{C_2, C_3, B_0, \tilde{J}^2, J_3\}$ corresponding to the eigenvalues: $\{0, 0, b, l(l+1), m\}$ for atypical representations or $\{p^2 - q^2, p(p^2 - q^2), b, l(l+1), m\}$ for typical ones. The atypical representations are

- $(0, 0, \lambda)$ the trivial representation contains the state $|(0, 0, \lambda)000\rangle$ with parity λ , and such that

$$J_{\pm 1/2}|(0, 0, \lambda)000\rangle = B_{\pm 1/2}|(0, 0, \lambda)000\rangle = 0. \quad (3.1)$$

- $(\pm q, q > 0, \lambda)$ that contains two multiplets: $[(\pm q, q, \lambda) \pm qm]$ with parity λ and $[(\pm q, q, \lambda) \pm q \pm 1/2, q - 1/2, m]$ with parity $\lambda + 1$.

The typical representations are

- $(p, 1/2, \lambda)$ that contains three multiplets: $[(p, 1/2, \lambda)p, 1/2, m]$ with parity λ and $[(p, 1/2, \lambda)p \pm 1/2, 0, 0]$ with parity $\lambda + 1$.
- $(p, q > 1/2, \lambda)$ that contains four multiplets: $[(p, q, \lambda)pqm]$, $[(p, q, \lambda)pq - lm]$ with parity λ and $[(p, q, \lambda)p \pm 1/2, q - 1/2, m]$ with parity $\lambda + 1$.

In the following, one assumes that $q > 0$ and when there is no ambiguity, the notation of the states will be abbreviated in the form $|bm\rangle$.

These representations can be written in terms of three non vanishing real numbers (α, β, δ) that appear in the actions of the generators $J_{\pm 1/2}, B_{\pm 1/2}$ on the basis states [2]: these numbers will be determined by the normalizations of the states. In the general case ($\pm p \neq q$, $q > 1/2$), the actions of the latter generators can be written

$$\begin{aligned} J_{\pm 1/2}|p + 1/2, q - 1/2, m\rangle &= B_{\pm 1/2}|p + 1/2, q - 1/2, m\rangle = \frac{1}{2\alpha} \\ &\times \left[\frac{p+q}{2q} \sqrt{q \pm m + 1/2} |pq, m \pm 1/2\rangle \pm \alpha\delta \sqrt{q \mp m - 1/2} |p, q - 1, m \pm 1/2\rangle \right]. \end{aligned} \quad (3.2)$$

$$\begin{aligned} J_{\pm 1/2}|p - 1/2, q - 1/2, m\rangle &= -B_{\pm 1/2}|p + 1/2, q - 1/2, m\rangle = -\frac{1}{2\beta} \\ &\times \left[\frac{p-q}{2q} \sqrt{q \pm m + 1/2} |pq, m \pm 1/2\rangle \pm \alpha\delta \sqrt{q \mp m - 1/2} |p, q - 1, m \pm 1/2\rangle \right]. \end{aligned} \quad (3.3)$$

$$\begin{aligned} J_{\pm 1/2}|pqm\rangle &= \pm \frac{1}{2} \sqrt{q \mp m} [\alpha |p + 1/2, q - 1/2, m \pm 1/2\rangle \\ &+ \beta |p - 1/2, q - 1/2, m \pm 1/2\rangle], \end{aligned} \quad (3.4)$$

$$\begin{aligned} B_{\pm 1/2}|pqm\rangle &= \mp \frac{1}{2} \sqrt{q \mp m} [\alpha |p + 1/2, q - 1/2, m \pm 1/2\rangle \\ &- \beta |p - 1/2, q - 1/2, m \pm 1/2\rangle], \end{aligned} \quad (3.5)$$

$$J_{\pm 1/2}|p, q - 1, m\rangle = -\frac{1}{2\delta} \sqrt{q \pm m} \left[\frac{p-q}{2q} |p + 1/2, q - 1/2, m \pm 1/2\rangle \right]$$

