Exact Yangian Symmetry
in the classical Euler-Calogero-Moser Model

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Abstract
We compute the r-matrix for the elliptic Euler-Calogero-Moser model. In the trigonometric limit we show that the model possesses an exact Yangian symmetry.
1 Introduction

The Euler-Calogero-Moser model was defined in [1, 2]. In [3] we considered the rational case and we derived the $r$-matrix. In this paper we are interested in its trigonometric and elliptic generalizations. In the elliptic case we compute the $r$-matrix and show that the usual elliptic Calogero-Moser model and its $r$-matrix are obtained by Hamiltonian reduction. In the trigonometric case we show that the current algebra symmetry exhibited by Gibbons and Hermsen [1] in the rational case, is deformed into a Yangian symmetry algebra.

We consider a system of $N$ particles on a line with pairwise interactions. The degrees of freedom consist of the positions and momenta $(p_i, q_i); i = 1 \ldots N$ and of antisymmetric additional variables $(f_{ij} = -f_{ji}) i, j = 1 \ldots N$, with the Poisson brackets

\[
\{p_i, q_j\} = \delta_{ij} \tag{1}
\]

\[
\{f_{ij}, f_{kl}\} = \frac{1}{2} (\delta_{il} f_{jk} + \delta_{ki} f_{lj} + \delta_{jk} f_{il} + \delta_{ij} f_{kl}). \tag{2}
\]

The Poisson brackets of the $f_{ij}$ just reproduce the $O(N)$ Lie algebra. The Hamiltonian will be taken of the form

\[
H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{i,j=1}^{N} f_{ij} f_{ji} V(q_{ij}) , \quad q_{ij} = q_i - q_j \tag{3}
\]

with an even potential $V(-x) = V(x)$.

The equations of motion are easily derived:

\[
\dot{q}_i = p_i ,
\]

\[
\dot{p}_i = \sum_{j \neq i}^N f_{ij} f_{ji} V'(q_{ij}) ,
\]

\[
\dot{f}_{ij} = \sum_{k \neq i,j}^N f_{ik} f_{jk} [V(q_{ik}) - V(q_{jk})].
\]

Such a system admits a Lax representation only for specific potentials. Indeed writing the following ansatz for the Lax pair

\[
L(\lambda) = \sum_{i=1}^{N} p_i \epsilon_{ii} + \sum_{i,j=1 \neq i}^{N} l(q_{ij}, \lambda) f_{ij} \epsilon_{ij} \tag{4}
\]

\[
M(\lambda) = \sum_{i,j=1 \neq i}^{N} m(q_{ij}, \lambda) f_{ij} \epsilon_{ij} \tag{5}
\]

where $\epsilon_{ij}$ is the $N \times N$ matrix $(\epsilon_{ij})_{kl} = \delta_{ik} \delta_{jl}$ and $\lambda \in \mathbb{C}$ is the spectral parameter, we find that the equations of motion can be written in the Lax form

\[
\dot{L}(\lambda) = [M(\lambda), L(\lambda)] \tag{6}
\]

if and only if the following equalities are satisfied:

\[
m(x, \lambda) = -\frac{\partial}{\partial x} l(x, \lambda) = -l'(x, \lambda) \tag{7}
\]

\[
l'(x, \lambda) l(y, \lambda) - l'(y, \lambda) l(x, \lambda) = l(x + y, \lambda) [V(x) - V(y)] \tag{8}
\]

\[
l(x) \sim -\frac{1}{x} \text{ when } x \to 0. \tag{9}
\]
Eq. (8) is the famous functional equation of Calogero. Its general solution is [4, 5]

\[
l(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(x) \sigma(\lambda)} \quad V(x) = \varphi(x)
\]

(10)

where \( \sigma \) and \( \varphi \) are Weierstrass elliptic functions, the relevant properties of which are recalled in the appendix.

The elliptic \( O(N) \) Euler-Calogero-Moser model is precisely defined by eq.(3) with \( V(x) = \varphi(x) \) together with the Poisson brackets (1,2).

