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$SO(5)_q$ and Contraction :Chevalley Basis Representations for q Generic and Root of Unity

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ABSTRACT

Representations of $SO(5)_q$ are constructed explicitly on the Chevalley basis. The classical representations, once obtained on this basis, are very simply deformed for arbitrary q , generic and root of unity. This is in sharp contrast to the classical canonical Gelfand-Zetlin representations which lead to problems, studied elsewhere, in q - deformations for orthogonal (though not for unitary) algebras. Our results are limited to irreducible representations labelled by 3 variable parameters, the maximal number being 4. Within this restriction we show how to construct all representations for all q 's. The contraction of the Chevalley generators (e_2, f_2) leads to an easily obtained complete Hopf algebra. The structure of the contracted algebra is quite different, even at the classical level, from the familiar $E(4)$, the 4 dim. Eucl. algebra, obtained on contracting the generator J_{45} in the G-Z formalism. The contracted representations are given. The $SO(5)_q$ representations have their own interest. Moreover they lead to a consistent Hopf algebra and representations for the non-simisimple contracted case, for all q .

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1 - Introduction :

Long ago I presented parallel treatments of construction of representations of contracted classical unitary and orthogonal Lie algebras [1]. For the *unitary* case the q - deformation of such non-semisimple Lie algebras was carried out quite generally and satisfactorily. This was done first for generic q [2]. Then, for q a root of unity, we introduced the formalism of “fractional parts” to be reconsidered here. This solved *completely* the problem of explicit construction of representations of unitary algebras (q -deformed) for q a root of unity for both the homogeneous [3] and the non-semisimple inhomogenous [4] cases. A brief review talk [5] presents the “what-why-how” of this approach. All this remained limited to unitary algebras. There the starting point was Jimbo’s “minimal” q - deformation [6] of classical Gelfand-Zetlin representations [7]. By “minimal” we mean introduction of q - brackets for each classical factor in the G-Z matrix elements. The same starting point is *not* good for orthogonal algebras. The difficulties one encounters in trying to q -deform classical GZ matrix element for $SO(n)$ and the reasons which make $SO(n)$ so different from $SU(n)$ in this respect have been discussed in detail for $SO(4)$ [8] and also briefly for $SO(5)$ [9]. Certain efforts of other authors [10], however ingenious, also reveal, in the opinion of the present author, the basic unsuitability of $SO(n)$ GZ representations concerning q - deformations.

So we make a fresh start by constructing explicitly representations on the Chevalley basis for the simplest non-trivial case $SO(5)$. *The basic idea is that since the Drinfeld-Jimbo Hopf algebra provides a simple prescription for all q - deformed simple Lie algebras using the Chevalley generators, classical representations constructed on this basis would turn out to be consistently q -deformable without problem, minimally on nearly so.* This hopeful conjecture turned out to be correct. But first the classical representations had to be constructed. *For unitary algebras this problem does not exist.* The Chevalley generators (e_i, f_i) correspond to (A_i^{i+1}, A_{i+1}^i) of the canonical generators directly and the Cartan generators h_i to $(A_i^i - A_{i+1}^{i+1})$, the latter being already *diagonalized* in the GZ representations. The basic, minimal set of generators necessary for building the whole algebra, is thus the same in each formalism and the same subset is diagonalized. Both these features are absent for $SO(n)$. The question of diagonalisation is particularly crucial since from h_i we have to go over to $q^{\pm h_i}$ for $q \neq 1$. In the GZ formalism only $h_1 = J_{12}$ is diagonalized. The problems due to a non-diagonalized Cartan generator in q - deformation is studied in detail in [8,9]. In our conventions, which we found to be the most convenient concerning the correspondance of the canonical and the Chevalley bases, *the G Z generator J_{45} has to be diagonalized* to exhibit the equivalence of the GZ with our representations. The matrix diagonalizing J_{45} is too difficult to construct explicitly. But awareness of this relation, even if implicit, can be helpful. See the discussion in the Appendix leading to (A.15) and what follows.

Having constructed the classical solutions, the q - deformation, as anticipated, turned out to be easy. The differences with the “minimal” q - deformation for the unitary case are in the appearances of $[x]_2$ q -brackets defined in (2.1) along with $[x]$ due to the existence of unequal roots. There are also external constant factors of $(q+q^{-1})$ as in (2.13), (2.15) and (2.16). The q -deformation being simple, we have chosen to present the representations directly in a form valid for all q ($q=1$, $q \neq 1$ and real, $q = e^{i2\pi/N}$). *It has been possible to unify the presentation including the root of unity case thanks to our formalism of “fractional part” discussed in detail in Sec.4.* In fact, for q a root of unity, we have already presented

before periodic representations of $SO(5)_q$ [11]. Here our aim, realized for a restricted class, is a formalism unifying all the q 's.

As emphasized in Sec.5 our prescription for contraction (of the “top-most” (e,f) pair) gives automatically a simple prescription for the Hopf algebra of the contracted non-semisimple structure, not only with $SO(5)_q$ as starting point but any $SO(n)_q$ (or $SU(n)_q$). But the contracted algebra, even at the classical level is quite different from that obtained by contracting $J_{n-1,n}$ (J_{45} for $SO(5)$). This aspect is examined in Sec.5.

For the contracted algebra also our formalism unifies the treatment of generic and root of unity q 's.

On the one hand one has “kappa-deformations” in which q is finally unity but the ratio of $\ln q$ and a contraction parameter (say $\epsilon \rightarrow 0$) being finite introduces a deformation [12, 13]. Root of unity q 's have no place there. On the otherhand Schnizer has constructed representations [14] which are finite dimensional only q a root of unity, not for a generic q .

Our formalism works in all cases and, in spite of fundamental, spectacular differences between generic and root of unity q 's, unifies the treatment maximally.

As already mentioned in the Abstract, we have constructed representations involving 3 variable parameters. For the general case one needs 4 variable ones (see Appendix). Moreover, particularly concerning partially periodic representations for q a root of unity, detailed enumeration of all variants and subcases has not been attempted. In spite of such limitations we consider the content rich enough. In a following paper we hope to present most general representations for $SO(5)$ and also to explore more thoroughly partially periodic cases. The q - deformations of the classical deformations ones of [1] (which lead to $U(n,1)_q$ in [2]) are also deferred for $SO(n)_q$. The ultimate goal is for $SO(n)_q$ all n . But $SO(5)_q$ and $SO(6)_q$ should be well explored first.

2 - Representations of $SO(5)_q$

The Drinfeld-Jimbo Hopf algebra of the Chevalley generators ($h_1, e_1, f_1, h_2, e_2, f_2$) is well-known. In our notations and conventions, with the Cartan matrix

$$\begin{vmatrix} 2 & -2 \\ -1 & 2 \end{vmatrix}$$

$$\text{and } [x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}, [x]_2 \equiv \frac{q^{2x} - q^{-2x}}{q^2 - q^{-2}} \quad (2.1)$$

the q - commutators and the Serre relations are

$$q^{\epsilon_1 h_1} q^{\epsilon_2 h_2} = q^{\epsilon_2 h_2} q^{\epsilon_1 h_1} \quad (\epsilon_1 = \pm 1, \epsilon_2 = \pm 1)$$

$$\begin{aligned}
q^{\epsilon_1 h_1} e_1 &= q^{\epsilon_1} e_1 q^{\epsilon_1 h_1}, q^{\epsilon_1 h_1} f_1 = q^{-\epsilon_1} f_1 q^{\epsilon_1 h_1} \\
q^{2\epsilon_2 h_2} e_1 &= q^{-\epsilon_2} e_1 q^{2\epsilon_2 h_2}, q^{2\epsilon_2 h_2} f_1 = q^{\epsilon_2} f_1 q^{2\epsilon_2 h_2}
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
q^{\epsilon_1 h_1} e_2 &= q^{-\epsilon_1} e_2 q^{\epsilon_1 h_1}, q^{\epsilon_1 h_1} f_2 = q^{\epsilon_1} f_2 q^{\epsilon_1 h_1} \\
q^{\epsilon_2 h_2} e_2 &= q^{\epsilon_2} e_2 q^{\epsilon_2 h_2}, q^{\epsilon_2 h_2} f_2 = q^{-\epsilon_2} f_2 q^{\epsilon_2 h_2} \\
[e_1, f_1] &= [2h_1], [e_1, f_2] = 0 \\
[e_2, f_2] &= [2h_2], [e_2, f_1] = 0
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
e_1^3 e_2 - (q^2 + q^{-2} + 1) e_1^2 e_2 e_1 + (q^2 + q^{-2} + 1) e_1 e_2 e_1^2 - e_2 e_1^3 &= 0 \\
f_1^3 f_2 - (q^2 + q^{-2} + 1) f_1^2 f_2 f_1 + (q^2 + q^{-2} + 1) f_1 f_2 f_1^2 - f_2 f_1^3 &= 0 \\
e_2^2 e_1 - (q^2 + q^{-2}) e_2 e_1 e_2 + e_1 e_2^2 &= 0 \\
f_2^2 f_1 - (q^2 + q^{-2}) f_2 f_1 f_2 + f_1 f_2^2 &= 0
\end{aligned} \tag{2.4}$$

Coproducts, counits and antipodes are discussed in the context of contraction (Sec.3). Here they are the standard ones.

