Viscosities of Quark-Gluon Plasmas

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The quark and gluon viscosities are calculated in quark-gluon plasmas to leading orders in the coupling constant by including screening. For weakly interaction QCD and QED plasmas dynamical screening of transverse interactions and Debye screening of longitudinal interactions controls the infrared divergences. For strongly interacting plasmas other screening mechanisms taken from lattice calculations are employed. By solving the Boltzmann equation for quarks and gluons including screening the viscosity is calculated to leading orders in the coupling constant. The leading logarithmic order is calculated exactly by a full variational treatment. The next to leading orders are found to be very important for sizable coupling constants as those relevant for the transport properties relevant for quark-gluon plasmas created in relativistic heavy ion collisions and the early universe.
I. INTRODUCTION

Transport and relaxation properties of quark and gluon (QCD) plasmas are important in a number of different contexts. They determine the time that it takes a quark-gluon plasma formed in a heavy-ion collision to approach equilibrium, and they are of interest in astrophysical situations such as the early universe, and possibly neutron stars.

The basic difficulty in calculating transport properties of such plasmas, as well as of relativistic electron-photon (QED) plasmas, is the singular nature of the long-range interactions between constituents, which leads to divergences in scattering cross sections similar to those for Rutherford scattering. This makes the problem of fundamental methodological interest, in addition to its possible applications. The first approaches to describe the transport properties of quark-gluon plasmas employed the relaxation time approximation [1-3] for the collision term. This approximation simplifies the collision integral enormously and transport coefficients are related directly to the relaxation time. The latter is typically estimated from a characteristic cross section times the density of scatterers. In Refs. [2,3] the divergent part of the total cross section at small momentum transfers was assumed to be screened at momentum transfers less than the Debye momentum. However, Debye screening influences only the longitudinal (electric) part of the QED and QCD interactions, and the transverse (magnetic) part is unscreened in the static limit.

Recently it has been shown that the physics responsible for cutting off transverse interactions at small momenta is dynamical screening [4]. This effect is due to Landau damping of the exchanged gluons or photons. Within perturbative QCD and QED rigorous analytical calculations of transport coefficients to leading order have been made for temperatures high [4,5] as well as low [6] compared with the chemical potentials of the constituents.

Transport processes depend on a characteristic relaxation time, $\tau_{nr}$, of the particular transport process considered. For example, in high temperature plasma the viscosities, $\eta_i = u_i \tau_{nr}/5$, of particle type $i$ are proportional to the characteristic times for viscous relaxation, $\tau_{nr}$, which were first calculated in [4] to leading order in the coupling constant. More generally one finds that the typical transport relaxation rates, that determines momentum stopping, thermal and viscous relaxation, is in a weakly interacting QCD plasma

$$\frac{1}{\tau_{nr}} \propto \alpha_s^2 \ln(1/\alpha_s) T + O(\alpha_s^2).$$

where the expansion is in terms of the fine structure constant $\alpha_s = g^2/4\pi$. The coefficients of proportionality to the leading order in $\alpha_s$ (in the following called the leading logarithmic order) has been calculated analytically for a number of transport processes in high temperature plasmas [4,5]. Likewise in a QED plasma the typical transport relaxation rates for viscous processes, momentum stopping, thermal and electrical conduction has same dependence as (1) on the QED fine structure constant $\alpha [5]$.

The dependence of the transport rates on the coupling constants is very sensitive to the screening. Besides factor $\alpha_s^2$ from the matrix element squared of the quark and gluon interactions, the very singular QCD interactions for small momentum transfers lead to a logarithmic term in the high temperature quark-gluon plasma. The typical particle lifetime, $\tau_{min} \propto 1/\alpha_s$, for small momentum transfers of order $q_{min} \sim q_D$. This gives the leading logarithmic order in the coupling constant, $\ln(T/q_D) \sim \ln(1/\alpha_s)$, to the transport (1).

The calculations in [4,5] were brief and dealt only the leading logarithmic order in the coupling constant, with a given ansatz for the distribution function. More detailed calculations of the quark and gluon viscosities in the high temperature quark-gluon plasma presented. The leading logarithmic order is calculated exactly by a variational method and the next to leading order - the $\alpha_s^2$ term in (1) - is calculated as well. Both $\alpha_s$ is not exponentially small, the next to leading order is important in many realistic physical situations - relativistic heavy ion collisions and the early universe. Thermore when the Debye screening length is large, the interparticle screening, which occurs when $\alpha_s \geq 1$, we shall see below, Debye and dynamical screening is minimal. Instead lattice gauge calculations have found quark-gluon plasmas seem to develop a constant screening mass, $m_{pl} \simeq 1.1T$, for temperatures $T \gtrsim 2 - 3T_c$. It is important to see what effects this alternative screening mechanism has in strongly interacting plasmas.

We shall first describe in section II the transport theory we use, namely the Boltzmann equation, and the screening of long range QCD and QED interactions. In section III, we describe the process of shear flow and the additional calculation necessary in order to find the viscosity. In section IV we then evaluate the collision term to leading logarithmic order with a simplifying ansatz for trial function and refer to Appendix A for a full actual variational calculation. In section V we calculate viscosity to higher orders in the coupling constant and discuss strongly interacting plasmas. Finally, in section VI we give a summary and discuss generalizations of methods developed here to other transport coefficients.

II. TRANSPORT THEORY

Transport processes are most easily described by the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + v_1 \cdot \nabla_x + F \cdot \nabla p_i\right) n_1 = 2 \pi v_2 \sum_{234} |M|^2$$

where $v_1$ are the fluid velocities, $F$ the forces, $n_1$ the equilibrium distribution functions, and $M$ the scattering matrices. The sum is over all processes that contribute to the transport coefficients. The coupling constant is $\alpha_s = g^2/4\pi$. The energy and momentum conserving collisions are described by $M$ and the energy dependent cross sections $v_2$. The distribution function is normalized to $\int n_1 d^3x = N$, where $N$ is the number of particles. The time evolution of the distribution function is determined by the collision term.

In the high temperature limit, the collision term is dominated by small momentum transfers, $q_{min} \sim q_D$.

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\[ M_{jj}^2 = \frac{g^4}{4} \left( 3 - \frac{us}{t^2} - \frac{st}{u^2} - \frac{ut}{s^2} \right), \]

where \( s, t, \) and \( u \) are the usual Mandelstam variables. In Eq. (3) the double counting of final states has been corrected for by inserting a factor 1/2. For quark-gluon scattering

\[ |M_{qg}|^2 = g^4 \left( u^2 + s^2 \right) \left( \frac{1}{t^2} - \frac{4}{9us} \right), \]

and for scattering of two different quark flavors

\[ |M_{q_1 q_2}|^2 = \frac{4}{9} g^4 \frac{s^2 + u^2}{t^2}. \]

The matrix element for scattering of the same quark flavors or quark-antiquark scattering is different at large momentum transfer but the same as (5) at small momentum transfers.