2 The \( r \)-matrix

From eq.(6) it follows that \( trL^n(\lambda) \) is a set of conserved quantities. In particular

\[
trL(\lambda) = \sum_{i=1}^{N} p_i, \quad trL^2(\lambda) = 2H + \varphi(\lambda).
\]

The involution property of these quantities \( trL^n(\lambda) \) will follow from the existence of an \( r \)-matrix which we now calculate [6, 7]. Introducing the notations \( L_1(\lambda) = L(\lambda) \odot 1 \) and \( L_2(\lambda) = 1 \odot L(\lambda) \) we show that the Poisson brackets of the Lax matrix elements can be recast as

\[
\{ L_1(\lambda), L_2(\mu) \} = [r_{12}(\lambda, \mu), L_1(\lambda)] = [r_{21}(\mu, \lambda), L_2(\mu)].
\]

(11)

Following [3] we assume that \( r \) is of the form

\[
r_{12}(\lambda, \mu) = a(\lambda, \mu) \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i,j=1}^{N} b_{ij}(\lambda, \mu) e_{ij} \otimes e_{ij} + \sum_{i,j=1}^{N} c_{ij}(\lambda, \mu) e_{ij} \otimes e_{ij}.
\]

Requiring that \( r_{12}(\lambda, \mu) \) be independent of the \( p_i \) variables we obtain

\[
b_{ij}(\lambda, \mu) = -b_{ji}(\mu, \lambda)
\]

(12)

\[
c_{ij}(\lambda, \mu) = c_{ij}(\mu, \lambda).
\]

(13)

Moreover assuming that \( r_{12}(\lambda, \mu) \) is independent of the \( f_{ij} \) variables yields the following system:

\[
a(\lambda, \mu) l(q_{ij}, \lambda) - b_{ij}(\lambda, \mu) l(q_{ij}, \mu) + c_{ij}(\lambda, \mu) l(q_{ij}, \mu) = -l(q_{ij}, \lambda)
\]

(14)

\[
b_{ij}(\lambda, \mu) l(q_{jk}, \lambda) - b_{ik}(\lambda, \mu) l(q_{kj}, \mu) = -\frac{1}{2} l(q_{ik}, \lambda) l(q_{ji}, \mu)
\]

(15)

\[
c_{ij}(\lambda, \mu) l(q_{jk}, \lambda) + c_{ik}(\lambda, \mu) l(q_{kj}, \mu) = \frac{1}{2} l(q_{ik}, \lambda) l(q_{ji}, \mu)
\]

(16)

\[
c_{ij}(\lambda, \mu) l(q_{ki}, \lambda) + c_{kj}(\lambda, \mu) l(q_{ik}, \mu) = \frac{1}{2} l(q_{kj}, \lambda) l(q_{ij}, \mu).
\]

(17)

A solution to these equations is

\[
a(\lambda, \mu) = \frac{1}{2} [\zeta(\lambda + \mu) + \zeta(\lambda - \mu)]
\]

\[
b_{ij}(\lambda, \mu) = \frac{1}{2} l(q_{ij}, \lambda - \mu)
\]

\[
c_{ij}(\lambda, \mu) = \frac{1}{2} l(q_{ij}, \lambda + \mu).
\]

(2)
Indeed substituting the preceding expressions in eq.(15,16,17) leads to the same relation:
\[ l(q_{ij}, \lambda - \mu) l(q_{jk}, \lambda) + l(q_{ki}, \mu - \lambda) l(q_{jk}, \mu) + l(q_{ik}, \lambda) l(q_{ji}, \mu) = 0 \]
which upon setting \( x = \frac{1}{2} (\lambda + q_{ij}), \ y = \frac{1}{2} (2\mu - \lambda + q_{ij}), \ z = \frac{1}{2} (\lambda - q_{ki} + q_{kj}) \) is a direct consequence of relation (55). The expression for \( a(\lambda, \mu) \) is then given by eq.(14), and is simplified using eq.(51) and (55). Finally the \( r \)-matrix reads
\[
r_{12}(\lambda, \mu) = \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{N} l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{N} l(q_{ij}, \lambda + \mu) e_{ij} \otimes e_{ij} \\
- \frac{1}{2} [\zeta(\lambda + \mu) + \zeta(\lambda - \mu)] \sum_{i=1}^{N} e_{ii} \otimes e_{ii}. \tag{18}
\]