For reasons soon to become evident, we define

$$q^M = q^{h_1}, q^{K-M} = q^{2h_2} \tag{2.5}$$

Three classes of representations with *three variable parameters* have been obtained explicitly. The classical representations are characterized by two invariant parameters, both integers or half integers, say

$$n_1 \text{ and } n_2 \text{ with } n_1 \geq n_2.$$

The classical limits of our representations can be shown to correspond to (see Appendix) $n_2 = 0$, $n_2 = n_1$ and $n_2 = \frac{1}{2}$ respectively.

More generally one can have *four variables* labelling states $|j \ m \ k \ l \rangle$, say. We consider restricted cases with fixed l .

Exhibiting only the variable parameters in the states $|j \ m \ k \rangle$

we start with the ansatz

$$\begin{aligned} q^M |j\ m\ k\rangle &= q^m |j\ m\ k\rangle \\ q^K |j\ m\ k\rangle &= q^k |j\ m\ k\rangle \end{aligned} \quad (2.6)$$

$$\begin{aligned} e_1 |j\ m\ k\rangle &= ([j - m] [j + m + 1])^{1/2} |j\ m + 1\ k\rangle \\ f_1 |j\ m\ k\rangle &= ([j + m] [j - m + 1])^{1/2} |j\ m - 1\ k\rangle \end{aligned} \quad (2.7)$$

$$\begin{aligned} e_2 |j\ m\ k\rangle &= ([j - m + 1] [j - m + 2])^{1/2} a(j, k) |j + 1\ m - 1\ k + 1\rangle \\ &+ ([j + m] [j + m - 1])^{1/2} b(j, k) |j - 1\ m - 1\ k + 1\rangle \\ &+ ([j + m] [j - m + 1])^{1/2} c(j, k) |j\ m - 1\ k + 1\rangle \end{aligned} \quad (2.8)$$

$$\begin{aligned} f_2 |j\ m\ k\rangle &= ([j + m + 1] [j + m + 2])^{1/2} b(j + 1, k - 1) |j + 1\ m + 1\ k - 1\rangle \\ &+ ([j - m] [j - m - 1])^{1/2} a(j - 1, k - 1) |j - 1\ m + 1\ k - 1\rangle \\ &+ ([j - m] [j + m + 1])^{1/2} c(j, k - 1) |j\ m + 1\ k - 1\rangle \end{aligned} \quad (2.9)$$

The reduced elements (a, b, c) are now determined from the Serre relations (2.4) and with (2.3) written as

$$[e_2, f_2] = [K - M]_2 \quad (2.10)$$

The three classes of solutions obtained are the following ones. (See Appendix)

Case I : $(n_2 = 0, n_1 = n)$

$$\begin{aligned} a(j, k) &= (q + q^{-1})^{-1} \left(\frac{[n - j - k] [n + j + k + 3]}{[2j + 1] [2j + 3]} \right)^{1/2} \\ b(j, k) &= (q + q^{-1})^{-1} \left(\frac{[n + j - k + 1] [n - j + k + 2]}{[2j - 1] [2j + 1]} \right)^{1/2} \\ c(j, k) &= 0 \end{aligned} \quad (2.11)$$

The relative, sign of a and b is not fixed by the constraints. We have chosen this convention.

The domains of the parameters, *for generic q*, are (note the double spacing for k)

$$\begin{aligned} j &= 0, 1, \dots, n \\ k &= n - j, n - j - 2, \dots, -n + j \\ m &= j, j - 1, \dots, -j \end{aligned} \quad (2.12)$$

(For the generic cases the dimensions are easily counted. When q is a root of unity the domains of the parameters and the invariants have to be discussed again. For all the three case it will be done in the following section.)

Case 2 : $(n_2 = n_1 = n)$

$$\begin{aligned}
 a(j,k) &= (q + q^{-1})^{-1} \left(\frac{[n-j]_2 [n+j+2]_2 [j+k+1] [j+k+2]}{[2j+3] [2j+1] [j+1]_2^2} \right)^{1/2} \\
 b(j,k) &= (q + q^{-1})^{-1} \left(\frac{[n-j+1]_2 [n+j+1]_2 [j-k] [j-k-1]}{[2j+1] [2j-1] [j]_2^2} \right)^{1/2} \\
 c(j,k) &= (q + q^{-1})^{-1} [n+1]_2 \frac{([j-k] [j+k+1])^{1/2}}{[j+1]_2 [j]_2}
 \end{aligned} \tag{2.13}$$

For generic q ,

$$\begin{aligned}
 j &= n, n-1, \dots, 0 \ (1/2) \\
 k &= j, j-1, \dots, -j \\
 m &= j, j-1, \dots, -j
 \end{aligned} \tag{2.14}$$

Case 3 : $(n_1 = n, n_2 = 1/2)$. The conventions chosen below ensure, for $q = 1$, most direct equivalence with the GZ representations [7] for $n_2 = \frac{1}{2}$, inspite of the apparent radical differences.

Here n, j, k, m are all half integers. (Following [7] we include half integers in $SO(n)$ at the level of algebra.)

(i) For $n + \frac{1}{2} - j - k = 2p \ (p = 0, 1, 2, \dots)$

$$\begin{aligned}
 a(j,k) &= (q + q^{-1})^{-2} ([j+1]_2)^{-1} \left([n + \frac{1}{2} - j - k] [n + \frac{1}{2} + j + k + 3] \right)^{1/2} \\
 b(j,k) &= - (q + q^{-1})^{-2} ([j]_2)^{-1} \left([n + \frac{1}{2} - j + k + 1] [n + \frac{1}{2} + j - k] \right)^{1/2} \\
 c(j,k) &= (q + q^{-1})^{-2} ([j+1]_2 [j]_2)^{-1} \left([n + \frac{1}{2} - j - k] [n + \frac{1}{2} - j + k + 1] \right)^{1/2}
 \end{aligned} \tag{2.15}$$

(ii) For $n + \frac{1}{2} - j - k = 2p + 1 \ (p = 0, 1, \dots)$

$$\begin{aligned}
 a(j,k) &= (q + q^{-1})^{-2} ([j+1]_2)^{-1} \left([n + \frac{1}{2} - j - k - 1] [n + \frac{1}{2} + j + k + 2] \right)^{1/2} \\
 b(j,k) &= - (q + q^{-1})^{-2} ([j]_2)^{-1} \left([n + \frac{1}{2} - j + k + 2] [n + \frac{1}{2} + j - k + 1] \right)^{1/2} \\
 c(j,k) &= (q + q^{-1})^{-2} ([j+1]_2 [j]_2)^{-1} \left([n + \frac{1}{2} + j + k + 2] [n + \frac{1}{2} + j - k + 1] \right)^{1/2}
 \end{aligned} \tag{2.16}$$

The domains, for generic q , are

$$\begin{aligned} j &= n, n-1, \dots, 1/2 \\ k &= n + \frac{1}{2} - j, \dots, -(n + \frac{1}{2} - j) \\ m &= j, \dots, -j \end{aligned} \quad (2.17)$$

We have preferred to start by exhibiting the parallel structures of (i) and (ii) separately. They can be unified as follows.

$$\text{Define} \quad \delta \equiv \frac{1}{2} (1 - (-1)^{n+\frac{1}{2}-j-k}) \quad (2.18)$$

Then

$$\begin{aligned} a(j,k) &= (q + q^{-1})^{-2} [j+1]_2^{-1} ([n + \frac{1}{2} - j - k - \delta] [n + \frac{1}{2} + j + k + 3 - \delta])^{1/2} \\ b(j,k) &= - (q + q^{-1})^{-2} [j]_2^{-1} ([n + \frac{1}{2} - j + k + 1 + \delta] [n + \frac{1}{2} + j - k + \delta])^{1/2} \end{aligned} \quad (2.19)$$

$$c(j,k) = (q + q^{-1})^{-2} [j+1]_2^{-1} [j+1]_2^{-1} ([n + \frac{1}{2} + (2\delta - 1)j - k + \delta] [n + \frac{1}{2} + (2\delta - 1)j + k + 1 + \delta])^{1/2}$$

3 - Casimir operator and q -deformed canonical generators

$$\begin{aligned} \text{Define} \quad e_3^{(\pm)} &= (q^{\pm 1} e_1 e_2 - q^{\mp 1} e_2 e_1) \\ f_3^{(\pm)} &= (q^{\pm 1} f_2 f_1 - q^{\mp 1} f_1 f_2), \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} e_4 &= (q^{-1} e_1 e_3^{(+)} - q e_3^{(+)} e_1) = (q e_1 e_3^{(-)} - q^{-1} e_3^{(-)} e_1) \\ f_4 &= (q^{-1} f_3^{(+)} f_1 - q f_1 f_3^{(+)}) = (q f_3^{(-)} f_1 - q^{-1} f_2 f_3^{(-)}). \end{aligned} \quad (3.2)$$