The \( u^{-2} \) and \( v^{-2} \) singularities in Eq. (3-5) lead to diverging transport cross sections and therefore vanishing transport coefficients. Including screening, it was shown in [4-6] that finite transport coefficients are obtained. In fact, the leading contribution to transport coefficients comes from these singularities. In the \( t = \omega^2 - q^2 \) channel the singularity occurs for small momentum \( q \) and energy \( q \) transfers (see Fig. (1)).

For small momentum transfer, \( q \ll \epsilon_1, \epsilon_2 \sim T \), energy conservation implies that \( \omega = \epsilon_1 - \epsilon_2 \simeq v_1 \cdot q = -v_2 \cdot q \) where \( v_1 = p_i \). Therefore the velocity projections transverse to \( q \) have lengths \( |v_{1,T}| = |v_{2,T}| = \sqrt{1 - \mu^2} \), where \( \mu = \omega/q \). Consequently \( v_{1,T} \cdot v_{2,T} \simeq (1 - \mu^2) \cos \phi \), where \( \phi \) is the angle between \( v_{1,T} \) and \( v_{2,T} \). For \( q \ll T \) we thus have

\[
s \simeq -u \simeq 2 p_1 p_2 (1 - \cos \theta_{12}) \simeq 2 p_1 p_2 (1 - \mu) (1 - \cos \phi),
\]

and the interactions splits into longitudinal and transverse ones, [8]

\[ |M_{gg}|^2 = \frac{g^4}{4} \left( \frac{1}{q^2 + \Pi_L} - \frac{1}{q^2 - \omega^2 + \Pi_T} \right)^2. \]

The interactions are modified by the self-energies, \( \Pi_L \) and \( \Pi_T \) [8] (see also Fig. 1). In the random-phase approximation the polarization functions in the long wavelength limit are

\[ \Pi_L(q, \omega) = q^2 D \left( 1 - \frac{\mu}{2} \ln \left( \frac{\mu + 1}{\mu - 1} \right) \right), \]

\[ \Pi_T(q, \omega) = q^2 D \left( \frac{\mu^2}{2} + \frac{\mu(1 - \mu)}{4} \ln \left( \frac{\mu + 1}{\mu - 1} \right) \right), \]

where \( \mu = \omega/q \) and \( q_D = 1/\lambda_D \) is the Debye wave number.

In a weakly-interacting high temperature QCD plasma [8,9]

\[ q_D^2 = 4 \pi (1 + N_f/6) \alpha_s T^2, \]

where \( \alpha_s = g^2/4\pi \) is the fine structure constant for quark-gluon interactions, the factor \( (1 + N_f/6) \) is the sum of contributions from gluon screening, the "1, -" and light quark self energies, which give the Debye and dynamical screening lengths.

One should keep in mind the self energies and (9) are only valid in the long wavelength limit for \( q \ll T \). When \( q \sim T \) other contributions of \( \alpha_s q T \) enter (see, e.g., [8]) which may be gauge dependent [14]. However, as long as \( \alpha_s \) is small all contributions from the self energies can be ignored in gluon screening, where the gauge matrix element already carry the order \( \alpha_s^2 \).

In the above derivations we have consistently assumed that the screening was provided in RPA by the self energies which give the Debye and dynamical screening lengths for longitudinally and transverse interactions respectively. Both effects provide a natural cut-off for the effective current momentum transfer less than \( q_{\text{min}} \sim q_D \). These perturbative ideas must, however, break down when the screening length becomes as short as the interparticle spacing, i.e. \( q_D \sim T \), or in terms of the coupling constant \( \alpha_s \sim 4 \pi (1 + N_f/6))^{-1} \sim 0.1 \) according to Eq. (10). Notice gauge calculations of quark-gluon plasma above the phase transition, \( T_c \approx 170 \) MeV, one finds strong nonperturbative effects in the plasma that the typical screening mass is \( m_{qg} \approx 1.1 T \) [12] and one may argue [13] that perturbation theory still applies. Large momentum transfers so that the matrix elements are given by the simple Feynman tree diagrams, but perturbation theory does not apply for small momentum transfers of order \( q \sim q_D \) and then one should rather use the effective cutoff found by lattice gauge calculations

\[ \Pi_L \approx \Pi_T \approx m_{qg}, \quad \alpha_s \geq 0.1. \]

The phenomenological screening mass of (11) provides with a method to extend our calculations of transport coefficients to larger values for \( \alpha_s \) and it can be combined with the Debye and dynamical screening in weakly interacting quark-gluon plasmas.
III. THE VISCOSITY

With screening included in the interaction we can now proceed to calculate transport properties as the viscosity. In the presence of a small shear flow, $u(y)$, in the $x$-direction we obtain from the Boltzmann equation

$$p_1 x y \frac{\partial n_1}{\partial y} - \frac{\partial n_1}{\partial x} = 2\pi n_2 \sum_{\frac{3\pi}{2}} |M|^2 \left[ n_1 n_2 (1 + n_3) (1 + n_4) - (1 + n_1) (1 + n_2) n_3 n_4 \right] \times \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \delta_{p_1 + p_2 p_3 + p_4}. \quad (12)$$

For small $u$ we can furthermore linearize the quasiparticle distribution function

$$n_i = n_i^{\text{LE}} + \frac{\partial n_1}{\partial x} \Phi \frac{\partial n_1}{\partial y}, \quad (13)$$

where the local equilibrium distribution is

$$n_i^{\text{LE}} = (\exp(\epsilon_i - u \cdot p)/T) \mp 1)^{-1}, \quad (14)$$

and $\Phi$ is an unknown function that represents the deviations from local equilibrium. By symmetry $\Phi$ has to be on the form

$$\Phi = p_2 y f(p/T), \quad (15)$$

where now the function $f$ must be determined from the Boltzmann equation. Inserting (13) in the Boltzmann equation we find

$$p_1 x y \frac{\partial n_1}{\partial y} - \frac{\partial n_1}{\partial x} = 2\pi n_2 \sum_{\frac{3\pi}{2}} |M|^2 \left[ n_1 n_2 (1 + n_3) (1 + n_4) - (1 + n_1) (1 + n_2) n_3 n_4 \right] \times \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \delta_{p_1 + p_2 p_3 + p_4} \times (\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4). \quad (16)$$

It is very convenient to define a scalar product of two real functions by:

$$\langle \psi_1 | \psi_2 \rangle = -n_2 \sum_p \psi_1(p) \psi_2(p) \frac{\partial n}{\partial p}. \quad (17)$$

Thus Eq. (16 may be written on the form $[X] = I|\Phi\rangle$ where $[X] = p_2 y$ and $I$ is the integral operator acting on $\Phi$. The viscosity is given in terms of $\Phi$ [10] and can now be written as

$$\eta = -n_2 \sum_p p_2 y \frac{\partial n}{\partial p} \Phi = \langle X | \Phi \rangle. \quad (18)$$

Equivalently, the viscosity is given from (16 as

$$\eta = \frac{\langle X | \Phi \rangle^2}{\langle \Phi | I | \Phi \rangle}. \quad (19)$$

Since $\langle X | \Phi \rangle$ defines an inner product, the quantity $\langle X | \Phi \rangle^2/\langle \Phi | I | \Phi \rangle$ is minimal for $X = \Phi$ with the minimal value $\eta$. Equation (19 is therefore convenient for a functional treatment, which will be carried out in Appendix A.