3 The \( sl(N) \) model

The above \( O(N) \) model can be obtained from the more general \( sl(N) \) model by a mean procedure \([8, 9, 10]\). The \( sl(N) \) elliptic Euler-Calogero Moser model is defined by the Hamiltonian
\[
H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{N} f_{ij} f_{ji} \varphi(q_{ij}) \tag{19}
\]
and the Poisson brackets
\[
\{p_i, q_j\} = \delta_{ij} \tag{20}
\]
\[
\{f_{ij}, f_{kl}\} = \delta_{jk} f_{il} - \delta_{il} f_{kj}. \tag{21}
\]
For this model we define a Lax matrix as
\[
L(\lambda) = \sum_{i=1}^{N} (p_i - \zeta(\lambda) f_{i} e_{ii} + \sum_{i,j=1 \atop i \neq j}^{N} l(q_{ij}, \lambda) f_{ij} e_{ij}. \tag{22}
\]
The Hamiltonian is given by \( H = \frac{1}{2} \int \frac{d^2}{d^2 x} tr L^2(\lambda) \). A direct calculation gives
\[
\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\mu, \lambda), L_2(\mu)] \\
- \sum_{i,j=1 \atop i \neq j}^{N} l'(q_{ij}, \lambda - \mu) (f_{ii} - f_{jj}) e_{ij} \otimes e_{ij} \tag{23}
\]
with the beautifully simple \( r \)-matrix
\[
r_{12}(\lambda, \mu) = -\zeta(\lambda - \mu) \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i,j=1 \atop i \neq j}^{N} l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji}. \tag{24}
\]
At this point let us make two remarks:

- Because of the third term in the right member of eq.(23) the integrals of motion \( tr L^n(\lambda) \) are not in involution. However we can restrict ourselves to the manifolds \( (f_{ii} = \text{constant})_{i=1...N} \) since \( tr L^n(\lambda) \) Poisson-commute with \( f_{ii} \). On these manifolds \( tr L^n(\lambda) \) are in involution.
• The $r$-matrix for the $O(N)$ model eq.(24) is immediately seen to be of the form

$$ r_{12}^{O(N)} = \frac{1}{2} (1 + \sigma \otimes 1) r_{12}^{H(N)} $$

where $\sigma$ is the involutive automorphism

$$ \sigma : \lambda^n e_{ij} \longmapsto -(-\lambda)^n e_{ij}. $$

This is typical of a mean construction.

In the following we will restrict the $f_{ij}$ to a symplectic leaf of the Poisson manifold (21). Introducing vectors

$$(\xi_i)_{i=1\ldots N} \quad \text{with} \quad \xi_i = (\xi^a_i)_{a=1\ldots r}$$

$$(\eta_i)_{i=1\ldots N} \quad \text{with} \quad \eta_i = (\eta^a_i)_{a=1\ldots r}$$

with the Poisson brackets

$$\{\xi^a_i, \xi^b_j\} = 0, \quad \{\eta^a_i, \eta^b_j\} = 0, \quad \{\xi^a_i, \eta^b_j\} = -\delta_{ij} \delta_{ab}, \quad (25)$$

we parametrize the $f_{ij}$ as follows:

$$f_{ij} = \langle \xi_i | \eta_j \rangle = \sum_{a=1}^r \xi^a_i \eta^a_j. \quad (26)$$

The phase space now becomes a true symplectic manifold.

4 The $r$-matrix of the elliptic Calogero model

We show here that the $r$-matrix for the elliptic Calogero model [6, 12] can be obtained from eq.(23) by a Hamiltonian reduction procedure [8, 9, 10].