IV Characterization of the states by means of $osp(1|2)$

Instead of the inclusion $sl(1|2) \supset gl(1) \oplus sl(2)$, we now use the embedding $sl(1|2) \supset osp(1|2)$ to characterize the representation states. These states, denoted by $|(p, q, \lambda)(j)lm\rangle$, are eigenstates of the commuting operator set $\{C_2, C_3, J^2, J^2, J_3\}$ corresponding to the eigenvalues: $\{0, 0, j(j+1/2), l(l+1), m\}$ for atypical representations or $\{p^2 - q^2, p(p^2 - q^2), j(j+1/2), l(l+1), m\}$ for typical ones. In order to prove that such a characterization is possible, let us consider the action of the operator J^2 on the states $|lm\rangle$ of the previous section, in the general case ($q > 1/2$, $|p| \neq q$). By using Eqs. (2.6) and (3.2,3,4,6), one gets

$$J^2|pqm\rangle = q(q + \frac{1}{2})|pqm\rangle, \quad (4.1)$$

$$J^2|p, q-1, m\rangle = q(q - \frac{1}{2})|p, q-1, m\rangle, \quad (4.2)$$

$$J^2|p + 1/2, q - 1/2, m\rangle = (q^2 + \frac{1}{2}p)|p + 1/2, q - 1/2, m\rangle + \frac{\beta(q+p)}{2\alpha}|p - 1/2, q - 1/2, m\rangle, \quad (4.3)$$

$$J^2|p - 1/2, q - 1/2, m\rangle = \frac{\alpha(q-p)}{2\beta}|p + 1/2, q - 1/2, m\rangle + (q^2 - \frac{1}{2}p)|p - 1/2, q - 1/2, m\rangle. \quad (4.4)$$

The first two states are eigenstates of J^2 corresponding to the super-spin $j = q$ and $j = q - 1/2$, while both latter equations yield a two dimensional eigenvalues problem. The eigenvalues are again q and $q - 1/2$, so, one defines the $osp(1|2)$ basis states as

$$|(p, q, \lambda)(q)q, m\rangle = |(p, q, \lambda)pqm\rangle, \quad (4.5)$$

$$|(p, q, \lambda)(q)q - 1/2, m\rangle = -[\alpha|(p, q, \lambda)p + 1/2, q - 1/2, m\rangle + \beta|(p, q, \lambda)p - 1/2, q - 1/2, m\rangle], \quad (4.6)$$

$$|(p, q, \lambda)(q - 1/2)q - 1/2, m\rangle = \frac{1}{2q\alpha\delta}[(p - q)\alpha|(p, q, \lambda)p + 1/2, q - 1/2, m\rangle + (p + q)\beta|(p, q, \lambda)p - 1/2, q - 1/2, m\rangle], \quad (4.7)$$

$$|(p, q, \lambda)(q - 1/2)q - 1, m\rangle = |(p, q, \lambda)p, q - 1, qm\rangle, \quad (4.8)$$

where the eigenstates are built so that they form a standard $osp(1|2)$ basis [3]

$$J_{\pm 1/2}|(p, q, \lambda)(j)jm\rangle = \mp \frac{1}{2}\sqrt{j \mp m}|(p, q, \lambda)(j)j - 1/2, m \pm 1/2\rangle \quad (4.9)$$

$$J_{\pm 1/2}|(p, q, \lambda)(j)j - 1/2, m\rangle = -\frac{1}{2}\sqrt{j \pm m + 1/2}|(p, q, \lambda)(j)j, m \pm 1/2\rangle. \quad (4.10)$$

The one-to-one correspondence between both state bases guarantees that there is no missing label in the $sl(1|2) \supset osp(1|2)$ embedding. The $osp(1|2)$ basis states constitute two multiplets of $osp(1|2)$: one with $j = q$ and parity λ , the other one with $j = q - 1/2$ and parity $\lambda + 1$. Hence, the j -multiplet has parity $\lambda + 2(q - j)$ while the states $|(p, q, \lambda)(j)lm\rangle$ have the Grassman degree $\lambda + 2(q - l)$.

$$+ \frac{(p+q)\beta}{2q\alpha}|p - 1/2, q - 1/2, m \pm 1/2\rangle, \quad (3.6)$$

$$B_{\pm 1/2}|p, q - 1, m\rangle = \frac{1}{2\delta}\sqrt{q \pm m}\left[\frac{p-q}{2q}|p + 1/2, q - 1/2, m \pm 1/2\rangle - \frac{(p+q)\beta}{2q\alpha}|p - 1/2, q - 1/2, m \pm 1/2\rangle\right]. \quad (3.7)$$