At this stage these are just *auxiliary* operators (in the closed algebra already defined by $q^{\pm i}$, e_i , f_i ; $i = 1, 2$) which help in constructing the Casimir, displaying relevant structures in the most convenient fashion. They can however be given a more intrinsic role in the context of the q -deformed Cartan-Weyl basis of $SO(5)_q$ or $Sp(4)_q$ [13, 15]. *If one wants to avoid the “doubling” of (e_3, f_3) one can use relations such as (with upper or lower signs)*

$$\begin{aligned} e_3^{(+)} \pm e_3^{(-)} &= (q \pm q^{-1}) (e_1 e_2 \mp e_2 e_1) \\ f_3^{(+)} \pm f_3^{(-)} &= (q \pm q^{-1}) (f_2 f_1 \mp f_1 f_2) \end{aligned} \quad (3.3)$$

to eliminate one set ($e_3^{(-)}$, $f_3^{(-)}$), say (or any linear combination) from all the formulae to follow. The resulting structures will then be less elegant. Useful relations are

$$\begin{aligned} [e_1, e_2] &= (q + q^{-1})^{-1} (e_3^{(+)} + e_3^{(-)}) \\ [e_1, e_4] &= 0 \end{aligned}$$

$$\begin{aligned} [e_1, f_4] &= - (f_3^{(+)} q^{2M} + f_3^{(-)} q^{-2M}) \\ [e_1, f_3^{(\pm)}] &= -(q + q^{-1}) f_2 q^{\mp} (2M + 1) \end{aligned} \quad (3.4)$$

$$\begin{aligned} e_2 e_3^{(\pm)} &= q^{\mp 2} e_3^{(\pm)} e_2 \\ [e_4, f_3^{(\pm)}] &= f_1 q^{\pm(2K - 2M + 1)} \\ [e_2, e_4] &= 0 = [e_2, f_4] \end{aligned} \quad (3.5)$$

Corresponding sets of commutators can be obtained for f_1, f_2 through formal hermitian conjugations, since (for real q at least) $f_i = e_i^{\dagger}$ ($i = 1, 2$).

Thus, for exemple,

$$\begin{aligned} [f_1, e_3^{(\pm)}] &= (q + q^{-1}) e_2 q^{\mp(2M - 1)} \\ [f_2, e_3^{(\pm)}] &= - e_1 q^{\pm(2K - 2M - 1)} \end{aligned} \quad (3.6)$$

and so on.

Using such relations one can verify that the following operator A is central (adjusted to have zero eigen value for $n_1 = n_2 = 0$). Let

$$\begin{aligned} A &= (q + q^{-1})^{-1} \left\{ (f_1 e_1 + [M] [M + 1]) \left(\frac{q^{2K+3} + q^{-2K-3}}{q + q^{-1}} \right) + [K + 1] [K + 2] \right\} \\ &+ f_2 e_2 + (q + q^{-1})^{-2} \left\{ q f_3^{(+)} e_3^{(+)} q^{2M} + q^{-1} f_3^{(-)} e_3^{(-)} q^{-2M} + f_4 e_4 \right\} - 1 \end{aligned} \quad (3.7)$$

$$\begin{aligned} &= (q + q^{-1}) \left\{ (e_1 f_1 + [M] [M - 1]) \left(\frac{q^{2K-3} + q^{-2K+3}}{q + q^{-1}} \right) + [K - 1] [K - 2] \right\} \\ &+ e_2 f_2 + (q + q^{-1})^{-2} \left\{ q^{-1} e_3^{(+)} f_3^{(+)} q^{2M} + q e_3^{(-)} f_3^{(-)} q^{-2M} + e_4 f_4 \right\} - 1 \end{aligned} \quad (3.8)$$

This should be compared with equation (15) of [16]. The compact forms given above, directly verifiable using (3.4), (3.5), (3.6), are suitable for our representations.

This, the only one considered here, is the most useful invariant. $2A$ is the q - deformation of the classical canonical invariant

$$J_{(2)} = \sum_{i < j} J_{ij}^2 \quad (3.9)$$

of the generators satisfying

$$[J_{ab}, J_{cd}] = i (\delta_{ac} J_{bd} + \delta_{bd} J_{ac} - \delta_{ad} J_{bc} - \delta_{bc} J_{ad}) \quad (3.10)$$

For $SO(5)$, with the standard invariant parameters (n_1, n_2) labelling irreducible representations,

$$J_{(2)} \approx n_1 (n_1 + 3) + n_2 (n_2 + 1) \quad (3.11)$$

Here for (2.11)

$$n_1 = n, n_2 = 0$$

$$A |jmk\rangle = \frac{1}{(q + q^{-1})} [n] [n + 3] |jmk\rangle \quad (3.12)$$

For (2.13),

$$n_1 = n_2 = n$$

$$A |jmk\rangle = [n]_2 [n+2]_2 |jmk\rangle \quad (3.13)$$

For (2.15) and (2.16)

$$n_1 = n, n_2 = \frac{1}{2}$$

$$A |jmk\rangle = \frac{1}{(q+q^{-1})} \left\{ \left(\frac{q^2+q^{-2}}{q+q^{-1}} \right) [n] [n+3] + (q^2+q^{-2}-1) \left[\frac{1}{2} \right] \left[\frac{3}{2} \right] \right\} |jmk\rangle \quad (3.14)$$

For $q \rightarrow 1$, in each case 2A gives (3.11) with the corresponding n_2 . (These values are relevant to the remarks concerning irreducibility at the end of Sec.4).

For $q = 1$, the generators J_{ab} of (3.9) and (3.10) are given by, with

$$e_3 = [e_1, e_2], e_4 = [e_1, e_3] \quad (3.15)$$

and the corresponding f 's,

$$\begin{aligned} J_{12} &= M, J_{23} = \frac{i}{2} (e_1 - f_1), J_{13} = \frac{1}{2} (e_1 + f_1) \\ J_{34} &= \frac{1}{2} (e_3 + f_3) \\ J_{24} &= -\frac{1}{2} (e_2 + f_2) + \frac{1}{4} (e_4 + f_4) \\ J_{14} &= \frac{i}{2} (e_2 - f_2) + \frac{i}{4} (e_4 - f_4) \end{aligned} \quad (3.16)$$

$$\begin{aligned} J_{45} &= K, J_{35} = -\frac{i}{2} (e_3 - f_3) \\ J_{25} &= \frac{i}{2} (e_2 - f_2) - \frac{i}{4} (e_4 - f_4) \\ J_{15} &= \frac{1}{2} (e_2 + f_2) + \frac{1}{4} (e_4 + f_4) \end{aligned}$$

In q -deforming this set one has first to note the doubling of (e_3, f_3) according to (3.1). This is analogous to the doubling of K_0 to $K_0^{(\pm)}$ in the q -deformation of $SO(4)$ to $SO(4)_q$ [8, 9]. This leads to the doubling of J_{34} and J_{35} . But as pointed out, using (3.3) to eliminate one set, one can avoid doubling at the cost of complicating formulae somewhat more. To obtain the q -deformed commutators of the J 's one can use, along with results already given,

$$[e_3^{(+)}, f_3^{(-)}] = [e_3^{(-)}, f_3^{(+)}] = [2K] \quad (3.17)$$

$$[e_3^{(+)}, f_3^{(+)}] q^{2M} = (q^2 e_1 f_1 - q^{-2} f_1 e_1) q^{2K} + (q + q^{-1}) (q^{-2} e_2 f_2 - q^2 f_2 e_2)$$

$$[e_3^{(-)}, f_3^{(-)}] q^{-2M} = (q^{-2} e_1 f_1 - q^2 f_1 e_1) q^{-2K} + (q + q^{-1}) (q^2 e_2 f_2 - q^{-2} f_2 e_2) \quad (3.18)$$

The doubling disappears at the level of (e_4, f_4) as seen from (3.2). The commutator $[e_4, f_4]$, more complicated, can be obtained directly by comparing (3.7) and (3.8). In this commutator again, $e_3^{(+)}$, $f_3^{(+)}$ appear parallelly. It is not particularly useful here to display the commutators of q -deformed J 's

systematically. Contraction (Sec.5) will simplify them somewhat. Such features are typical of the difficulties of q-deforming the canonical basis. How far one can go with q-deformed G-Z matrix elements for SO(5) has been discussed in [9]. Here the coproducts, such as

$$\Delta e_3^{(+)} = e_3^{(+)} \otimes q^K + q^{-K} \otimes e_3^{(+)} + (q^2 - q^{-2}) e_2 q^{-M} \otimes e_1 q^{K-M} \quad (3.19)$$

all follow from the standard Hopf algebra (Drinfeld-Jimbo) for SO(5)_q. This is more fully discussed in Sec.5. Here we briefly note that the concept of “antipode-extended Cartan-Weyl basis” of [13] is relevant to $(e_3(\pm), f_3(\pm))$, since the antipode operation (S), gives

$$S e_3(\pm) = -q^3 e_3(\mp), S f_3(\pm) = -q^{-3} f_3(\mp) \quad (3.20)$$

For the unitary case we already introduced such a structure defining $P_{n+1(\epsilon)}^j$ ($\epsilon = \pm$) [2,5].

$$4 - q = e^{i2\pi/N} \quad (N = 3, 4, \dots)$$

After the remarks in Sec.1 and the sources cited there [3, 4, 5, 8, 9, 11] concerning the formalism of “fractional parts” introduced by us in previous papers let us start again by studying the q-deformation of the classical SO(3) or SU(2) sub-algebra of SO(5) for q a root of unity. The detailed discussion at this stage will display the essential points of this approach simply.