We choose $r = 1$ in eq.(26). On the manifold $f_{ij} = \xi_i \eta_j$ acts an Abelian Lie group

$$\xi_i \longrightarrow \lambda_i \xi_i, \quad \eta_i \longrightarrow \lambda_i^{-1} \eta_i. \quad (27)$$

Remark that the group acts on $L(\lambda)$ as conjugation by the matrix $\text{diag}(\lambda_i)_{i=1\ldots N}$ and therefore all the Hamiltonians $tr L^a(\lambda)$ are invariant. Thus one can apply the method of Hamiltonian reduction. The infinitesimal generator of this action is

$$H_\alpha = \sum_{i=1}^N \epsilon_i \; f_{ii}, \quad \lambda_i = 1 + \epsilon_i. $$

We fix the momentum by choosing

$$f_{ii} = \alpha.$$

To compute the reduced Poisson brackets of the Lax matrix, we remark that the matrix

$$L^{\text{Cal}}(\lambda) = g^{-1} L(\lambda) g \quad \text{with} \quad g = \text{diag}(\xi_i)_{i=1\ldots N}$$

$$= \sum_{i=1}^N \left[ p_i - \alpha \zeta(\lambda) \right] e_{ii} + \alpha \sum_{i,j=1}^N L(q_{ij,\lambda}) e_{ij} \quad (28)$$

is invariant under the isotropy group $G_\alpha$ of $\alpha$ (which is the whole group itself since it is Abelian) and we can compute the Poisson brackets of its matrix elements directly. We find

$$\{L^{\text{Cal}}_1(\lambda), L^{\text{Cal}}_2(\mu)\} = [r_{12}^{\text{Cal}}(\lambda, \mu), L^{\text{Cal}}_1(\lambda)] - [r_{21}^{\text{Cal}}(\mu, \lambda), L^{\text{Cal}}_2(\mu)] \quad (29)$$

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with

\[ r_{12}^{C;i}(\lambda, \mu) = g_1^{-1} g_2^{-1} \left[ r_{12}(\lambda, \mu) - \{ g_1, L_2(\mu) \} g_1^{-1} + \frac{1}{2} [u_{12}, L_2(\mu)] \right] g_1 g_2 \]

where \( u_{12} = \{ g_1, g_2 \} g_1^{-1} g_2^{-1} \) is here equal to zero. Redefining

\[ r_{12}^{C;i}(\lambda, \mu) \longrightarrow r_{12}^{C;i}(\lambda, \mu) + \left[ \frac{1}{2\alpha} \sum_{i=1}^{N} e_{ii} \otimes e_{ii}, L_2(\mu) \right] \]

does not change eq.(29) and yields exactly the \( r \)-matrix found in \([11, 12]\)

\[ r_{12}^{C;i}(\lambda, \mu) = \sum_{i,j=1}^{N} l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{i,j=1}^{N} l(q_{ij}, \mu) (e_{ii} + e_{jj}) \otimes e_{ij} \]

\[ -[\zeta(\lambda - \mu) + \zeta(\mu)] \sum_{i=1}^{N} e_{ii} \otimes e_{ii}. \] (30)

5 Yangian symmetry in the trigonometric case

The parametrization (26) of \( f_{ij} \) introduces a \( sl(r) \) symmetry into the theory. The transformation

\[ \eta_i^a \longrightarrow \sum_{h=1}^{r} u^{a}_{bh} \eta_i^b \]

\[ \xi_i^a \longrightarrow \sum_{h=1}^{r} (u^{-1})^{ab} \xi_i^b \]

leaves the \( f_{ij} \) invariant and therefore also the Hamiltonians. This symmetry is generated by a set of conserved currents

\[ F_0^{ab} = \sum_{i=1}^{N} e_i^b \eta_i^a. \] (31)

It is remarkable that this current was shown, in the rational case \([1]\), to be the first of a hierarchy building a current algebra commuting with the Hamiltonian — and more generally with a subset of the commuting Hamiltonians.