To find the viscosity we must solve the integral equation (16 for which we have to evaluate

$$\langle \Phi | I | \Phi \rangle = 2\pi n_2 \sum_{\frac{3\pi}{2}} |M|^2 \left[ n_1 n_2 (1 + n_3) (1 + n_4) \right] \times (\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4)^2 \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4). \quad (12)$$

Momentum conservation requires that $p_3 = p_1 + p_4 = p_2 - q$ where $q$ is the momentum transfer. Introducing an auxiliary integral over energy transfers, the delta-function in energy can be written

$$\delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) = \int dq_1 \int dq_2 \int d\omega \frac{p_3}{p_1 q} \delta(\cos \theta_1 - \mu - \frac{1}{2p_1 q} \times \frac{p_3}{p_2 q} \delta(\cos \theta_2 - \mu + \frac{1}{2p_2 q}).$$

where $\theta_1$ is the polar angle between $q$ and $p_1$ and $\theta_2$ the corresponding one between $q$ and $p_2$ (see Fig. 1). Consequently, we find

$$\langle \Phi | I | \Phi \rangle = \frac{1}{4\pi T} \int dq \int d\omega \frac{dp_1}{-q} \left. \int dp_2 \int dp_3 \int dp_4 \right. \times \frac{1}{2\pi} \frac{1}{2\pi} \left. \right. \frac{1}{2\pi} \left. \right. \delta(\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4)^2. \quad (12)$$

This integral equation for $\Phi$ has been solved in several cases under simplifying circumstances. For ex., in liquids the sharp Fermi surface restricts all particles near the Fermi surface and with a simplified model for the scattering matrix element techniques have been developed to calculate a number of transport coefficients exactly [10]. For the QCD and QED plasmas the regular interaction can, once screened, be exploited as an expansion at small momentum transfers. An analytical calculation of the transport coefficients allows an expansion at small momentum transfers.

IV. VISCOSITY TO LEADING LOGARITHMIC ORDER

In Ref. [4], through solution of the Boltzmann equation, the first viscosity of a quark-gluon plasma derived to leading logarithmic order in the QCD coupling constant [4-6].

\[ \text{Ref. [4]} \]
The total viscosity, to leading order, is an additive sum of the gluon and quark viscosities, $\eta = \eta_g + \eta_q$.

The leading logarithmic order comes from small momentum transfers because of the very singular matrix element (7) dominates. For small $q$ the kinematics simplify enormously and, as we will now show, the integrals separate allowing almost analytical calculations. First, we can set the lower limits on the $p_1$ and $p_2$ integrals to zero, however, then replacing the upper limit on $q$ by the natural cutoff from the distribution functions which is $q_{max} \sim T$. Thus we find from (22)

$$
\langle \Phi | i | \Phi \rangle = \frac{1}{2\pi^3 T} \int_0^\infty d p_1 p_1^2 \sigma_1 (1 \pm n_1) 
\times \int_0^\infty d p_2 p_2^2 \sigma_2 (1 \pm n_2) 
\times \int_{q_{min}}^{q_{max}} d q q \int_0^1 d \mu \int_0^{2\pi} d \phi \frac{1}{2 \pi}
\times |M|^2 (\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4)^2.
$$

(23)

to leading logarithmic order

The solution to the integral equation or equivalently the variational calculation of (19) is quite technical and is for that reason given in Appendix A. A much simpler calculation is to make the standard assumption in viscous processes, i.e., to take the trial function as

$$
\overline{f(p/T)} = (p/T)^2.
$$

(24)

As will be shown in Appendix A this turns out to be a very good approximation. It is accurate to more than 99% for reasons also explained in the appendix. $f$ can be defined up to any constant which cancels in (19) and therefore never enters in the viscosity.

The quantity $\langle \Phi_1 + \Phi_2 - \Phi_3 - \Phi_4 \rangle^2$ can be averaged over $x$- and $y$-direction while keeping $\mu$ and $\phi$ fixed. This corresponds to keeping the relative positions of the three vectors $q$, $p$, and $p'$ fixed relative to each other and rotating this system over the three Euler angles (see also Appendix A). Consequently, we obtain

$$
\langle \Phi_1 + \Phi_2 - \Phi_3 - \Phi_4 \rangle^2 = \frac{q^2}{15T^4} \left[ 3(p_2 - p_1)^2 + (q \cdot (p_2 - p_1))^2 \right]
\times 2p_1 p_2 (4\mu^2 + 3(1 - \mu^2) \cos \phi) \right].
$$

(25)

The integrals over $p_1$ and $p_2$ in (23) are elementary. Next we perform the integrations over $\mu$ and $\phi$ required in (23). We note in passing that the term in (25) proportional to $p_1 p_2$ vanishes and that $\mu^2$ effectively can be replaced by $1/3$ (see Appendix A). Let us first consider the case of gluon-gluon scattering inserting $|M_{gg}|^2$ from (3). We thus find

$$
\langle \Phi | i | \Phi \rangle = \frac{2\pi^3}{45} \alpha_s^2 \int_0^{q_{max}} q^2 d q \int_0^1 d \mu \frac{1}{|q^2 + 2\pi^2 T^2(\mu)|^{1/2} + 1/2}
$$

This integral is discussed in detail in appendix B. The longitudinal interactions $\Pi_L \approx q_D^2$ due to screening and the leading term is a logarithm of maximum ratio of minimum momentum transfer $\ln(q_{max}/4m) \sim \ln(T/qD)$. Likewise for the transverse interactions $\Pi_T \approx \ln(T/qD)$ due to Landau screening and the dependence on $\mu = \omega/q$ provides suitable screening to render the integral finite and the leading term is the same logarithm as for the longitudinal interactions. Whereas the details of the screening are important for the leading logarithmic order, they are less important for the higher orders and they are calculated in detail in Appendix B. The final result is thus to leading logarithmic order

$$
\langle \Phi | i | \Phi \rangle \approx \frac{2\pi^3}{15} \alpha_s^2 \ln(T/qD) T^3.
$$

Since $\langle \Phi | X | \Phi \rangle = (64\pi^5/27) T^3$, we find from (19) and (3)

$$
\eta_{gg} = \frac{2\pi^5}{\pi^2} \alpha_s^2 \ln(T/qD)
\simeq 0.342
\alpha_s^2 \ln(1/\alpha_s),
$$

to leading logarithmic order in $\alpha_s$.