Consider first fully periodic representations for q a root of unity. For generic q (not a root of unity) in (2.7) j and m satisfy

$$(j \mp m) \in \mathbb{Z} \text{ and } j \geq |m| \quad (4.1)$$

Now let j and m have “fractional parts” (which for brevity will refer both to real proper fractions and to imaginary parts) denoted as f.p such that, for $q^N = 1$,

$$(j \mp m) \notin \mathbb{Z}/2 \text{ for } N \text{ odd} \quad (4.2)$$

and

$$(j \mp m) \notin \mathbb{Z} \text{ for } N \text{ even}$$

That is, they cannot be integers for even N and not even half integers for odd N.

Then neither e_1 nor f_1 can be nilpotent. Concerning periodicity one can consider two possible prescriptions.

$$\begin{aligned} \text{(I)} \quad e_1 |j, m, k\rangle &= \gamma ([j - m] [j + m + 1])^{1/2} |j, m + 1, k\rangle \\ f_1 |j, m, k\rangle &= \gamma^{-1} ([j + m] [j - m + 1])^{1/2} |j, m - 1, k\rangle \end{aligned} \quad (4.3)$$

$$\text{and} \quad |j, m + N, k\rangle = |j, m, k\rangle \quad (4.4)$$

where γ is a non-zero finite constant. It can be complex. The identification (4.4) gives the periodicity, quotienting the initial infinite dimensional space.

Or

$$(II) \quad \begin{aligned} e_1 |j \ m \ k\rangle &= ([j - m] [j + m + 1])^{1/2} |j \ m + 1 \ k\rangle \\ f_1 |j \ m \ k\rangle &= ([j + m] [j - m + 1])^{1/2} |j \ m - 1 \ k\rangle \end{aligned} \quad (4.5)$$

$$\text{with} \quad |j, \ m + N, \ k\rangle = \gamma^N |j \ m \ k\rangle \quad (4.6)$$

$$\text{In both cases } q^{\pm 1} |j \ m \ k\rangle = q^{\pm M} |j \ m \ k\rangle = q^{\pm m} |j \ m \ k\rangle \quad (4.7)$$

The constant γ is now in the periodicity condition, the *matrix elements are formally unchanged* as compared with the generic q - case. The fundamental change is implicit in the respective domains of the parameters.

The constant γ can depend on N (γ being $\gamma(N)$) to fix suitably the eigenvalues of the central operators e_1^N, f_1^N .

From (4.5) and (4.6), for odd N ,

$$e_1^N |j \ m \ k\rangle = \gamma^N \frac{(-1)^{\frac{N-1}{2}}}{2^{N-1} (\sin \frac{2\pi}{N})^N} (\sin 2\pi (j - m) \sin 2\pi (j + m))^{1/2} |j \ m \ k\rangle \quad (4.8)$$

and for even N ,

$$e_1^N |j \ m \ k\rangle = \gamma^N \frac{(-1)^{\frac{N-2}{2}}}{2^{N-2} (\sin \frac{2\pi}{N})^N} (\sin \pi (j - m) \sin \pi (j + m)) |j \ m \ k\rangle \quad (4.9)$$

with similar expressions for f_1^N .

The representation of the $(e_1, f_1, q^{\pm M})$ sub-algebra now depends on 3 invariant parameters (γ, j and $\text{frac.part of } m$). The dimension is N instead of the familiar $(2j + 1)$ for generic q .

The representations are now, in general complex, through supplementary restrictions, to be mentioned later, can be introduced to maintain hermitian conjugacy between e_1 and f_1 .

The convention (II) can be directly related, for $Sl(2, c)_q$, to that of Roche and Arnaudon [17] who constructed the first explicit periodic representations.

For convenience of comparison we reproduce the construction of [17] in the authors' notation (with minor changes) as follows

$$\begin{aligned} (q^h V_p =) k V_p &= q^{\mu/2 - p} V_p \quad (p = 0, \dots, N - 1) \\ (1 \leq p \leq N-1) \ e_1 V_p &= ([p] [\mu + 1 - p] + \alpha \beta)^{1/2} V_{p-1} \\ (0 \leq p \leq N-2) \ f_2 V_p &= ([p + 1] [\mu - p] + \alpha \beta)^{1/2} V_{p+1} \end{aligned} \quad (4.10)$$

$$e_1 V_0 = \beta V_{N-1}, f_1 V_{N-1} = \alpha V_0$$

This leads to

$$V_p = (\beta/\alpha)^{1/2} V_{p+N} \quad (4.11)$$

Comparing (4.5) and (4.6) with (4.11) one obtains (for any k)

$$\begin{aligned} m &= \mu/2 - p \\ [j] [j+1] &= [\mu/2] [\mu/2+1] + \alpha \beta \\ \gamma^N &= (\alpha/\beta)^{1/2} \end{aligned} \quad (4.12)$$

Our approach, however, is directly generalized to corresponding deformations of higher dimensional Lie algebras.

One first has to obtain the representations for generic q. Then one introduces, for q a root of unity, fractional parts for the parameters with suitable constraints on them to be determined from the forms of the initial matrix elements. Such constraints avoid divergences or unwanted zeroes. Examples can be found in [3] for unitary algebras, where the equivalence of the “f.p.” approach to another one due to Arnaudon [18] was established in detail for $SU(3)_q$. The generalizations of (I) and (II) both work for the deformed unitary algebras. *But (II) is better suited for orthogonal algebras.* Indeed, suitably generalized, this leads to a perfectly general prescription for all algebras (q-deformed) unifying the formal expressions of the matrix elements for generic q and root of unity. All the spectacular differences are then contained in the, suitably defined, respective domains of the parameters involved.

We have already considered periodic representations of $SO(5)_q$ elsewhere [11]. Here we present the unified approach by starting with the matrix elements of Sec.2 obtained there for generic q with ordered integer or half integer parameters.

Rather than enumerating and classifying systematically all the possibilities, cases and subcases, we indicate the essential features and the scope of the formalism. Moreover to simplify the discussion we assume in the following N to be odd. Modifications for even N we easily obtained.

(1) Redefinitions of the domains of the parameters

Apart from (4.2) one now needs further restrictions to assure finite, non-vanishing matrix elements. Thus for exemple, as sufficient constraints covering all the cases from (2.11) to (2.16) one can set ($\epsilon_1, \epsilon_2, \epsilon_3$ being each ± 1)

$$\{j, (j + \epsilon_1 k), (n + \epsilon_2 j), (n + \epsilon_3 (j + \epsilon_1 k))\} \notin \mathbb{Z}/2 \quad (4.13)$$

(2) Quotientings leading to periodicity

$$\text{Set } \gamma_1^{-1} |j + N, m, k\rangle = \gamma_2^{-1} |j, m + N, k\rangle = \gamma_3^{-1} |j, m, k + N\rangle = |j, m, k\rangle \quad (4.14)$$

The γ 's being finite constructants, in general complex and can depend on N .

(3) The centre and invariant parameters

For simplicity, consider first the case $N = 3$.

$$e_2^3 |j\ m\ k\rangle = \sum_{i=-3}^3 M_i |j + i, m-3, k+3\rangle \quad (4.15)$$

where the matrix elements M_i can be obtained by repeatedly using (2.11) or ((2.13) or (2.15) and (2.16) respectively. It can be shown laboriously and explicitly that

$$M_{\pm 1} = M_{\pm 2} = 0 \quad (4.16)$$

and hence using (4.14)

$$e_2^3 |j\ m\ k\rangle = M |j\ m\ k\rangle \quad (4.17)$$

where the eigenvalue M can also be obtained explicitly. Identities assuring analogues of (4.16) were studied in [8] for $SO(4)_q$ and indeed for $SU(n)_q$ for arbitrary n in [3]. But all this is not really necessary. They are all inevitable consequences of the initial algebra (here (2.2)) along with the condition

$$q^N = 1$$

It is well-known that e_2^N , for example, commutes with all the generators, the same result holding for all generators raised to the N -th power. They are all central, *this fact, indeed, can be used as a machine to derive, sometimes surprising, identities wick are necessary to guarantee commutativity.* (See, for example, the Lemma in p.466 of [3], where however, a independent proof was sketched.)