We now extend this result to the trigonometric case, and we will show that the hierarchy of currents form a Yangian symmetry in this case. Taking the trigonometric limit (\( \omega_1 = \infty \) and \( \omega_2 = i \frac{\pi}{2} \)) in the above formulas, we see that the Lax matrix can be taken of the form

\[ L(\lambda) = L_0 - \coth(\lambda) \frac{F}{\sinh^2(q_{ij})} \] (32)

with

\[ L_0 = \sum_{i=1}^{N} p_i e_{ii} - \sum_{i,j=1}^{N} \coth(q_{ij}) f_{ij} e_{ij}, \quad F = \sum_{i,j=1}^{N} f_{ij} e_{ij}. \] (33)

By a straightforward calculation, or taking the limit of the elliptic case, we find

\[ \{ L_1(\lambda), L_2(\mu) \} = \{ r_{12}^{C;i}(\lambda), \mu) - \{ r_{12}^{C;i}, L_2(\mu) \} \]

\[ \sum_{i,j=1}^{N} (f_{ii} - f_{jj}) \frac{1}{\sinh^2(q_{ij})} \] (34)
where
\[ r_{12}^0 = - \sum_{i \neq j}^N \coth(q_{ij}) \epsilon_{ij} \otimes \epsilon_{ji} \]  
(35)
and \( C \) is the Casimir element of \( sl(N) \)
\[ C = \sum_{i,j=1}^N \epsilon_{ij} \otimes \epsilon_{ji}. \]  
(36)

In spite of the unusual second term in eq.(34) the quantities \( tr L^n(\lambda) \) are still in involution on the manifolds \( \Sigma_a : (f_{ij} = \alpha)_{i=1, \ldots, N} \). Indeed,
\[ \{ tr L^n(\lambda), tr L^m(\mu) \} = n m \sum_{i,j=1}^N \frac{f_{ii} - f_{jj}}{\sinh^2(q_{ij})} [L^{n-1}(\lambda)]_{ij} [L^{m-1}(\mu)]_{ji} \]
\[ - \frac{n m}{2} (1 - \coth(\lambda) \coth(\mu)) \left( tr L^n(\lambda) tr L^m(\mu) [C, F_1 - F_2] \right) \]

and since \( tr_2 ((1 \otimes A) C) = A \), we obtain
\[ \{ tr L^n(\lambda), tr L^m(\mu) \} = n m \sum_{i,j=1}^N \frac{f_{ii} - f_{jj}}{\sinh^2(q_{ij})} [L^{n-1}(\lambda)]_{ij} [L^{m-1}(\mu)]_{ji} \]
\[ - \frac{n m}{2} (1 - \coth(\lambda) \coth(\mu)) tr \{ L^{n-1}(\lambda) [L^{m-1}(\mu), F] - L^{m-1}(\mu) [L^{n-1}(\lambda), F] \}. \]

If we notice that
\[ F = - \frac{L(\lambda) - L(\mu)}{\coth(\lambda) - \coth(\mu)} \]
we immediately get the involution property.

We consider now the subset \( tr(L^n) = tr(L_0 + F)^n \) of commuting Hamiltonians; notice that \( H \) belongs to this subset, since \( H = \frac{1}{L} tr L^2 - \alpha tr L + \frac{1}{2} N \alpha^2 \).

We introduce the following quantities:
\[ J_{ab}^n = tr(L^n F^{ab}), \quad a, b = 1, \ldots, r \quad n = 0, 1, \ldots, \infty \]  
(37)
where \( F^{ab} \) is the \( N \times N \) matrix of elements
\[ (F^{ab})_{ij} = f_{ij}^{ab} = \xi^a_i \eta^b_j. \]  
(38)

We define the generating functional of the currents \( J_{ab}^n \). It is the \( r \times r \) matrix \( T(z) \) of elements
\[ T^{ab}(z) = - \frac{1}{2} \delta^{ab} - \sum_{n \in \mathbb{N}} \frac{1}{z^{n+1}} J_{n}^{ab} = - \frac{1}{2} \delta^{ab} + tr \left( \frac{1}{L - z} F^{ab} \right). \]  
(39)