To obtain the full gluon viscosity we must add scattering on quarks and antiquarks which is calculated gonsly and only has a few factors different. Firstly (3) and (4) we see that the matrix element square a factor $4/9$ smaller. Secondly, the statistical factor $p_2 = 12N_f$ instead of 16. Thirdly, in integrating the factor $(p_1^2 + p_2^2)$ in Eq. (25) we note that the distribution function, $n_2$, in Eq. (23) is now a fermion. Consequently, the $p_1$ and $p_2$ integrations give a factor $(1/2 + 7/8)/2$ less for gluon-quark collisions as compared to gluon-gluon collisions and we find

$$
\eta_g = \eta_{gg} \alpha_s^2 \ln(T/qD).
$$

In [4] the slightly different result $\eta_g = \eta_{gg}/(1 + \eta)$ was obtained.

The quark viscosity can be obtained analogously to the gluon one. The quark viscosity due to collisions with quarks only, $\eta_{qg}$, deviates from $\eta_{gg}$ by a factor $(4/9)$ the matrix elements and differences in having Fermi-Bose integrals. By comparing to (19,18,20) we find

$$
\eta_{qg} = \eta_{gg} \frac{(15/16)^2}{(4/9)^2 (7/8)/(1/2)} = \eta_{gg} \frac{5^2}{28^2}.
$$
Note that the statistical factors $\nu$ cancel in $\eta_{gg}$ and $\eta_{q0}$. Including quark scatterings on gluons lead to similar factors in (\Phi/\Phi^2, namely a factor (9/4) from the matrix element, a factor $16/12N_f$ from statistics, and a factor $(8/7 + 2)/2$ from Bose instead of Fermi integrals. Thus

$$\eta_0 = \frac{\eta_{gg}}{1 + 11N_f/48} \approx 0.429, \quad (31)$$

which for $N_f = 2$ results in $\eta_0 = 4.4\eta_{gg}$, a quark viscosity that is larger than the gluon one partly because the gluons generally interact stronger than the quarks and partly because of differences between Bose and Fermi distribution functions.

V. Viscosity to Higher Orders in $\alpha_s$

The leading logarithmic order dominates at extremely high temperatures, where the running coupling constant is small, but it is insufficient at lower temperatures. The next to leading order correction to the viscous rate in the coupling constant is of order $\alpha_s^2$. It may be significant because the leading logarithm is a slowly increasing function. In the derivation of the leading logarithmic order, Eq. (27), we have been very cavalier with any factors entering in the logarithm, which are of order $\alpha_s^2$. It was only argued that the leading logarithmic order $\ln(q_{\text{max}}/q_{\text{min}}) \sim \ln(T/q_D)$ because $q_{\text{max}}$ and $q_{\text{min}}$ were of order $\sim T$ and $\sim q_D$ respectively. Finally, if thermal quark-gluon plasmas are created in relativistic heavy ion collisions at CERN and RHIC energies, the temperatures achieved will probably be below a GeV. We can thus estimate the interaction strength from the running coupling constant $\alpha_s \approx 6\pi/(33 - 2N_f)\ln(T/A)$ which, with $A \approx 150$MeV and $T \approx 1$GeV, gives $\alpha_s \approx 0.4$. For such large coupling constants Delye and dynamical screening is replaced by an effective screening mass, $m_{\text{pl}}$, as discussed above which will affect the viscosity considerably.

To calculate the viscosity to order $\alpha_s^2$ exactly, the 5-dimensional integral of (22) must be evaluated numerically and at the same time a variational calculation of $\Phi$ must be performed. This is a very difficult task and we shall instead use the information obtained in the previous section, that the trial function $f \propto p^2$ is expected to be an extremely good approximation. With that ansatz for the trial function, it is then straightforward to calculate the integral of (21) numerically and find the viscosity to order $\alpha_s^2$ for the given screening mechanism. The 5-dimensional numerical evaluation of the collision integral of (21) is a complicated function of the coupling constant. It is convenient to write it in terms of the function $Q$

$$\langle \Phi/\Phi \rangle_{gg} = \frac{2^7\pi^3}{15} \alpha_s^2 Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) T^3, \quad (32)$$

where the index $gg$ refers to gluon-gluon scattering but the analogous definitions apply to gluon-quark and quark-quark scattering. The function $Q$ and the effective maximum and minimum momentum transfer and $q_{\text{min}}$, are given in Appendix B. In weakly interacting plasmas, where the screening is provided by both photonic and dynamical screening, the function $Q$ is basically a logarithm of the ratio of the maximum and minimum momentum transfer, i.e.,

$$Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = \ln \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right), \quad \alpha_s \lesssim 0.1$$

By numerical integration we find that the distribution functions lead to an effective cutoff of $q_{\text{max}} \approx 3T$. This is because the distribution functions are weighted several powers of particle momenta and thus count the most for $p \approx 3T$. The effective cutoff is slightly larger for quark-gluon and quark-quark scattering because the Fermi distribution functions emphasize large momenta more than the Bose ones. Delye and dynamical screening lead to $q_{\text{min}} \approx 1.26q_D$ as described in Eq. (B6) and so

$$Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = \ln \left( \frac{0.44}{\alpha_s(1 + 6/7)} \right), \quad \alpha_s \lesssim 0.1$$

The numerical factor inside the logarithm, which is of the order $\alpha_s^2$, is discussed in more detail in Appendix B.

In the other limit, $q_D \lesssim T$ or equivalently $\alpha_s \lesssim 0.2$, the perturbative ideas breaks down and we assume an effective screening mass taken from lattice calculations, $q_D \approx 1.1T$, as described by Eq. (11). Thus we find (see also Eq. (B10))

$$Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = 0.626, \quad \alpha_s \lesssim 0.1,$$

and similarly for quark-gluon and gluon-gluon scattering $Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = 0.819$ and $Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = 0.819$, respectively.