It is known from general theorems [19], that the maximal number of invariant parameters in periodic representations of $SO(5)_q$ is 10, the dimension of the classical $SO(5)$ Lie algebra. Indeed had we constructed, to start with for generic q , the most general representations with 4 variable parameters labelling states as

$$|j\ m\ k\ l\rangle_{(n_1, n_2)} \quad (4.18)$$

Then after introducing “f.p.”s and the 4 periodicity constraints

$$\gamma_1^{-1} |j+N, m, k, l\rangle = \gamma_2^{-1} |j, m+N, k, l\rangle = \gamma_3^{-1} |j, m, k+N, l\rangle = \gamma_4^{-1} |j, m, k, l+N\rangle = |jmk l\rangle \quad (4.19)$$

there would have been the following 10 invariant parameters

$$n_1, n_2 ; \text{f.p.'s of } j, m, k \text{ and } l ; \gamma_1, \gamma_2, \gamma_3, \gamma_4 \quad (4.20)$$

But for our restricted classes with states

$$|j\ m\ k\rangle_{(n_1, n_2)} \quad (4.21)$$

There are finally 8 *invariant parameters*

$$n_1, n_2 ; \text{f.p.'s of } j, m, k ; \gamma_1, \gamma_2, \gamma_3 \quad (4.22)$$

We could like to add that adopting the point of view (II) with conditions analogous to (4.19) the maximal number (10) of invariant parameters can be introduced (immediately without further work) in the formalism of [11] where the point of view (analogous to (I)) adopted led to 8 such parameters. Schnizer's formalism [14] also gives 10 parameters for this case.

(4) *Dimensions and partial periodicity*

For odd N the maximal dimension for periodic $SO(5)_q$ can be N^4 on the space of states (4.18) with (4.19). But our realized restricted representations corresponding to (4.21) and (4.22) can have at most N^3 dimensions. But one can also obtain lower dimensional “partially periodic” representations.

For $SU(n)_q$ we have, elsewhere [3], systematically displayed such possibilities, counting the dimension in each case. Finally there we arrived at highest weight representations (for $q^N = 1$) by suitably restricting all the f.p.'s. Here again one can start, relaxing (4.13) by stages, to build partially periodic representations where some generators become nilpotent (as for generic q) while others still remain injective. One may further note that, if instead of conserving (4.5), (which corresponds to $e_1^\dagger = f_1$ for generic q) one sets (since now representations can be permitted to be complex)

$$e_1 |jmk\rangle = [j - m] |jm + 1 k\rangle$$

$$f_1 |jmk\rangle = [j + m] |jm - 1 k\rangle \quad (4.23)$$

one has still

$$[e_1, f_1] |jmk\rangle = [2m] |jmk\rangle \quad (4.24)$$

but now giving the same f.p. to j and m (to j and $-m$) one can separately make e_1 (f_1) nilpotent while f_1 (e_1) still has a periodic action. One can also similarly consider the effect of setting

$$\begin{aligned} e_1 |jmk\rangle &= [j - m] [j + m + 1] |jm + 1 k\rangle \\ f_1 |jmk\rangle &= [j m - 1 k] \end{aligned} \quad (4.25)$$

Similar variants can also be considered for (e_2, f_2) . Here we stop with only such brief remarks concerning partial periodicity. A systematic study will be attempted when most general representations are obtained for generic q to start with.

Finally we note that though for q a root of unity the representations are in general, complex, *when the f.p.'s are real proper fractions* (no imaginary parts) and *the constants γ* (introduced in the periodicity conditions) *are phases* one can maintain $e_1^\dagger = f_1$, $e_2^\dagger = f_2$, while $q^{\pm h_1}$, $q^{\pm h_2}$ are unitary.

(5) Irreducibility

It is convenient to discuss the irreducibility of our representations for both cases (q generic and root of unity) together. The domains of the parameters and the dimensions of the representations are very different for the two cases however. Thus for generic q one has for $n_2 = 0$, n_1 and $\frac{1}{2}$ the ranges of j , m , k given respectively by (2.12), (2.14) and (2.17). But for $q^N = 1$ (say with odd N) one has for each of the domains

$$(f.p.) + r \quad (r = 0, 1, \dots, N-1)$$

(f.p.) being the frac.part of j , m and k by turn. Yet it seems to be a general feature of our formalism that the introduction of f.p.'s do not spoil the irreducibility of an initial generic- q representation so constructed. One indication of this is that *the formal expressions of the eigenvalues of the Casimir operators remain unchanged as one passes from generic to root of unity q .*

Here we have studied only one (A , evaluated in (3.12), (3.13) and (3.14)). But this is true in general, simply because in the evaluation no reference is made to the specific values of the variables (here j , m , k) and the formal expressions of the matrix elements remain the same. This is yet another agreeable feature of our method. For q a root of unity there are, of course, supplementary invariants to which corresponds precisely our f. p.'s and the factors (γ) in the periodicity constraints.

For our representations here irreducibility is particularly easy to verify. One can easily construct projectors corresponding to the parameters (j , m , k). Thus with

$$B \equiv f_1 e_1 + [M] [M+1]$$

$$P(j') = \prod_{j \neq j'} \frac{(B - [j] [j + 1])}{([j'] [j' + 1] - [j] [j + 1])} \quad (4.26)$$

projects on the state $|j' m k\rangle$ for any (m , k). The projectors for m and k are even simpler in terms of q^M and q^K respectively. Using repeatedly the generators (e_1, f_1, e_2, f_2) and the projectors in suitable order and combinations one can connect any initial to any final state and, indeed, any initial linear combination of the states $|j m k\rangle$ to any final linear combination. There is no invariant subspace. This is the fundamental definition of irreducibility. Non-decomposable representations may arise in *products* of root of unity representations. But that is a different aspect.

At the classical level ($q=1$) our representations correspond to the diagonalization of the generator J_{45} of the Gelfand-Zetlin formalism (see the remarks in Appendix). The well-known irreducibility of the GZ-representation is conserved when in our basis one conveniently q -deforms for generic q , since one to one correspondance is maintained. Then introduction of f.p.'s for root of unity q still conserves irreducibility in spite of fundamental differences elsewhere. This line of thought will be relevant for more general cases. That is one more reason for pointing it out here.

5. Contraction

From the set of generators

$$(q^{\pm h_1}, q^{\pm h_2}, e_1, f_1, e_2, f_2) \quad (5.1)$$

we go over to the set

$$(q^{\pm h_1}, q^{\pm h_2}, e_1, f_1, \hat{e}_2, \hat{f}_2) \quad (5.2)$$

the only change being in the contraction

$$(e_2, f_2) \rightarrow (\hat{e}_2, \hat{f}_2) \quad (5.3)$$

such that instead of (as in (2.3))

$$[e_2, f_2] = [2 h_2]_2 = [K - M]_2 \quad (5.4)$$

one has

$$[\hat{e}_2, \hat{f}_2] = 0 \quad (5.5)$$

All other relations are obtained by replacing (e_2, f_2) in (2.2) to (2.4) by $(\hat{e}_2, \hat{f}_2)$, including the Serre relations.

One now has (along with (2.6), (2.7) for $q^{\pm M}, q^{\pm K}, e_1, f_1$)

$$\begin{aligned} \hat{e}_2 |j m k\rangle &= ([j-m+1] [j-m+2])^{1/2} \hat{a}(j, k) |j+1 m-1 k+1\rangle \\ &+ ([j+m] [j+m-1])^{1/2} \hat{b}(j, k) |j-1 m-1 k+1\rangle \\ &+ ([j+m] [j-m+1])^{1/2} \hat{c}(j, k) |j m-1 k+1\rangle \end{aligned} \quad (5.6)$$

$$\text{with } \langle j' m' k' | \hat{f}_2 | j m k \rangle = \langle j m k | \hat{e}_2 | j' m' k' \rangle \quad (5.7)$$

as before, and (starting with generic q) $(\hat{a}, \hat{b}, \hat{c}) \approx$ the coeff. of the highest power of $q^{\pm n_1}$, (for $q \gtrless 1$) of (a, b, c) respectively. (5.8)

Any constant factor common to all three of $(\hat{a}, \hat{b}, \hat{c})$ can be discarded. This corresponds to the limiting process $(\hat{a}, \hat{b}, \hat{c}) = \mu \left(\lim_{n_1 \rightarrow \infty} q^{\mp n_1} (a, b, c) \right)$ for $q \gtrless 1$ respectively (5.9)

where μ is a real constant for generic q .

This is our “contraction” prescription. The representation thus obtained (all, of course, of infinite dimensions for *generic* q for the non-compact contracted case) are as follows.

case I (corresponding to (2.11))

$$\begin{aligned} \hat{a}(j) &= \mu ([2j+1] [2j+3])^{-1/2} \\ \hat{b}(j) &= \mu ([2j-1] [2j+1])^{-1/2} \\ \hat{c}(j) &= 0 \end{aligned} \quad (5.10)$$

This is a *degenerate case* (k dependence disappears from these matrix elements and they no longer restrict the range of k). The states can be labelled still as

$$|j m k\rangle_{(\mu)} \quad (5.11)$$

where $j = 0, 1, \dots$; $|m| \leq j$ and q^k is the eigenvalue of q^K . Maintaining the full algebra makes the situation more directly comparable to the other cases

Case 2 (corresponding to (2.13))

$$\begin{aligned}\hat{a}(j, k) &= \frac{\mu}{[j+1]_2} \left(\frac{[j+k+1][j+k+2]}{[2j+3][2j+1]} \right)^{1/2} \\ \hat{b}(j, k) &= \frac{\mu}{[j]_2} \left(\frac{[j-k][j-k-1]}{[2j+1][2j-1]} \right)^{1/2} \\ \hat{c}(j, k) &= \mu \frac{([j-k][j+k+1])^{1/2}}{[j+1]_2 [j]_2}\end{aligned}\tag{5.12}$$

The matrix elements depend on all the three variables and in

$$|jmk\rangle_{(\mu)}$$

as before $-j \leq (k, m) \leq j$, all the three being integer (half-integer) while j can go to infinity.