Assuming that the states are orthogonal, the constants (α, β, δ) are related to the squares of the states

$$g = \langle pqm|pqm\rangle, \quad (3.8)$$

$$g_{\pm} = \langle p \pm 1/2, q - 1/2, m|p \pm 1/2, q - 1/2, m\rangle, \quad (3.9)$$

$$g_1 = \langle pq - 1|m|pq - 1m\rangle, \quad (3.10)$$

for a grade star representation, as follows ($\phi = \lambda + \epsilon + 1$)

$$g_+ = (-1)^{\phi}\frac{q+p}{2q\alpha^2}g, \quad (3.11)$$

$$g_- = (-1)^{\phi}\frac{q-p}{2q\beta^2}g, \quad (3.12)$$

$$g_1 = \frac{p^2 - q^2}{4q^2\alpha^2\delta^2}g. \quad (3.13)$$

Finally, let us consider the particular cases

- for $(q = 1/2, |p| \neq 1/2)$, Eqs.(3.6,7,13) are suppressed and $\delta = 0$.
- for $p = q$, Eqs.(3.3,6,7,12,13) are suppressed and $\beta = \delta = 0$.
- for $p = -q$, Eqs. (3.2,6,7,11,13) are suppressed and $\alpha = \delta = 0$.

By using spherical components J_m ($i = 1/2, 1$) of the J generators

$$J_{i0} = J_z, \quad J_{i\pm 1} = \mp \frac{1}{\sqrt{2}}(J_i \pm iJ_2), \quad J_{1/2,\pm 1/2} = J_{\pm 1/2}, \quad (4.11)$$

the action of the super-spin on the states can be written by means of the Clebsch-Gordan coefficients of $osp(1|2)$ [2, 4, 6], as

$$J_m |p, q, \lambda\rangle_{jm} = -\sqrt{j(j+1/2)} \sum_{i=j-1/2}^j (1i; jm|j^i m + n) |p, q, \lambda\rangle_{j^i m + n}, \quad (4.12)$$

where the condition $i + l + l'$ integer implies that there is only one term in the sum. In a similar way, the action of the B generators in spherical components B_m ($i = 0, 1/2$)

$$B_{00} = B_0, \quad B_{1/2,\pm 1/2} = B_{\pm 1/2}, \quad (4.13)$$

can be written by means of the Clebsch-Gordan coefficients of $osp(1|2)$, as

$$B_m |p, q, \lambda\rangle_{jm} = \sum_{j'=q-1/2}^q \sum_{l'=j'-1/2}^{j'} (j' \| B \| j) (1/2, i; j' m | j^i m + n) |p, q, \lambda\rangle_{j^i m + n}, \quad (4.14)$$

where the reduced matrix elements $(j' \| B \| j)$ are

$$(q \| B \| q) = -p \sqrt{\frac{2q+1}{2q}}, \quad (q-1/2 \| B \| q-1/2) = p \sqrt{\frac{2q-1}{2q}}, \quad (4.15)$$

$$(q-1/2 \| B \| q) = \sqrt{2q}\alpha\delta, \quad (q \| B \| q-1/2) = \frac{p^2 - q^2}{2q} \frac{1}{\sqrt{2q}\alpha\delta}. \quad (4.16)$$

Let us note that the condition $i + l + l'$ integer implies that there is only two terms in the double sum (4.14).

For the particular representation cases, the $osp(1|2)$ state bases can be defined as

- $(0, 0, \lambda)$ one $osp(1|2)$ singlet

$$|(0, 0, \lambda)(0)00\rangle = |(0, 0, \lambda)000\rangle. \quad (4.17)$$

- $(q, q > 0, \lambda)$ one $osp(1|2)$ multiplet

$$|(q, q, \lambda)(q)qm\rangle = |(q, q, \lambda)qqm\rangle, \quad (4.18)$$

$$|(q, q, \lambda)(q)q-1/2, m\rangle = -\alpha |(q, q, \lambda)q+1/2, q-1/2, m\rangle. \quad (4.19)$$

- $(-q, q > 0, \lambda)$ one $osp(1|2)$ multiplet

$$|(-q, q, \lambda)(q)qm\rangle = |(-q, q, \lambda)-qqm\rangle, \quad (4.20)$$

$$|(-q, q, \lambda)(q)q-1/2, m\rangle = -\beta |(-q, q, \lambda)-q-1/2, q-1/2, m\rangle. \quad (4.21)$$

- $(p, 1/2, \lambda)$ with $|p| \neq 1/2$, one singlet and one triplet of $osp(1|2)$

$$|(p, 1/2, \lambda)(0)00\rangle = |p^2 - \frac{1}{4}|^{-1/2} [(p-1/2)\alpha |(p, 1/2, \lambda)(p+1/2)00\rangle + (p+1/2)\beta |(p, 1/2, \lambda)(p-1/2)00\rangle], \quad (4.22)$$

$$|(p, 1/2, \lambda)(1/2)00\rangle = -[\alpha |(p, 1/2, \lambda)(p+1/2)00\rangle + \beta |(p, 1/2, \lambda)(p-1/2)00\rangle], \quad (4.23)$$

$$|(p, 1/2, \lambda)(1/2)1/2, m\rangle = |(p, 1/2, \lambda)p1/2, m\rangle. \quad (4.24)$$