Adding gluon-gluon and gluon-quark scatterings, we obtain the gluon viscosities

$$\eta_g = \frac{2^7\pi^3}{15} \alpha_s^2 \left[ Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) + \frac{11N_f}{48} \right]^{-1},$$

which extends Eq. (37) to higher orders. In weakly interacting plasmas (36) reduces to

$$\eta_g \approx 0.342 \frac{T^3}{\alpha_s^2} \ln \left( \frac{0.44}{\alpha_s(1 + 6/7)} \right) + \frac{11N_f}{48} \ln \left( \frac{0.72}{\alpha_s(1 + 6/7)} \right)^{-1}, \quad \alpha_s \lesssim 0.1$$

to leading orders in $\alpha_s$. In strongly interacting plasmas, we obtain by inserting (B10) in (32)

$$\eta_g \approx 0.55 \frac{T^3}{\alpha_s^2} \left[ 1 + \frac{11N_f}{48} \right]^{-1}, \quad \alpha_s \lesssim 0.1.$$
In Fig. (3) we show the gluon viscosity with the various assumptions for screening. With dash-dotted curve the result of Eq. (38) assuming a constant screening mass, \( m_{p1} = 1.17 \), is shown. With dashed curve the numerical result assuming Debye and dynamical screening of Eqs. (8) and (9) is shown. For \( \alpha_s \gtrsim 0.05 \) it is given by Eq. (37) to a good approximation whereas for \( \alpha_s \gtrsim 0.1 \) the result of Eq. (B4) is better. The final viscosity shown by full curve is obtained by combining the two limits, i.e., applying Debye and dynamical screening in weakly interacting plasmas when \( qD \gtrsim T \) or equivalently \( \alpha_s \lesssim 0.1 \) but an effective screening mass \( m_{p1} = 1.17 \) as given by Eq. (11) when \( \alpha_s \gtrsim 0.1 \). This corresponds to choosing the smallest value of the viscosities as seen in Fig. (3), i.e., the two limits of Eqs. (37) and (38).

Similarly, adding quark-quark and quark-gluon scatterings we find the quark viscosity

\[
\eta_q = \frac{5\pi^3\zeta(5)^2}{2^33^2\pi^7} N_f \frac{\alpha_s^2}{\alpha_s^2} \left[ Q \left( \frac{\langle qg \rangle}{q_{\text{max}}} \right) + \frac{7N_f}{33} Q \left( \frac{\langle qg \rangle}{q_D} \right) \right]^{-1},
\]

which in weakly interacting plasmas gives

\[
\eta_q \simeq 0.752 N_f \frac{\alpha_s^2}{\alpha_s^2} \left[ \ln \left( \frac{0.72}{\alpha_s(1 + N_f/6)} \right) + \frac{7N_f}{33} \ln \left( \frac{1.15}{\alpha_s(1 + N_f/6)} \right) \right]^{-1}, \alpha_s \lesssim 0.1,
\]

and in the strongly interacting plasmas

\[
\eta_q \simeq 0.92 N_f \frac{\alpha_s^2}{\alpha_s^2} \left[ 1 + 1.25 \frac{7N_f}{33} \right]^{-1}, \alpha_s \gtrsim 0.1.
\]

The quark viscosity increases with the number of quark flavors, \( N_f \), whereas the gluon viscosity decreases as can be seen in Fig. (4) and (5), where the viscosities are shown for two and three flavors respectively. The total viscosity of a quark-gluon plasma, \( \eta = \eta_g + \eta_q \), is dominated by the quark viscosity.

From the definition of the viscosity in terms of the collision integral (18) and (20), which only contains positive quantities, it follows trivially that the viscosity is positive as is a physical necessity. The resulting viscosities of Eqs. (36) and (39) are positive quantities whereas the \( \alpha_s \lesssim 0.1 \) expansions of Eqs. (37) and (40) are not when extended to the region \( \alpha_s \gtrsim 0.5 \). This explains the results found in [11], where it was claimed that estimates of the next to leading order \( \alpha_s^2 \) could lead to a negative viscosity.

Contributions from vertex corrections should also be considered. In fact for the calculation of the quasiparticle damping rate, \( \gamma_p \), Braaten and Pisarski [15] found that vertex corrections contributed to leading order \( \gamma_p^{(g)} \approx 6.6 \alpha_s \) for zero gluon momenta, \( p \). Vertex corrections do also contribute to order \( \alpha_s \) for large quasiparticle momenta, \( p \gg gT \), but they can here be ignored since the leading order is \( \gamma_p^{(g)} = 3\alpha_s \ln(1/\alpha_s) \) as expected [16]. For the viscosity, however, vertex corrections can be ignored since the extra vertices adds a factor \( \alpha_s^2 \), though integration over soft momenta may cancel a factor \( \alpha_s \) the result is still of higher order in the coupling constant.

Writing each of the viscosities \( \eta_i \) (\( i = q,g \)) in terms of the viscous relaxation time, \( \tau_{\eta,i} \), as

\[
\eta_i = \frac{\alpha_s^2}{\alpha_s^2} \left[ \frac{W_q^{(g)}}{\tau_{\eta,g}} + \frac{W_g^{(g)}}{\tau_{\eta,g}} \right], \quad \tau_{\eta,g} = \frac{\pi^3}{3^35^2\zeta(5)} T^4 \eta_q,
\]

and for gluons and quarks respectively, we can write the viscous relaxation rate for gluons

\[
\frac{1}{\tau_{\eta,g}} = \frac{W_g^{(g)}}{\tau_{\eta,g}} = \frac{W_g^{(g)}}{\tau_{\eta,g}} = \frac{\pi^3}{3^35^2\zeta(5)} T^4 \eta_q,
\]

where \( W_g^{(g)} = (32\pi^3/45) T^4 \) and \( W_q^{(g)} = (N_f T^2/15) \), the gluon and quark enthalpies respectively, we can write the viscous relaxation rate for quarks

\[
\frac{1}{\tau_{\eta,q}} = \frac{W_q^{(g)}}{\tau_{\eta,q}} = \frac{W_q^{(g)}}{\tau_{\eta,q}} = \frac{\pi^3}{3^35^2\zeta(5)} T^4 \eta_q,
\]

The viscous relaxation times, \( \tau_{\eta,g} \), \( \tau_{\eta,q} \) and \( \tau_{\eta} = 1/\tau_{\eta,g} \) are thus very similar to the corresponding viscous times (times temperature) as well as when divided by a factor of \( T^4 \). The curves on (4) and (5) therefore applies to the viscous relaxation times (times temperature) as well when divided a factor of ~ 1.4 and ~ 0.92 \( N_f \) for gluons and quarks respectively according to Eq. (42).