Case 3 (corresponding to (2.15) and (2.16))

The limiting procedure is subtler here. One possible choice of conventions satisfying all the required constraint (see Appendix) is

(i) for $(j+k)$ even

$$\hat{a}(j) = \frac{\mu q}{[j+1]_2}, \quad \hat{b}(j) = -\frac{\mu}{[j]_2}, \quad \hat{c}(j) = \frac{\mu q^{-j}}{[j+1]_2 [j]_2}\tag{5.13}$$

(ii) and for $(j+k)$ odd

$$\hat{a}(j) = \frac{\mu}{[j+1]_2}, \quad \hat{b}(j) = -\frac{\mu q}{[j]_2}, \quad \hat{c}(j) = \frac{\mu q^{j+1}}{[j+1]_2 [j]_2}\tag{5.14}$$

It has been verified *directly* that these prescriptions indeed solve all the constraints.

Note that though k does not appear explicitly in $(\hat{a} \hat{b} \hat{c})$ two sets are necessary. (thus $\hat{c}(j, k)$ and $\hat{c}(j, k+1)$ are different for the same j . And similarly for $\hat{a}$ and $\hat{b}$.) Interchanging the prescriptions for even and odd $(j+k)$ another possible form of solution is obtained.

Note also that here the symmetry $q \rightarrow q^{-1}$ has been lost in $(\hat{a}, \hat{b}, \hat{c})$. But $q \rightarrow q^{-1}$ in $(\hat{a}, \hat{b}, \hat{c})$ again gives another possible form of solution.

Now in $|jmk\rangle_{(\mu)}$

as before $|m| \leq j, j = \frac{1}{2}, \frac{3}{2}, \dots$

and the range of k is again no longer restricted by the matrix elements.

Corresponding to (2.8) and (2.9) one can unify (5.13) and (5.14) as

$$\hat{a}(j,k) = \frac{\mu q^{1-\hat{\delta}}}{[j+1]_2}, \quad \hat{b}(j,k) = -\frac{\mu q^{\hat{\delta}}}{[j]_2}, \quad \hat{c}(j,k) = \frac{\mu q^{\hat{\delta}+(2\hat{\delta}-1)j}}{[j+1]_2 [j]_2} \quad (5.15)$$

where k appears only through

$$\hat{\delta} \equiv \frac{1}{2} (1 - (-1)^{j+k}) \quad (5.16)$$

Of our three classes only case 2 of (2.13) retains a full-fledged k -dependence, as shown in (5.12).

As compared to (3.7) and (3.8) the Casimir is now

$$\begin{aligned} \hat{A} &= \hat{f}_2 \hat{e}_2 + (q+q^{-1})^{-2} \{ (q \hat{f}_3^{(+)} \hat{e}_3^{(+)} q^{2M} + q^{-1} \hat{f}_3^{(-)} \hat{e}_3^{(-)} q^{-2M}) + \hat{f}_4 \hat{e}_4 \} \\ &= \hat{e}_2 \hat{f}_2 + (q+q^{-1})^{-2} \{ (q^{-1} \hat{e}_3^{(+)} \hat{f}_3^{(+)} q^{2M} + q \hat{e}_3^{(-)} \hat{f}_3^{(-)} q^{-2M}) + \hat{e}_4 \hat{f}_4 \} \end{aligned} \quad (5.17)$$

where, in evident notations (compare (3.1), (3.2))

$$\begin{aligned} \hat{e}_3^{(\pm)} &= q^{\pm 1} e_1 \hat{e}_2 - q^{\mp 1} \hat{e}_2 e_1 \\ \hat{e}_4 &= q^{-1} e_1 \hat{e}_3^{(+)} - q \hat{e}_3^{(+)} e_1 = q e_1 \hat{e}_3^{(-)} - q^{-1} \hat{e}_3^{(-)} e_1 \end{aligned} \quad (5.18)$$

and similarly for $\hat{f}_3^{(\pm)}, \hat{f}_4$.

For the cases 1, 2 and 3 (see 5.10, 5.12, 5.13, 5.14, 5.15) the eigenvalue of $\hat{A}$ turns out to be

$$\hat{A} \approx (q+q^{-1})\mu^2, (q+q^{-1})^2 \mu^2, q (q+q^{-1})^2 (q^2+q^{-2}) \mu^2 \quad (5.19)$$

on the respective spaces spanned by the states indicated. Our choice of simplest expressions for the matrix elements (5.10 to 5.15) introduces q -factors in (5.19). It is easy to modify the constant factors in the matrix elements to obtain, for each case

$$\hat{A}|jmk\rangle_{(\mu)} = \mu^2 |jmk\rangle_{(\mu)} \quad (5.20)$$

For q a root of unity we maintain the formal expressions of the matrix elements (for $q^{\pm M}, q^{\pm K}, e_1, f_2, \hat{e}_2, \hat{f}_2$) unchanged but introduce appropriately fractional parts (f.p.) in the parameters. This is exactly as explained in detail in Sec. 4. A major new consequence, however, is that infinite dimensional contracted representations for generic q can be made, now, finite dimensional by invoking periodicity and quotienting the representation space as before. The corresponding situation for ISU(n) has been discussed elsewhere [4]. One can count the number of parameters, study partially periodic subcases and so on. In this paper, after these brief remarks we turn to the contracted Hopf algebra. Again our task is easy. Contraction of the “top-most” pair (ef) (here e_2, f_2) by retaining the coefficient of the largest parameter (here n_1) leaves the formal structure of the Hopf algebra intact. This was pointed out already for the unitary case

[2,5]. Here for completeness we write down the coproducts, counits and the antipodes of the contracted algebra. Choosing, for example, the symmetric convention,

$$\Delta q^{\pm h_i} = q^{\pm h_i} \otimes q^{\pm h_i} \quad (i=1,2 ; h_i = M, 2h_2 = K-M) \quad (5.21)$$

$$\begin{aligned} \Delta e_1 &= e_1 \otimes q^M + q^{-M} \otimes e_1 \\ \Delta f_1 &= f_1 \otimes q^M + q^{-M} \otimes f_1 \end{aligned} \quad (5.22)$$

$$\begin{aligned} \Delta \hat{e}_2 &= \hat{e}_2 \otimes q^{K-M} + q^{-K+M} \otimes \hat{e}_2 \\ \Delta \hat{f}_2 &= \hat{f}_2 \otimes q^{K-M} + q^{-K+M} \otimes \hat{f}_2 \end{aligned} \quad (5.23)$$

$$\begin{aligned} \varepsilon(q^{\pm h_i}) &= 1 \quad (i = 1,2) \\ \varepsilon(e_1) &= \varepsilon(f_1) = \varepsilon(\hat{e}_2) = \varepsilon(\hat{f}_2) = 0 \\ s q^{\pm h_i} &= q^{\mp h_i} \end{aligned} \quad (5.24)$$

$$\begin{aligned} s e_1 &= -q^M e_1 q^{-M} = -q e_1, \quad s f_1 = -q^M f_1 q^{-M} = -q^{-1} f_1 \\ s \hat{e}_2 &= -q^{K-M} \hat{e}_2 q^{-K+M} = -q^2 \hat{e}_2, \quad s \hat{f}_2 = -q^{K-M} \hat{f}_2 q^{-K+M} = -q^{-2} \hat{f}_2 \end{aligned} \quad (5.25)$$

The prescriptions for $(\hat{e}_2, \hat{f}_2)$ are formally the same as those for (e_2, f_2) . Thus exactly as

$$\begin{aligned} [\Delta e_2, \Delta f_2] &= [e_2, f_2] \otimes q^{2(K-M)} + q^{-2(K-M)} \otimes [e_2, f_2] \\ \text{one has} \quad [\Delta \hat{e}_2, \Delta \hat{f}_2] &= [\hat{e}_2, \hat{f}_2] \otimes q^{2(K-M)} + q^{-2(K-M)} \otimes [\hat{e}_2, \hat{f}_2]. \end{aligned} \quad (5.26)$$

But whereas

$$[\Delta e_2, \Delta f_2] = [\Delta (K-M)]_2$$

Now, since

$$[\hat{e}_2, \hat{f}_2] = 0$$

consistently

$$[\Delta \hat{e}_2, \Delta \hat{f}_2] = 0 \quad (5.27)$$

All formal constraints such as

$$M (s \otimes \text{id}) \Delta = M (\text{id} \otimes s) \Delta = \varepsilon \quad (5.28)$$

can be verified easily.