**Proposition.** On the manifolds \( \Sigma_a \) we have the following two properties:

1. The currents \( J_{ab}^n \) Poisson commute with all the quantities of the form \( tr L^n \).

2. The generating functional \( T(z) \) satisfies the defining relation of a \( (\text{classical}) \) Yangian algebra:
\[ \{ T(y) \otimes T(z) \} = [R(y, z), T(y) \otimes T(z)] \]  
(40)
with
\[ R(y, z) = -2 \frac{\Pi}{y - z}, \quad \Pi = \sum_{a,b=1}^r \epsilon_{ab} \otimes \epsilon_{ba}. \]  
(41)
Proof. To prove this proposition we need the Poisson brackets

\[ \{ L_1, L_2 \} = [r_0^1, L_1] - [r_0^2, L_2] + \sum_{i,j=1}^N \left( f_{ii} - f_{jj} \right) \frac{1}{\sinh^2(q_{ij})} c_{ij} \tag{42} \]

\[ \{ L_1, F_{ab}^{a} \} = [-r_0^0 + C, F_{ab}^{a}] \tag{43} \]

\[ \{ F_{ac}^{a}, F_{bd}^{b} \} = \left( \delta_{ab} F_{cd}^{b} - \delta_{cb} F_{ad}^{a} \right) C. \tag{44} \]

Remark that the currents \( J_{\alpha}^{ab} \) and the Hamiltonians \( tr(L^n) \) are invariant under the symmetry

\[ \xi_{\alpha}^a \rightarrow \lambda_{\alpha} \xi_{\alpha}^a, \quad \eta_{\alpha}^a \rightarrow \lambda_{\alpha}^{-1} \eta_{\alpha}^a. \]

Therefore we can compute their Poisson brackets on the reduced phase space straightforwardly; restricting ourselves to the manifolds \( f_{ii} = \alpha \), the last term in eq.(42) vanishes, and we will systematically drop its contribution in intermediate calculations.

We emphasize that in eqs.(42,43) the same \( r \)-matrix appears. Moreover it is the term \([C, F_{ab}^{a}]\) in eq.(43) which is responsible for the quadratic form of eq.(40), as we shall see in what follows.

Introducing the generating functional \( H(z) = tr(\frac{I}{L^2}) \) of the Hamiltonians \( tr(L^n) \) we compute

\[ \{ \frac{1}{L_1 - y} r_0^1 \left( \frac{1}{L_2 - z} \right)^2 \} = - \left[ \frac{1}{L_2 - z} r_0^1 \frac{1}{L_2 - z} + \frac{1}{L_1 - y} F_{1}^{ab} \right] + \left[ \frac{1}{L_1 - y} r_0^1 \frac{1}{L_1 - y} F_{1}^{ab} \right] \]

\[ + \frac{1}{L_1 - y L_2 - z} \left[ [C, F_{1}^{ab}] \right] \frac{1}{L_2 - z}. \]

Taking the trace we obtain

\[ \{ T_{ab}^{ac}, H(z) \} = tr \left( F_{1}^{ab} \left[ \frac{1}{L - y}, \frac{1}{(L - z)^2} \right] \right) = 0. \]

This proves the first part of the proposition. To prove the second part we evaluate

\[ \{ \frac{1}{L_1 - y} r_0^1 \left( \frac{1}{L_2 - z} \right)^2 \} = - \left[ \frac{1}{L_2 - z} r_0^1 \frac{1}{L_2 - z} + \frac{1}{L_1 - y} F_{1}^{ab} \right] \]

\[ + \left[ \frac{1}{L_1 - y} r_0^1 \frac{1}{L_1 - y} F_{1}^{ab} \right] \frac{1}{L_2 - z} \]

\[ + \frac{1}{L_1 - y L_2 - z} \left( \delta_{ab} F_{1}^{cd} - \delta_{cd} F_{1}^{ab} \right) C \]

\[ + \frac{1}{L_1 - y L_2 - z} \left( [C, F_{1}^{ab}] \right) - \frac{1}{L_2 - z} F_{1}^{cd} \]

\[ + \frac{1}{L_1 - y L_2 - z} \left[ [C, F_{1}^{ab}] \right] \frac{1}{L_2 - z} F_{1}^{cd} \] \]