V Grade star representations of $sl(1|2)$

To derive the grade star representations of $sl(1|2)$, one uses a non positive definite scalar product which is a generalization of that defined by Minnaert and Mozzyzmas [5] for $osp(1|2)$. For the previous basis the scalar product of two states is defined as

$$\langle (p, q, \lambda)(j)lm | (p, q, \lambda)(j')l'm' \rangle = (-1)^{k(\phi+\psi)+\psi} \delta_{j,j'} \delta_{ll'} \delta_{mm'}, \quad (5.1)$$

with,

$$k = 2(j - l), \quad h = 2(q - j), \quad \phi = \lambda + \epsilon + 1, \quad \text{mod } 2, \quad (5.2)$$

and where the binary variable ψ only depends on the representation, i.e. on the numbers p, q and λ . First, one considers the general representation case ($|p| \neq q, q > 1/2$) and one adopts the abbreviated notation $|(j)lm \rangle$ for the states. Then, the inverse relations of Eqs.(4.5-8) between both bases read

$$|pqm \rangle = |(q)qm \rangle, \quad |p, q - 1, m \rangle = |(q - 1/2)q - 1, m \rangle, \quad (5.3)$$

$$|p + 1/2, q - 1/2, m \rangle = \frac{p + q}{2q\alpha} |(q)q - 1/2, m \rangle + \delta |(q - 1/2)q - 1/2, m \rangle, \quad (5.4)$$

$$|p - 1/2, q - 1/2, m \rangle = -\frac{p - q}{2q\beta} |(q)q - 1/2, m \rangle - \frac{\alpha\delta}{\beta} |(q - 1/2)q - 1/2, m \rangle. \quad (5.5)$$

Then, the orthogonality of the states $|p \pm 1/2, q - 1/2, m \rangle$ requires that

$$\alpha^2 \delta^2 = (-1)^\phi \frac{q^2 - p^2}{4q^2}. \quad (5.6)$$

Hence, one can have a grade star representation only if

$$\phi = \theta(q^2 - p^2), \quad (5.7)$$

where θ is the Heaviside function: $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$; there are three possible classes of representations

$$G^\pm \text{ for } \pm p > q, \quad (\phi = 1); \quad G^0 \text{ for } |p| < q, \quad (\phi = 0).$$

To determine completely the representations, one calculates the squares of the states $|blm \rangle$; by using the notations (3.8-10), one gets

$$g = (-1)^\psi, \quad (5.8)$$

$$g_1 = -(-1)^{\phi+\psi}, \quad (5.9)$$

$$g_+ = (-1)^{\phi+\psi} \frac{p+q}{2q\alpha^2}, \quad (5.10)$$

$$g_- = (-1)^{\phi+\psi} \frac{q-p}{2q\beta^2}. \quad (5.11)$$

Of course, equations (5.6,8,9) are compatible with previous equations (3.11-13). Then, for all classes, one can choose

$$\alpha = \sqrt{\frac{|p+q|}{2q}}, \quad \beta = \delta = \sqrt{\frac{|p-q|}{2q}}, \quad (5.12)$$

so that the squares of the states $|blm \rangle$ read for each class

$$G^\pm : g = \mp g_+ = \pm g_- = g_1 = (-1)^\psi, \quad (5.13)$$

$$G^0 : g = g_+ = g_- = -g_1 = (-1)^\psi. \quad (5.14)$$

Hence, the non diagonal reduced matrix elements (4.16) of the B generators read for all classes

$$(q - 1/2 \| B \| q) = -(-1)^\phi q \| B \| B - 1/2) = \sqrt{\frac{|p^2 - q^2|}{2q}}, \quad (5.15)$$

that achieves the construction of grade star representations in the general case.

The values (5.11) of the coefficients α, β, δ are the same as that adopted by Scheunert et al. [2], to their classes $S^\pm$ of star representations and to their grade star representations which is only defined for $q = 1/2$. This shows that our grade star classes $G^\pm$ are equivalent to their star classes $S^\pm$ and our class G^0 generalizes their grade star one.