Thus our approach delivers the Hopt algebra and representations (for both generic and root of unity q) for the contracted algebra, all effortlessly.

Let us now study the effect of our contraction on the canonical generators J_{ij} of (3.16). In a

previous paper [1] we studied systematically, for classical $U(n)$ and $SO(n)$, the consequences of a particular type of contraction and that of a subsequent (classical) deformation. For $SO(5)$ that corresponds to contracting the generators J_{i5} ($i = 1, 2, 3, 4$) in the Gelfand-Zetlin representation leading to

$$SO(5) \rightarrow E(4) \text{ (Eucl. gr. in 4 dim.)}$$

But even at the classical level *the effect of the present contraction* ($e_2, f_2 \rightarrow \hat{e}_2 \hat{f}_2$) *in the Chevalley basis is spectacularly different*. One obtains the following, quite different structure. In the contracted version of (3.16) one now has

$$(i) \text{ an } SO(3) \text{ subalgebra } (J_{12}, J_{23}, J_{13}) \quad (5.29)$$

$$(ii) \text{ two "translation" triplets (with } \hat{J}_{ij} \text{ denoting contraction of } J_{ij}) \\ (\hat{J}_{14}, \hat{J}_{24}, \hat{J}_{34}) \text{ and } (\hat{J}_{15}, \hat{J}_{25}, \hat{J}_{35}) \quad (5.30)$$

each set forming an $E(3)$ subalgebra with (i)

$$(iii) \text{ and with } J_{45} \text{ linking these triplets as} \\ [J_{45}, \hat{J}_{i4}] = -i \hat{J}_{i5}, [J_{45}, \hat{J}_{i5}] = i \hat{J}_{i4} \quad (i = 1, 2, 3) \quad (5.31)$$

The "translations" commute at this stage. The q -deformation induced in this contracted structure through our prescription concerning (e_2, f_2) can now be examined.

We point out the essential features without going into full details. In q -deforming the right hand sides of the contracted version of (3.16), one first has to make a suitable choice for J_{12} and J_{45} .

One can replace M and K by $[M]$ and $[K]$ respectively, for example.

$$\text{There one encounters the doubling of } \hat{J}_{34} \text{ and } \hat{J}_{35} \text{ as} \\ \hat{J}_{34}^{(\pm)} = \frac{1}{2} (\hat{e}_3^{(\pm)} + \hat{f}_3^{(\pm)}), \hat{J}_{35}^{(\pm)} = -\frac{i}{2} (\hat{e}_3^{(\pm)} - \hat{f}_3^{(\pm)}) \quad (5.32)$$

But corresponding to (3.3) one now has

$$\hat{e}_3^{(+)} \pm \hat{e}_3^{(-)} = (q \pm q^{-1}) (\hat{e}_1 \hat{e}_2 \mp \hat{e}_2 \hat{e}_1) \\ \hat{f}_3^{(+)} \pm \hat{f}_3^{(-)} = (q \pm q^{-1}) (\hat{f}_2 \hat{f}_1 \mp \hat{f}_1 \hat{f}_2) \quad (5.33)$$

So again one can eliminate one set (say $\hat{J}_{34}^{(-)}, \hat{J}_{35}^{(-)}$), (or any linear combination) using alongwith (5.32), (5.33), the inverse relations such as

$$\hat{e}_2 = \frac{1}{2} \{ (\hat{J}_{15} - \hat{J}_{24}) - i (\hat{J}_{14} + \hat{J}_{25}) \} \\ \hat{f}_2 = \frac{1}{2} \{ (\hat{J}_{15} - \hat{J}_{24}) + i (\hat{J}_{14} + \hat{J}_{25}) \} \quad (5.34)$$

But, as noted before, using the minimal set one has more complicated looking results.

The Abelian structure of the triplets (ii) is also deformed.

Thus, for example,

$$[\hat{J}_{14}, \hat{J}_{24}] = [\hat{J}_{15}, \hat{J}_{25}] = \frac{i}{2} [e_4, f_4]$$

$$\begin{aligned}
&= \frac{1}{2} (q - q^{-1}) (\hat{f}_3^{(+)} \hat{e}_3^{(+)} q^{2M} - \hat{e}_3^{(-)} \hat{f}_3^{(-)} q^{-2M}) \\
&= \frac{1}{2} (q - q^{-1}) (\hat{e}_3^{(+)} \hat{f}_3^{(+)} q^{2M} - \hat{f}_3^{(-)} \hat{e}_3^{(-)} q^{-2M})
\end{aligned} \tag{5.35}$$

$$\text{where } \hat{e}_3^{(\pm)} = \hat{J}_{34}^{(\pm)} + i \hat{J}_{35}^{(\pm)}, \hat{f}_3^{(\pm)} = \hat{J}_{34}^{(\pm)} - i \hat{J}_{35}^{(\pm)} \tag{5.36}$$

Instead of a long list of complicated commutators of the $\hat{J}$'s, we indicate some key relations which permit their systematic derivations. One has, apart from (5.35) for $[e_4, f_4]$, also the results

$$\begin{aligned}
[\hat{e}_3^{(+)}, \hat{f}_3^{(-)}] &= [\hat{e}_3^{(-)}, \hat{f}_3^{(+)}] = 0 \\
[\hat{e}_3^{(+)}, \hat{f}_3^{(+)}] q^{2M} &= - [\hat{e}_3^{(-)}, \hat{f}_3^{(-)}] q^{-2M} \\
&= - (q^2 - q^{-2}) (q + q^{-1}) \hat{e}_2 \hat{f}_2
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
[\hat{e}_3^{(+)}, \hat{e}_3^{(-)}] &= (q^2 - q^{-2}) \hat{e}_2 \hat{e}_4 \\
[\hat{f}_3^{(+)}, \hat{f}_3^{(-)}] &= - (q^2 - q^{-2}) \hat{f}_2 \hat{f}_4
\end{aligned}$$

Other useful results are quite evident consequences of contractions of (3.4), (3.5) and (3.6). The central lesson is that, apart from the question of possible interpretations of the q -deformed “translations”, such contracted non-remisimples algebras are best studied in the Chevalley basis.

6 - REMARKS

In this paper we have taken care to point out the limitations ($SO(5)_q$ representations with only three and not four variable parameters,...) and tried to be complete within such limits (all such representations for all q 's,...). Possible links with other formalisms remain to be explored. For q a root of unity it would be interesting to compare our ‘fractional part’ formalism closely with others [14, 20]. Cayley-Klein orthogonal algebras have been studied by several authors for classical [21] and quantum [22, 23] cases. Through efficient construction of $SO(5)_q$ representations our formalism can provide a suitable starting point for going beyond C-K $SU(2)$ or $SO(3)$ without sacrificing the parameter q (by making $q \rightarrow 1$ as in “kappa-deformation”). We particularly want to explore periodic representations for $q^N = 1$. Many important applications of q -groups are precisely for q a root of unity. From our point of view finite dimensional, periodic (lattice-like) representations of q -translations represent intriguing possibilities. They were studied elsewhere for contracted unitary algebra [4]. Here we have made it clear that our contraction does not lead to $E(4)$ but to a different non-semisimple structure with two classical triplets. But for the latter q -representations and the full Hopf algebra are obtained completely and simply.

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APPENDIX

Constructions of the reduced matrix elements $a(j,k)$, $b(j,k)$, $c(j,k)$ of equation (2.8).

The m -dependent factors are so chosen in (2.7), (2.8) and (2.9) that it is sufficient to use the constraints

$$e_2^2 e_1 - (q^2 + q^{-2}) e_2 e_1 e_2 + e_1 e_2^2 = 0 \quad (A.1)$$

$$[e_2, f_2] = [2h_2]_2 = [K-M]_2 \quad (A.2)$$

to determine a , b and c . All the others are then automatically satisfied.

From (A.1), inserting the matrix elements of e_1 and e_2 , one obtains the conditions

$$[j+2]_2 a(j,k) c(j+1, k+1) = [j]_2 a(j, k+1) c(j,k) \quad (A.3)$$

$$[j-1]_2 b(j,k) c(j-1, k+1) = [j+1]_2 b(j, k+1) c(j,k) \quad (A.4)$$

and

$$\begin{aligned} & [2j+3] a(j,k) b(j+1, k+1) - [2j-1] a(j-1, k+1) b(j,k) \\ & = - \left(\frac{q^{2j+1} + q^{-2j-1}}{q + q^{-1}} \right) c(j,k) c(j, k+1) \end{aligned} \quad (A.5)$$

(A.2) Leads to the constraints

$$a(j,k) b(j+2,k) = a(j+1, k-1) b(j+1, k-1) \quad (A.6)$$

$$\begin{aligned} & q^{\epsilon(j+2)} b(j+1, k-1) c(j+1, k-1) + q^{-\epsilon(j+2)} a(j,k) c(j+1,k) \\ & = q^{-\epsilon j} c(j, k-1) a(j, k-1) + q^{\epsilon j} c(j,k) b(j+1,k) \quad (\epsilon = \pm 1) \end{aligned} \quad (A.7)$$

$$\begin{aligned} & b^2(j+1, k-1) + a^2(j-1, k-1) - b^2(j,k) - a^2(j,k) \\ & = \left(\frac{q^{2j+1} + q^{-2j-1}}{q + q^{-1}} \right) (c^2(j, k-1) - c^2(j,k)) \end{aligned} \quad (A.8)$$

$$\left\{ \begin{aligned} & q^{2(2j+3)} b^2(j+1, k-1) + q^{-\epsilon(2j-1)} a^2(j-1, k-1) + q^{-\epsilon} c^2(j,k) \\ & - q^{\epsilon(2j-1)} b^2(j,k) - q^{-\epsilon(2j+3)} a^2(j,k) - q^{\epsilon} c^2(j, k-1) \end{aligned} \right\} = - \frac{\epsilon(q-q^{-1})}{(q+q^{-1})} q^{-\epsilon 2k} \quad (\epsilon = \pm 1) \quad (A.9)$$

(After contraction $(a,b,c) \rightarrow (\hat{a}, \hat{b}, \hat{c})$ of Sec 5. the r.h.s. of (A.9) is zero. Other equations remain formally the same.)