Hence taking the trace we get

\[ \{ T_{ab}^{ac}, T_{cd}^{de} \} = tr \left( \frac{1}{L - y} \frac{1}{L - z} \left( \delta_{ad} F_{1}^{bc} - \delta_{cd} F_{1}^{ab} \right) \right) \]

\[ + tr \left( \frac{1}{L - y} \left[ L_{1} - z \frac{1}{L_{2} - z} F_{1}^{cd} \right] \right) \]

\[ + tr \left( \frac{1}{L - y} \left[ L_{1} - y \frac{1}{L_{2} - z} F_{1}^{ab} \right] \right) \]

Using the cyclicity of the trace and

\[ \frac{1}{L - y L - z} = \frac{1}{y - z} \left( \frac{1}{L - y} - \frac{1}{L - z} \right) \]

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this becomes

$$\{ T^{ab}(y), T^{cd}(z) \} = \frac{1}{y-z} \left( \delta_{ad} T^{cb}(y) - T^{cb}(z) \right) - \delta_{cd} \left( T^{ad}(y) - T^{ad}(z) \right)$$

$$+ \frac{2}{y-z} tr \left( \frac{1}{L-y} f^{cd} \frac{1}{L-z} f^{ab} - \frac{1}{L-y} f^{ab} \frac{1}{L-z} f^{cd} \right).$$

Remark that

$$tr \left( \frac{1}{L-y} f^{cd} \frac{1}{L-z} f^{ab} \right) = \sum_{ij=1}^{N} \left( \frac{1}{L-y} \right)_{ij} \xi^d_j \eta^a_k \left( \frac{1}{L-z} \right)_{kl} \xi^b_l \eta^c_k$$

$$= \left( \sum_{ij=1}^{N} \left( \frac{1}{L-y} \right)_{ij} \xi^d_j \eta^a_k \right) \left( \sum_{kl=1}^{N} \left( \frac{1}{L-z} \right)_{kl} \xi^b_l \eta^c_k \right)$$

$$= \left( T^{ad}(y) + \frac{1}{2} \delta_{ad} \right) \left( T^{cb}(z) + \frac{1}{2} \delta_{cb} \right).$$

we prove the result (40). □

The rational limit is obtained by applying the canonical transformation

$$p_i \rightarrow \frac{1}{\epsilon} p_i$$

$$q_i \rightarrow \epsilon q_i$$

and sending $\epsilon$ to zero. In this limit

$$L_0 \rightarrow \frac{1}{\epsilon} L_{\text{rational}}$$

$$r_{12}^0 \rightarrow \frac{1}{\epsilon} r_{12}^{0 \text{ rational}}.$$  

The Casimir term drops therefore from eq.(43), leaving us with a linear Poisson algebra

$$\{ T(y) \otimes T(z) \} = -\frac{1}{2} [R(y, z), T(y) \otimes 1 + 1 \otimes T(z)]$$

which is the result found by Gibbons and Hermsen.

6 Conclusion

The Euler-Calogero-Moser model is becoming more and more interesting. On the one hand the computation of the classical $r$-matrix is made considerably easier by the existence of the extra variables $f_{ij}$, the more complicated $r$-matrix of the Calogero-Moser model following naturally from a Hamiltonian reduction procedure. On the other hand, this model exhibits an exact infinite symmetry which is just a current algebra symmetry in the rational case and becomes an exact Yangian symmetry in the trigonometric case. This structure is very much reminiscent of the one discovered in [13]. Actually the two currents $J_0$ and $J_1$ (which generate the full algebra) are identical in the two cases. Indeed, in our case we have

$$J_1^{ab} = \sum_{i=1}^{N} p_if_{ij}^{ab} - \sum_{i,j=1}^{N} \frac{q_{ij} \eta^{a} \eta^{b}}{\sinh(q_{ij})} f_{ij} f_{ij}^{ab}.$$  

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Setting \((X_i)^{ab} = f_i^{ab}, \Theta_{ij} = 2\frac{\varepsilon^{ij}}{\varepsilon_i - \varepsilon_j}\) and using eq. (26) we can rewrite

\[
f_1^{ab} = \sum_{i=1}^{N} m_i X_i^{ab} - \sum_{i,j \neq i}^{N} \Theta_{ij} (X_i X_j)^{ab}.
\]

This is exactly the current found in [13]. In fact the model considered in [14, 15, 13] is a quantum version of our model for a particular choice of orbit.