In weakly interacting plasma the viscous rates can be approximated by

\[
\frac{1}{\tau_{\eta,g}} \simeq 4.11 \alpha_s^2 \left[ \ln \left( \frac{0.44}{\alpha_s(1 + N_f/6)} \right) + \frac{11N_f}{48} \ln \left( \frac{0.72}{\alpha_s(1 + N_f/6)} \right) \right], \quad \alpha_s \lesssim 0.1
\]

and

\[
\frac{1}{\tau_{\eta,q}} \simeq 1.27 \alpha_s^2 \left[ \ln \left( \frac{0.72}{\alpha_s(1 + N_f/6)} \right) + \frac{7N_f}{33} \ln \left( \frac{1.15}{\alpha_s(1 + N_f/6)} \right) \right], \quad \alpha_s \lesssim 0.1,
\]

to leading orders in \( \alpha_s \).

\[\text{VI. SUMMARY}\]

By solving the Boltzmann equation for quarks and gluons the viscosities in quark-gluon plasmas were calculated to leading orders in the coupling constant. Inc
of dynamical screening of transverse interactions, which controls the infrared divergences in QED and QCD, is essential for obtaining finite transport coefficients in the weakly interacting plasmas. The solution of the transport process was extended to strongly interacting plasmas by assuming an effective screening mass of order $m_{\phi} = 1.1 T$, as found in lattice calculations, when the Debye screening length becomes larger than the interparticle distance or when $\alpha_s \geq 0.1$. The Boltzmann equation was solved exactly to leading logarithmic order numerically but the result only differed by less than a percent from an analytical result obtained by a simple ansatz for the deviation from local equilibrium, $\Phi \propto p_x p_y$. The next to leading orders was also calculated and found to be very important for the transport properties relevant for quark-gluon plasmas created in relativistic heavy ion collisions and the early universe. For $\alpha_s \geq 0.1$ we find $\eta = C_{i,j} T^3/\alpha_s^2 \ln(C_{i,j}/\alpha_s)$ whereas for $\alpha_s \geq 0.1$ we find $\eta = C_{i,j} T^3/\alpha_s^2$ with coefficients $C_{i,j}$ given above.

The viscosity in degenerate plasmas of quarks, i.e., for $T \ll \mu_q$ was calculated in [6]. Several differences were found. In the high temperature quark-gluon plasma the chemical potential can be ignored and the transport processes depend on two momentum scales only, namely $T$ and $q_D \sim g T$. In degenerate quark matter three momentum scales enter, namely $\mu_q$, $T$, and $q_D \sim g \mu_q$, and the transport process depends considerably on which of $q_D$ and $T$ is the larger. In fact for $T \ll q_D$ transverse interactions turn out to be dominant in contrast to the high temperature quark-gluon plasma where transverse and longitudinal interactions contribute by similar magnitude. Furthermore, the existence of a relative sharp Fermi surface allows an almost analytical calculation of both the leading (logarithmic) order as well as the next order $\alpha_s^2$.

The techniques for calculating the viscosities to leading orders in the coupling constants can be applied to other transport coefficients as well. The leading logarithmic orders to momentum stopping, electrical conductivities and thermal dissipation in QCD and QED plasmas have been estimated with simple ansätze for the distribution functions in [5]. Based on the experience with the viscosity studied here, we do not expect the leading logarithmic order for these transport coefficients to decrease by much when a full variational calculation is performed. The next to leading logarithmic order for these transport coefficients can also be estimated with the experience obtained above for the viscosity. Good estimates are obtained if one in (B5) replaces $q_{\text{max}}$ by the average particle momenta entering the collision integral for the relevant transport process and $q_{\text{min}} \sim q_D$. A few transport coefficients are, however, different. The second viscosity $\zeta$ is zero for a gas of massless relativistic particles [1] and one cannot define a thermal conductivity in a plasma of zero baryon number. One can, however, consider thermal dissipation processes [5] where the leading orders also can be calculated with the above methods. The effective soft cutoff will, however, be different for thermal dissipation processes as described because the transport of energy introduce dependence on $\omega$ which also is present in the transverse screening integral $\Pi_T(\omega/q)$.

All the transport processes discussed above depend only on momentum scales from the typical quark mass, $q_{\text{max}} \sim T$ down to the Debye screening wavenumber $q_{\text{min}} \sim q_D \sim g T$ which also is the momentum scale for dynamical screening. There is, however, a shorter momentum scale of order the magnetic moment cutoff $m_{\text{mag}} \sim g^2 T$, at which perturbative ideas of the quark-gluon plasma fail [17]. As shown in [16] the quark-gluon quasiparticle decay rates depend on this inner cutoff, $m_{\text{mag}}$. Furthermore, recent studies [19] find the color diffusion and conductivity also depend on this cutoff and therefore the rate of color relaxation is around $1/\alpha_s$ larger than Eq. (1).

**ACKNOWLEDGMENTS**

This work was supported by DOE grant No. AC03-76SF00098, NSF grant No. PHY 89-21021, and the Danish Natural Science Research Council. Discussions with Gordon Baym and Chris Pethick are gratefully acknowledged.

**APPENDIX A: EXACT VARIATIONAL CALCULATION TO LEADING LOGARITHMIC ORDER**

In this appendix we solve the Boltzmann equation to find the deviation from local equilibrium, $\Phi$, by a variational treatment of Eq. (18).

For a general function $\Phi = p_x p_y f(p/T)$ we have:

$$\Phi_1 + \Phi_2 - \Phi_3 = p_1 x p_2 y f(p_1) + p_2 x p_3 y f(p_2) - p_3 x p_4 y f(p_3) - p_4 x p_0 y f(p_4)$$

$$= -(q_x p_y + q_y p_x) f(p) - p_x p_y f_1(p) + (q_x p_y + q_y p_x) f(p') + p_x p_y f_1(p'),$$

where we have changed notation to $p = p_1 + q/2 = q/2$ and $p' = p_2 - q/2 = p_4 + q/2$. We have used the energy conserving $\delta$-functions of (21) implies $pq = \mu$. Furthermore, have defined the function

$$f_1(p) = p^2 d(f/p^2)/dp = p^2 - 2 f,$$

that vanishes when $f \propto p^2$ which was the case for the ansatz used in section V.

For small momentum transfer the matrix element depends only on energy and momentum transfer and the azimuthal angle $\phi$. The $\delta$-functions of (21) taking energy conservation fixes the polar angles $\theta_1$ and $\theta_2$ with respect to $q$. Thus all the angular integrals for $f$ and $\phi$ reduces to rotating the three vectors $q$, $p$
over all Euler angles keeping them fixed relatively to each other. Only \( (\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4)^2 \) depends on the Euler angles and the integration or averaging over the three Eulerian angles, while keeping the relative positions of the vectors \( q, p \) and \( p' \) fixed, i.e. keeping \( \mu \) and \( \phi \) fixed, gives

\[
(q_x p_y + q_y p_x)^2 = (q_x p'_y + q_y p'_x)^2 = \frac{3 + \mu^2}{15}, \tag{A3}
\]

\[
(q_x p_y + q_y p_x)(p_x p'_y + p_y p'_x) = \frac{2}{15} \mu^2, \tag{A4}
\]

\[
(q_x p'_y + q_y p'_x)(p'_x p'_y + p'_y p'_x) = \frac{2}{15} \mu^2, \tag{A5}
\]

\[
(q_x p_y + q_y p_x)(q'_x p'_y + q'_y p'_x) = \frac{1}{15}(3pp' + \mu^2). \tag{A6}
\]

\[
(q_x p_y + q_y p_x)(q'_x p'_y + q'_y p'_x) = \frac{1}{15}(3pp' + \mu^2), \tag{A7}
\]

\[
(p_x p'_y + p_y p'_x)(p'_x p'_y + p'_y p'_x) = \frac{\mu^2}{15}(3pp' - 1), \tag{A8}
\]

\[
(p_x p'_y + p_y p'_x)(p'_x p'_y + p'_y p'_x) = \frac{\mu^2}{15}(3pp' - 1). \tag{A9}
\]

where \( p = p/q \) and \( p' = p'/q' \). Since we assume that the plasma temperature is much larger than any of the particle masses, the particles are relativistic and \( p, p', q \) are unit vectors. The vector product of \( p \) and \( p' \) is most useful in terms of \( \mu \) and \( \phi \) (see Eq. (2))

\[
pp' = \mu^2 + (1 - \mu^2) \cos \phi. \tag{A10}
\]