The *three solutions* satisfying all the above constraints are given in (2.1.1.), (2.13) and (2.15), (2.16), (2.19). That there can be three and only three solutions can easily be understood by considering the classical limit ($q=1$) and comparing with the Gelfand-Zetlin formalism to which ours must then be equivalent.

The Gelfand-Zetlin states [7] are (in slightly changed notations)

$$\left| \begin{matrix} n_1 & n_2 \\ m_1 & m_2 \\ j & m \end{matrix} \right\rangle \quad (\text{A.10})$$

where there are in general, *four* variable parameters (j, m, m_1, m_2) and the invariant parameters (n_1, n_2) label the irreducible representation. The parameters are all integral or all half integral. They satisfy

$$\begin{aligned} n_1 &\geq m_1 \geq n_2 \geq m_2 \geq -n_2 \\ m_1 &\geq j \geq |m_2| \\ j &\geq |m| \end{aligned} \quad (\text{A.11})$$

So only *three* variable parameters are left when

$$\text{Case 1. } n_2 = 0 \text{ (} m_2 = 0 \text{)} \quad (\text{A.12})$$

$$\text{Case 2. } n_2 = n_1 \text{ (} m_1 = n_1 \text{)} \quad (\text{A.13})$$

These two cases correspond respectively our solutions (2.11) and (2.13). *Note that there the same parameter k replaces m_1 in Case 1 and m_2 in Case 2.*

$$\text{Case 3. } n_2 = \frac{1}{2}$$

This is more subtle. Here m_2 is not quite fixed. Two values are possible (apart from the variations of m_1). For each value of m_1 , $m_2 = \pm \frac{1}{2}$. (A.14)

In our Case 3, k again takes over alone, the repurcussion of the two values of m_2 in (A.14) is seen in the necessity of having the two sets (2.15) and (2.16) or in the presence of δ of (2.18) in (2.19).

Though this paper is restricted to the above-mentioned cases, let us here add some remarks concerning the role of k (th eigenvalue of q^K being q^k) in a more general context where an auxiliary variable (l) is needed to take care of *multiplicities* of states already labelled by (jmk).

We consider the classical case ($q=1$) when the passage from the G-Z to our Chevalley basis representations corresponds essentially to the *diagonalization of the G-Z matrix for the generator J_{45}* . The general transformation seems too difficult to obtain directly. But for $j = n_1$, when $m_1 = n_1$ (for arbitrary n_1 and n_2) it can be shown that (using (A.10)) and GZ matrix elements

$$J_{45} \left(\sum_{m_2} d_{km_2}^{n_2} (-\pi/2) \left| \begin{matrix} n_1 & n_2 \\ n_1 & m_2 \\ n_1 & m \end{matrix} \right\rangle \right) = k \left(\sum_{m_2} d_{km_2}^{n_2} (-\pi/2) \left| \begin{matrix} n_1 & n_2 \\ n_1 & m_2 \\ n_1 & m \end{matrix} \right\rangle \right) \quad (\text{A.15})$$

where d is the standard Wigner rotation matrix.

Thus for $k = n_1$, the range of our k is that of m_2 , namely

$$k = n_2, n_2 - 1, \dots, -n_2 \quad (\text{A.16})$$

and there is *no multiplicity*.

For $n_2 = 0$ there is no problem even afterwards. So we assume

$$n_2 > 0.$$

For $j = n_1 - 1$ (assuming $n_1 - n_2$ large enough)

$$m_1 = n_1, n_1 - 1 \text{ and } m_2 = n_2, \dots, -n_2, m = n_1 - 1, \dots, -(n_1 - 1) \quad (\text{A.17})$$

and hence there are now $2(2n_2 + 1)$ states *for each* m .

In the other scheme one has, for k , the range

$$k = n_2 + 1, n_2, \dots, -n_2, -(n_2 + 1) \quad (\text{A.18})$$

namely $(2n_2 + 1 + 2)$ values.

The discrepancy is (for each value of m)

$$2(2n_2 + 1) - (2n_2 + 3) = (2n_2 - 1)$$

For $j = n_1 - p > n_2$ the difference is

$$(p + 1)(2n_2 + 1) - (2n_2 + 1 + 2p) = p(2n_2 - 1) \quad (\text{A.19})$$

So one now needs a supplementary variable l in the states

$$|j m k l\rangle$$

except when $n_2 = \frac{1}{2}$ when the r. h. s. of (A.19) vanishes. This explains why no l is needed even for $n_2 = \frac{1}{2}$. For $n_2 > \frac{1}{2}$ the multiplicities are distributed in a particular fashion for $j > n_2$ and in another one for $j < n_2$. Consistency is needed at $j = n_2$. We will study this completely in a following paper in which we hope to present most general representations for all q 's. Here as a simple example we note that (for $q = 1$ and hence for generic q) for

$$n_1 = 2, n_2 = 1$$

only the states with $j = 1, k = 0$ has a multiplicity (two fold) for each m ($m = \pm 1, 0$).

A more interesting case, showing already some of the complexity to be expected is the following one.

For, $n_1 = 3, n_2 = 1$ (suppressing the $m = j, \dots, -j$ at each stage)

and $j = 3$, one has $k = \pm 1, 0$ ($m_1 = 3, m_2 = \pm 1, 0$)

$j = 2, k = \pm 2, \pm 1, (0)_2$ ($m_1 = 3, 2, m_2 = \pm 1, 0$)

$j = 1, k = \pm 3, \pm 2, (\pm 1)_2, 0$ ($m_1 = 3, 2, 1; m_2 = \pm 1, 0$)

$j = 0, k = \pm 2, 0$ ($m_1 = 3, 2, 1; m_2 = 0$) (A.20)

The multiplicities are indicated by subscripts, there are 2 states with $k = 0, j = 2$ (for each m) and similarly for $k = \pm 1, j = 1$. Note how the multiplicity jumps from $k = 0$ to $k = \pm 1$.

The generalizations of the constraints from (A.3) up to (A.9) are not difficult to write down for the general states

$$|j m k l\rangle$$

But solving them is another task. Here we have presented all the solutions for the states (with fixed l)

$$|j m k \rangle$$

Excellent checks on these solutions are provided by evaluating the Casimir A (of (3.8) and (3.9)) directly on an arbitrary $|j\ m\ k\rangle$ state instead of using particular (j, m, k) values to simplify the evaluation. One can show easily that the transition elements $(j \rightarrow j \pm 2, j \rightarrow j \pm 1)$ vanish while

$$\begin{aligned} & \langle jmk | \{ f_2 e_2 + (q+q^{-1})^{-2} f_4 e_4 + ((q+q^{-1})^{-2} (q^{2M+1} f_3^{(+)} e_3^{(+)} + q^{-2M-1} f_3^{(-)} e_3^{(-)}) \} | jmk \rangle \\ &= \{ [2j+3] [2j+2] a^2(j,k) + [2j] [2j-1] b^2(j,k) + [j]_2 [j+1]_2 (q^{2j+1} + q^{-2j-1}) c^2(j,k) \} \end{aligned} \quad (A.21)$$

and

$$\begin{aligned} & \langle jmk | \{ e_2 f_2 + (q+q^{-1})^{-2} e_4 f_4 + (q+q^{-1})^{-2} (q^{2M-1} e_3^{(+)} f_3^{(+)} + q^{-2M+1} e_3^{(-)} f_3^{(-)}) \} | jmk \rangle \\ &= \{ [2j] [2j-1] a^2(j,-1,k-1) + [2j+3] [2j+2] b^2(j+1, k-1) + [j]_2 [j+1]_2 (q^{2j+1} + q^{-2j-1}) c^2(j,k-1) \} \end{aligned} \quad (A.22)$$

Then one can inject the values of a, b, c from our solutions and finally this must give the values (3.12), (3.13) and (3.14) respectively. For the contracted case (Sec.5) one has again the results (A.21) and (A.22) but with $(\hat{a}, \hat{b}, \hat{c})$.

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