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Unitarity Constraints on Meson Wavefunctions*

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ABSTRACT

We consider the constraints imposed on meson wave functions by unitarity and by the favored values of the semileptonic decay constants. We show that they require unexpectedly large values of the intrinsic transverse momentum, especially in the case of the B meson. We also obtain a constraint on the heavy quark momentum fraction.

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In this letter, we show that nontrivial bounds from unitarity can be derived, relating meson decay constants to the transverse momentum distribution of the quark constituents.

A heuristic overview of our procedure is as follows. We form positive-definite integrals containing the two-particle wave function $\psi_{q\bar{q}}(x, k_\perp)$, whose integral corresponds to the meson decay constant f_h , and its square, whose integral is constrained by unitarity. We use the condition of positivity to derive constraints on the behavior of the wave function.

The decay constant is defined by

$$\frac{f_h}{2\sqrt{2}N_c} = \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} \psi_{q\bar{q}}(x, k_\perp; Q) + O(Q^{-2}) \quad (1)$$

The wave function $\psi_{q\bar{q}}$ is weakly dependent on the separation scale Q ; we will ignore this dependence, and assume that Q is much larger than a typical intrinsic transverse momentum. We will always assume that the decay constant is real and positive, thus fixing the phase of $\psi_{q\bar{q}}$. For pseudoscalar mesons, the decay constant f_h is entirely independent of Q .

A unitary wave function must also satisfy

$$\int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} |\psi_{q\bar{q}}|^2 \leq 1 \quad (2)$$

Now, consider a region R of $(x, k_\perp)$ -space. Define

$$A[R] = \int_R \frac{dx \, d^2 k_\perp}{16\pi^3},$$

$$F[R] = 2\sqrt{2N_c} \operatorname{Re} \left[\int_R \frac{dx \, d^2 k_\perp}{16\pi^3} \psi_{q\bar{q}}(x, k_\perp) \right].$$

Thus A is the “area” of the region R , and F is the part of the decay constant contributed by the region R . Then we have

$$0 \leq \int_R \frac{dx \, d^2 k_\perp}{16\pi^3} |\psi_{q\bar{q}}(x, k_\perp) - b|^2 \leq 1 - \frac{bF[R]}{\sqrt{2N_c}} + b^2 A[R],$$

so that

$$F[R] \leq \sqrt{2N_c} \frac{1 + b^2 A[R]}{b} \quad (3)$$

for all regions R and positive real b . This modest equation turns out to have significant consequences.

Setting $b = A[R]^{-1/2}$, we obtain

$$F[R] \leq 2\sqrt{2N_c A[R]}; \quad (4)$$

the contribution to the decay constant from the region R cannot exceed this bound without violating unitarity.

For example, consider the π meson, whose decay constant is $f_\pi = 133$ MeV.

Define

$$\delta\mathcal{M}^2 \equiv \frac{k_\perp^2}{x} + \frac{k_\perp^2}{1-x} - m_\pi^2 = \frac{k_\perp^2}{x\bar{x}} - m_\pi^2,$$

and let $R \equiv \{(x, k_\perp) : \delta\mathcal{M}^2 \leq \mathcal{M}_{\max}^2\}$. Then $A[R] = (\mathcal{M}_{\max}^2 + m_\pi^2)/96\pi^2$, and (with $N_c = 3$) we have

$$F[R] \leq \frac{\sqrt{\mathcal{M}_{\max}^2 + m_\pi^2}}{2\pi}. \quad (5)$$

Thus unitarity requires that half of the pion decay constant must arise from the region in which $\delta\mathcal{M}^2 > 0.15$ GeV², and at least 13% from the region where $\delta\mathcal{M}^2 > 0.5$ GeV². The contribution to the wave function from regions of sizable intrinsic transverse momentum is thus very substantial.