Next we integrate over \( \mu \) and \( \phi \). The \( \mu \) integration averages \( \mu^2 \) to \( 1/3 \) whereas the \( \phi \) integration is weighted by a factor \( (1 - \cos \phi)^2 \) from the matrix elements. Thus we find that (A7-A9) vanishes whereby all combinations mixing \( p \) and \( p' \) very conveniently disappear. After averaging over both Euler angles and \( \mu \) and \( \phi \) we obtain

\[
\langle (\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4)^2 \rangle = \frac{q^2}{p^2} \frac{1}{15}(10f^2 + 2j^2 + 4fj), \tag{B1}
\]

\[
= \frac{q^2}{p^2} \frac{2}{15}(f^2 + \frac{1}{6} p^2 j^2). \tag{B2}
\]

Let us first consider the pure gluon plasma for which (23) gives

\[
\langle \Phi \Phi \rangle = \frac{8\pi}{15} g^2 \ln(T/qD) T^3 \times \int_\infty^\infty \left[ f^2 + \frac{1}{6} p^2 j^2 \right] \left( -\frac{\partial n}{\partial \epsilon_p} \right) \, dp, \tag{A12}
\]

where \( n = (\exp(p/T) - 1)^{-1} \) is the gluon distribution function. Since

\[
\langle \Phi \rangle = \frac{8\pi}{15} g^2 \ln(T/qD) T^3 \times \int_\infty^\infty \left[ f^2 + \frac{1}{6} p^2 j^2 \right] \left( -\frac{\partial n}{\partial \epsilon_p} \right) \, dp.
\]

we find from (18)

\[
\eta = \frac{8\pi g^2 \ln(T/qD)}{15} \left( \int_0^\infty f(\epsilon_f + \frac{1}{6} x^2 \epsilon_f' n)^n \, d\epsilon_f \right)^2,
\]

where \( x = p/T \). As mentioned above, the function \( \epsilon \) is determined by minimizing (A14). A functional variation with respect to \( f \) results in a second order inhomogeneous differential equation for \( f \)

\[
f'' + \left( \frac{2}{x} + \frac{n''}{n} \right) f' - \frac{6}{x} f = -C \epsilon\n\]

where \( n''/n' = -(1+2n) \). \( C \) is an arbitrary constant by rescaling \( f \), which can be chosen as \( C = 2 \) for convenience.

For \( x \gg 1 \) we can approximate \( n \approx 0 \) and so we find a solution to (A15), that does not increase exponentially for \( x \to \infty \), to be

\[
f(x) = x, \quad x \gg 1.
\]

For \( x \ll 1 \) we can approximate \( n \approx 1/x \) and the solution to (A15) that is finite at the origin is

\[
f(x) = C^{-1} \ln(x), \quad 0 \leq x \ll 1,
\]

where \( C \approx 0.7 \) is a constant that can only be determined by finding the full solution to (A15) and matching (A17). This is done by a numerical Runge-Kutta integration and the result is shown in Fig. 6. The viscosity is now found by inserting \( f \) in (A14). The exact value \( \eta \) thus obtained is only 0.523% less than the approximate value, \( \eta_{AB} \), of Eq. (38). Since the exact value is a variational minimum, it has to be smaller than that obtained for \( x \approx 2 \) for large as \( n \) the ansatz of (24) and \( f \) is mainly sampled over values \( \approx p' \) because the integrals over \( p \) and \( p' \) have powers \( \approx \mu^2 \) to \( \approx \mu^2 \) times \( n_p (1 + n_p) \). In Fig. 6 the variational calculation with trial functions \( f(p) \) lead to a minimal viscosity for \( n = 2.104 \). This result is close to the quadratic power of (A16) but tends slightly towards the asymptotic form of (A17) (see also Fig. 7). It has almost the same slope and curvature as the solution around \( p = 5T \) (note that the absolute value is unimportant since it cancels in the viscosity). The corresponding viscosity was 0.364% smaller than that of quarks, i.e., in between the exact result and the ansatz for gluons. The above analysis was restricted to a pure gluon plasma. As mentioned above the distribution function are weighted with several powers of momentum and do not find much difference between fermions and bosons. Therefore the deviation from local equilibrium for \( \eta \) will be not much different from quarks as well as from gluons, and one can be confident that the ansatz, \( \Phi \propto p_f p_f \) of Eq. (24) will serve as a good approximation for quarks as well accurate within less than a percent.
APPENDIX B: SOFT AND HARD CONTRIBUTIONS

The essential contribution to \( \langle \Phi \rangle \) is the integral

\[
Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = \frac{1}{3} \int_{\frac{1}{2}}^{1} d \mu \int_{0}^{q_{\text{max}}} q^{3} dq \left[ \frac{1}{q_{\mu}^{2} + \Pi_{L}(\mu)} \right]^{1/2} + \frac{1}{q_{\mu}^{2} + \Pi_{T}(\mu)/(1 - \mu^{2})^{2}}. \tag{B1}
\]

For dimensional reasons the function \( Q \) can only depend on the ratio of \( q_{\text{max}} \) to the momentum scale, \( q_{\text{min}} \), which is provided by the screening. For Debye and dynamical screening \( q_{\text{min}} \sim q_{D} \) whereas lattice calculations of strongly interacting plasmas give \( q_{\text{min}} \sim m_{q} = 1.17 T \).