At this point two interesting problems arise. One is the understanding of the role of the \(r\)-matrix in the quantization of these models. The other is the hypothetical extension of these results to the elliptic case, which still remains quite mysterious.

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**Appendix**

The Weierstrass \(\sigma\) function of periods \(2\omega_1, 2\omega_2\) is the entire function defined by

\[\sigma(z) = z \prod_{m,n \neq 0} \left(1 - \frac{z}{\omega_{mn}}\right) \exp \left[\frac{z}{\omega_{mn}} + \frac{1}{2} \left(\frac{z}{\omega_{mn}}\right)^2\right] \tag{46}\]

with \(\omega_{mn} = 2m\omega_1 + 2n\omega_2\). The functions \(\zeta\) and \(\rho\) are

\[\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}, \quad \rho(z) = -\zeta'(z), \tag{47}\]

these functions having the symmetries

\[\sigma(-z) = -\sigma(z), \quad \zeta(-z) = -\zeta(z), \quad \rho(-z) = \rho(z). \tag{48}\]

Their behaviour at the neighbourhood of zero is

\[\sigma(z) = z + O(z^3), \quad \zeta(z) = z^{-1} + O(z^3), \quad \rho(z) = z^{-2} + O(z^3). \tag{49}\]

Setting

\[l(q, \lambda) = -\frac{\sigma(q + \lambda)}{\sigma(q)} \sigma(\lambda) \tag{50}\]

it is easy to check that

\[l(-q, \lambda) = -l(q, -\lambda), \quad l'(q, \lambda) = l(q, \lambda) [\zeta(\lambda + q) - \zeta(q)]. \tag{51}\]

We need several non trivial relations:

\[-\frac{\sigma(\lambda - \mu) \sigma(\lambda + \mu)}{\sigma(\lambda) \sigma(\mu)} = \rho(\lambda) - \rho(\mu), \tag{52}\]

\[\sigma(x - y)\sigma(x + y)\sigma(z - t)\sigma(z + t) + \sigma(y - z)\sigma(y + z)\sigma(x - t)\sigma(x + t) + \sigma(z - x)\sigma(z + x)\sigma(y - t)\sigma(y + t) = 0, \tag{53}\]

\[\frac{\sigma(2z)}{\sigma(x + z)} \frac{\sigma(x + y)}{\sigma(x - z)} \frac{\sigma(y + z)}{\sigma(y - z)} = \zeta(x + z) - \zeta(x - z) + \zeta(y - z) - \zeta(y + z), \tag{54}\]
this last equation becoming, in terms of the \( l(q, \lambda) \) function,

\[
\frac{l(q, \lambda) \cdot l(-q, \lambda - \mu)}{l(q, \mu)} = \zeta(\lambda) + \zeta(\mu - \lambda) + \zeta(q) - \zeta(\mu + q).
\] (55)

Choosing the periods \( \omega_1 = \infty \) and \( \omega_2 = i\frac{2\pi}{\eta} \), we obtain the hyperbolic case

\[
\sigma(z) = \sinh(z) \exp\left(-\frac{z^2}{6}\right), \quad \zeta(z) = \coth(z) - \frac{z}{3}, \quad \rho(z) = \frac{1}{\sinh^2(z)} + \frac{1}{3}
\] (56)

and

\[
l(q, \lambda) = -\frac{\sinh(\lambda + q)}{\sinh(\lambda)\sinh(q)} \exp\left(-\frac{\lambda q}{3}\right).
\] (57)

All these formulas were collected in [11].

References

Leningrad 1979.