Even more severe constraints can be derived for the B meson, due to the unexpectedly large decay constant $f_B \simeq 240$ MeV [2,3] and to the expectation that the heavy b quark should carry the bulk of the longitudinal momentum. In this case, we define

$$\delta\mathcal{M}^2 \equiv \frac{k_\perp^2}{1-x} + \frac{k_\perp^2 + m_b^2}{x} - m_B^2 = \frac{k_\perp^2}{x(1-x)} + \frac{m_b^2}{x} - m_B^2.$$

Defining R as above, we obtain the bound

$$F[R] \leq \frac{(\mathcal{M}_{\max}^2 + m_B^2 - m_b^2)^{3/2}}{2\pi(\mathcal{M}_{\max}^2 + m_B^2)}. \quad (6)$$

Using $m_B = 5.28$ GeV and $m_b = 5$ GeV, we find that half of f_B must be due to the region in which $\delta\mathcal{M}^2 > 5.8$ GeV², and 19% to $\delta\mathcal{M}^2 > 10$ GeV²! The numerically large value of f_B can only be consistent with unitarity if the B wave function in the $q\bar{q}$ state is greatly spread out in momentum space. For example, at $x = 0.9$ the value $\delta\mathcal{M}^2 = 10$ GeV² corresponds to $|k_\perp| = 0.95$ GeV, and at $x = 0.8$ to

Fig. 1. Contours of constant $\delta\mathcal{M}^2$ in the $xk_\perp^2$ -plane for the B meson. Along the dotted line, the z -component of velocity is the same for the quark and antiquark. Note that as $\delta\mathcal{M}^2$ increases, x tends to decrease.

$|k_\perp| = 1.03$ GeV; see Fig. 1. Such large transverse momenta, and sizable values of $1 - x$, must be typical of the B meson.

If we make the further assumption that the wave function depends only on the virtuality of the intermediate state, $\psi(x, k_\perp) = \psi(\delta\mathcal{M}^2)$ and the measure of integration over the invariant phase space is

$$\int_{m_b^2 - m_B^2}^{\delta\mathcal{M}^2 + m_B^2} \frac{(\delta\mathcal{M}^2 + m_B^2 - m_b^2)^2 (\delta\mathcal{M}^2 + m_b^2 + 2m_B^2) d\delta\mathcal{M}^2}{96\pi^2 (\delta\mathcal{M}^2 + m_b^2)^3} . \quad (7)$$

It is then a simple problem in the calculus of variations to maximize the functional

$$\langle x^1 \rangle \equiv \frac{\int x \psi(x, k_\perp)}{\int \psi(x, k_\perp)} , \quad (8)$$

the first moment of the distribution amplitude, subject to the constraints of unitarity and a fixed value of f_B . The extremal function has the form

$$\psi \propto (x_{\min} - x_0) \theta(x_{\min} - x_0) \quad \text{where} \quad x_{\min} \equiv \frac{m_b^2}{\delta\mathcal{M}^2 + m_B^2 - m_b^2} ,$$

and we obtain the rigorous bound for any positive function $\psi(\delta\mathcal{M}^2)$

$$\langle x_b^1 \rangle < 0.86 \quad \text{for} \quad f_B = 240 \text{ MeV} . \quad (9)$$

This should be compared with the estimate $\langle x_b^1 \rangle = 0.90$ obtained in [2], which is consistent with the estimate of f_B in the same reference but not with the currently preferred value. Note that the assumption $\psi > 0$ is crucial.

It is a simple matter to derive similar bounds on other moments of the distribution amplitude; for example, we obtain

$$\langle x^2 \rangle < 0.75 \quad , \quad \langle x^3 \rangle < 0.65 \quad , \quad \text{and} \quad \langle (1-x)^{-1} \rangle < 9.15 .$$

It must be emphasized that $\langle x^n \rangle$ represents a moment, rather than an expectation value, and that unitarity can provide no constraints on expectation values. However, the amplitudes for exclusive processes are determined by convolutions of wave functions, not of their squares; thus Eq. (9) makes a strong statement about the shape of the wave function $\psi_{B \rightarrow b\bar{q}}$.

In sum, the known meson decay constant f_B can only be reconciled with unitarity by allowing unexpectedly large values of k_L , and values of x far from unity, to make sizable contributions to the wave function.

Acknowledgment

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References

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