As described in connection with screening non-perturbative effects become important when \( q_{D} \gtrsim T \) which corresponds to \( \alpha_{s} \lesssim 0.1 \). We shall treat the two limits separately starting with the weakly interacting plasmas for which the gluon self-energies, \( \Pi_{L}(\mu) \), are given by Eqs. (8) and (9). It is straightforward to calculate \( Q \) numerically and the result will be given below, but let us first make a simple analytical estimate. The main contribution to this integral can be obtained by including the leading terms in the self-energies (8, 9)

\[
\Pi_{L}(q, \omega) \simeq q_{D}^{2}, \tag{B2}
\]

\[
\Pi_{T}(q, \omega) \simeq i \frac{\pi}{4} \mu q_{D}^{2}. \tag{B3}
\]

Thus we find for (B1)

\[
Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = \frac{1}{3} \ln \left( 1 + \frac{q_{\text{max}}^{2}}{q_{D}^{2}} \right) - \frac{q_{\text{max}}^{2}}{q_{D}^{2}} + \frac{q_{\text{max}}^{2}}{q_{D}^{2}} \right) + \frac{1}{4} \ln \left[ 1 + \frac{q_{\text{max}}^{2}}{q_{D}^{2}} \left( \frac{4}{\pi} \right) \right] + \frac{2}{q_{\text{max}}^{2}} \pi \mu q_{D}^{2} A_{T} \left( \frac{\pi}{4} \frac{q_{D}^{2}}{\mu q_{\text{max}}} \right) \right]. \tag{B4}
\]

Expanding in the limit \( q_{\text{max}} \gg q_{D} \) or equivalently for small \( \alpha_{s} \) we obtain the leading orders up to \( \alpha_{s}^{2} \) in the coupling constant

\[
Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) \simeq \ln \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right), \quad q_{\text{max}} \gg q_{D}, \tag{B5}
\]

where

\[
q_{\text{min}} = q_{D} \exp \left( \frac{1}{6} \left( 1 - \ln \frac{4}{\pi} \right) \right) \simeq 1.13 q_{D}. \tag{B6}
\]

The two terms in (B5) corresponding to \( \ln q_{\text{max}} \) and \( \ln q_{\text{min}} \) are often referred to as “hard” and “soft” contributions in the literature [11].

A numerical evaluation of (B1) with \( \Pi_{L}, \Pi_{T} \) given by Eqs. (8) and (9) instead of (B2) and (B3) gives a slightly larger value for the effective minimum momentum transfer because the additional terms in \( \Pi_{L}, \Pi_{T} \) lead to some dimensional screening besides the Debye screening and Landau damping of (B2) and (B3). This effective cutoff is determined by the screening only and is therefore the same for quark-quark, quark-gluon and quark-quark scattering. Whereas \( q_{\text{min}} \) may serve as an effective “cutoff” of momentum transfers, it is not a parameter put in by hand as discussed in [20]. Contrarily, it is caused and determined by Debye and dynamical screening.

If the transverse interactions are assumed to be screened like the longitudinal ones, i.e., \( \Pi_{T} = q_{D}^{2} \) instead of (B3), the result would have been \( q_{\text{min}} = q_{D} \exp (0.5) \approx 1.65 \).

This is because dynamical screening of Eq. (B3) is more effective than the Debye screening of (B2) and thus occurs in a smaller \( q_{\text{min}} \).

It is convenient to express the results in terms of the effective minimum screening masses of order \( q_{D} \) and \( q_{q} \) in weakly interacting plasmas we find

\[
Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) \simeq \ln \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = \frac{1}{2} \ln \left( \frac{q_{\text{max}}/T^{2}}{4 \pi^{2} \alpha_{s} (1 + N_{f}/6)} \right), \quad \alpha_{s} \lesssim 0.1. \tag{B7}
\]

The upper effective cutoff \( q_{\text{max}} \) is provided by the screening of quark and gluon distribution functions as discussed in connection with Eq. (22) and it therefore varies with particle type. Because Bose distributions emphasize smaller momenta than Fermi ones, the upper cutoff is larger for quarks. We find \( q_{\text{max}}^{(q)} = 3.0 T, q_{\text{max}}^{(g)} = 1.65 T \) and \( q_{\text{max}}^{(gg)} = 4.8 T \) for gluon-gluon, quark-gluon, quark-quark scattering respectively. The lower effective cutoff \( q_{\text{min}} \) is, however, the same for the three cases because it only depends on the screening in the gluon propagator. Furthermore, we find that the extra terms in the matrix elements of (3), (4) and (5) besides the \( T^{2} \) do not contribute much since they have varying signs and turn out to be partially cancelling. Thus the constant within the logarithms of (37) and (40) just reflects the different \( q_{\text{max}} \) for gluon-gluon, gluon-quark and quark-quark scattering.

Lacking screening of transverse interactions in a static limit, it has often been assumed that some anisotropy like Debye screening might lead to screening of transverse interactions as well, i.e., \( m_{q} \approx q_{D} \). Lattice gauge calculations of QCD plasmas have shown that effective screening masses of order \( m_{q} \approx 1.17 T \) near the phase transition point, \( T_{c} \approx 180 \text{ MeV} \). In both cases, thus assumed that

\[
\Pi_{L} = \frac{\Pi_{T}}{(1 - \mu^{2})} = \frac{m_{q}}{\mu q_{\text{max}}}, \tag{B8}
\]

in (B1) which leads to

\[
Q \left( \frac{q_{\text{max}}}{q_{\text{min}}} \right) = \frac{1}{2} \ln \left( 1 + \frac{q_{\text{max}}^{2}}{m_{q}^{2}} \right) - \frac{q_{\text{max}}^{2}}{m_{q}^{2} + q_{\text{min}}^{2}}. \tag{B9}
\]

\( q_{\text{min}} = 1.26 q_{D} \).
With \[ q_{\text{max}}^{(g)} = 3.0T, \ q_{\text{max}}^{(q)} = 3.8T, \ q_{\text{max}}^{(qq)} = 4.8T \] and \[ m_{\text{pl}} = 1.1T \] we find \[ Q(q_{\text{max}}^{(g)}/q_{\text{min}}) = 0.626, \ Q(q_{\text{max}}^{(q)}/q_{\text{min}}) = 0.819 \] and \[ Q(q_{\text{max}}^{(qq)}/q_{\text{min}}) = 1.024. \] These values enter the \( \alpha_s \geq 0.1 \) expressions of Eqs. (38) and (41).

FIG. 1. Feynman diagram for gluon-gluon scattering in the t-channel. The lines in the loops can be either quark or gluon propagators.

FIG. 2. The collision geometry. For small momentum transfer, $q \ll p_1, p_2$, energy and momentum conservation requires $\cos \theta_1 = \cos \theta_2 = \omega/q$.

FIG. 3. The gluon viscosity for $N_f = 3$ assuming Debye and dynamical screening (dashed curve), a constant screening mass $m_{\pi^*} = 1.1T$ (dashed-dotted curve) and the minimal one (full curve).

FIG. 4. The quark, gluon and total viscosities for $N_f = 3$.

FIG. 5. The quark, gluon and total viscosities for $N_f = 2$.

FIG. 6. The function $f / x^2$ as determined by (A15). Also shown are the limits of (A16), (A17) and the simple ansatz $f = x^2$. 

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