Now, it remains to study the particular cases of representations. For the case ($q = 1/2, |p| \neq 1/2$), the corresponding grade star representations are obtained by choosing

$$\alpha = \sqrt{|p + 1/4|}, \quad \beta = \sqrt{|p - 1/4|}, \quad (5.16)$$

so that the involved reduced matrix elements of the generators B are simply

$$(1/2 \| B \| 1/2) = -\sqrt{2}p, \quad (0 \| B \| 1/2) = -(-1)^\phi (1/2 \| B \| 0) = \sqrt{|p^2 - 1/4|}. \quad (5.17)$$

Finally let us consider the atypical representations. All of them are grade star representations : for both values of ϕ in the obvious case of the trivial representation ($p = q = 0$), and only for $\phi = 1$ in the other atypical ones ($|p| = q > 0$) where one can choose

$$(p = q > 0) : g = -g_+ = (-1)^\psi, \quad \alpha = 1, \quad (5.18)$$

$$(-p = q > 0) : g = -g_- = (-1)^\psi, \quad \beta = 1. \quad (5.19)$$

The following fundamental representation of $sl(1|2)$ provides an example of such a grade star atypical representation with ($p = q = 1/2, \lambda = \epsilon = 0$),

$$\bar{\mathcal{J}} = \frac{1}{2} \begin{pmatrix} 0 & \\ & \bar{\sigma} \end{pmatrix}, \quad \mathcal{J}_{+1/2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{J}_{-1/2} = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (5.20)$$

$$\mathcal{B}_0 = \frac{1}{2} \begin{pmatrix} 2 & \\ & I \end{pmatrix}, \quad \mathcal{B}_{+1/2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{B}_{-1/2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (5.21)$$

where the expressions of $\mathcal{J}$ correspond to the fundamental representation of $osp(1|2)$.

VI Conclusion

It was first shown that a general irreducible representation (p, q, λ) of $sl(1|2)$ contains two $osp(1|2)$ multiplets with $j = q$ parity λ and $j = q - 1/2$ parity $\lambda + 1$, and then three classes of grade star representations was built: $\mathcal{G}^\pm$ for $\pm p > q$ and $\mathcal{G}^0$ for $|p| > q$.

In several realizations of the $osp(1|2)$ super-algebra and of its representation spaces [7, 8, 9], we adopted a very simple convention, compatible with the results of Ref.[5]: we fix the parity of the j -representation to be $\lambda = 2j \pmod{2}$, so that the Grassman degree of the state $|j\rangle$ is simply $2j \pmod{2}$, i.e. 0 for bosonic states (spin integer) and 1 for fermionic states (spin half-integer). This convention can be extended to the grade star representations of $sl(1|2)$ by choosing the parity of its (p, q) -representation to be $\lambda = 2q \pmod{2}$. Then, the parity of the $osp(1|2)$ -multiplets contained in the (p, q) -representation becomes $2j \pmod{2}$ and the the Grassman degree of the states is $2l \pmod{2}$ as previously.

Two recent articles [10, 11] are devoted to the Harmonic Oscillator and the Coulomb Potential in the $E(1|2)$ super-space, the energy levels and the eigenstates are determined, using super-spin eigenfunctions. In a forthcoming publication it will be shown that, for both cases, the eigenfunctions can be gathered in $(p, 1/2)$ -representations of $sl(1|2)$ with the above parity convention, the number p being related to the energy levels of the system.

Acknowledgements

I would acknowledge Professor P. Minnaert for discussions about this subject.

References

- [1] M. Scheunert, W. Nahm, and V. Rittenberg, J. Math. Phys. **18**, 146 (1977).
- [2] M. Scheunert, W. Nahm, and V. Rittenberg, J. Math. Phys. **18**, 155 (1977).
- [3] A. Pais and V. Rittenberg, J. Math. Phys. **16**, 2062 (1975).
- [4] F. A. Berezin and V. N. Tolstoy, Commun. Math. Phys. **78**, 409 (1981).
- [5] P. Minnaert and M. Mozzyrmas, J. Math. Phys. **33**, 1582 and 1594 (1992).
- [6] M. Daumens, P. Minnaert, M. Mozzyrmas and S. Toshev, Europhys. Lett. **20**, 671 (1992) and J. Math. Phys. **34**, 2475 (1993).
- [7] M. Daumens, and Y. Noiro, J. Math. Phys. **34**, 1519 (1993).
- [8] M. Daumens, J. Math. Phys. **34**, 2508 (1993).
- [9] M. Daumens, *Boson fermion realizations for the super rotation algebra*, LPTB-93-13, to be published in J. Math. Phys. **35**.
- [10] M. Daumens, and Y. Noiro, *Harmonic oscillator in the $E(1|2)$ super-space*, LPTB-93-20.
- [11] M. Daumens, *Schrodinger equation with central potential in the $E(1|2)$ super-space*, LPTB-93-22.