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A New Invariance Possessed by the Superparticle Lagrangian

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It is shown that the Lagrangian of a superparticle has, in addition to the Lorentz invariance, an invariance which corresponds to the invariance under the rotation of the body frame of the non-relativistic rigid sphere. This invariance leads to a new conservation law which corresponds to the conservation law of the body frame component of the angular momentum of this sphere.

The superparticle has been discussed by many authors as the simplest example of the supersymmetry [1], [2], [3]. It is described by Lagrangian of the form^{*)}

$$L = p_\mu \{ \dot{x}^\mu - i(\xi^* \sigma^\mu \dot{\xi} - \dot{\xi}^* \sigma^\mu \xi) \} - \frac{1}{2} e (p_\mu p^\mu + m^2), \quad (1)$$

where ξ is a two-component Grassmann spinor and e is a Lagrangian multiplier.

From this Lagrangian momenta P_μ , π^α and $\pi^{*\dot{\alpha}}$ canonically conjugate to x^μ , ξ_α and ξ_α^* respectively are given by^{**)}

$$\left. \begin{aligned} P_\mu &\equiv \partial L / \partial \dot{x}^\mu = p_\mu, \\ \pi^\alpha &\equiv \partial L / \partial \dot{\xi}_\alpha = -i p_\mu (\xi^* \sigma^\mu)^\alpha, \\ \pi^{*\dot{\alpha}} &\equiv \partial L / \partial \dot{\xi}_\alpha^* = -i p_\mu (\sigma^\mu \xi)_{\dot{\alpha}}, \end{aligned} \right\} \quad (2)$$

and the Euler equations are written as

$$\dot{p}_\mu = 0, \quad (3)$$

$$p_\mu p^\mu + m^2 = 0, \quad (4)$$

$$(p_\mu \sigma^\mu) \dot{\xi} = 0. \quad (5)$$

Eq.(3) gives

$$p_\mu = \text{const.} \quad (6)$$

On the other hand, Eq.(4) is nothing but the mass shell condition for p_μ . For simplicity we assume $m \neq 0$. Then we have from Eq.(5)

$$\xi = \text{const.} \quad (7)$$

The Lagrangian (1) is invariant under the Lorentz transformation, with the conservation law of the four-dimensional angular momentum [1]:

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad (8)$$

where $S_{\mu\nu}$ is the “spin” angular momentum coming from the internal motion described by ξ which is given by

$$S_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} p^\rho (\xi^* \sigma^\sigma \xi), \quad (9)$$

where $\varepsilon_{\mu\nu\rho\sigma} = -1$ or $+1$ according to whether μ, ν, ρ and σ are even or odd permutation of 1, 2, 3 and 0.

In this note we would like to point out that the Lagrangian (1) is, in addition to this, invariant under the transformation

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^{*)} In case $m \neq 0$, L can be written, eliminating p_μ , as $-m\sqrt{-V_\mu V^\mu}$ with $V^\mu = \dot{x}^\mu - i(\xi^* \sigma^\mu \dot{\xi} - \dot{\xi}^* \sigma^\mu \xi)$, which is just the Casalbuoni's Lagrangian [1]. The case $m = 0$ is the Brink-Schwarz's one [2].

^{**)} Unless otherwise noted, we use always right derivative for Grassmann variables.

$$\xi \rightarrow a\xi - b\Sigma' i\sigma_2 \xi^*, \quad (10)$$

as can be verified easily noticing that ξ is a Grassmann spinor^{*}), where a and b are arbitray constants satisfying

$$|a|^2 + |b|^2 = 1, \quad (11)$$

and Σ' is given by $\sigma_2 \Sigma^T \sigma_2$ with $\Sigma = p_\mu \sigma^\mu / ip$ satisfying $\Sigma \Sigma' = \Sigma' \Sigma = 1$, where $p \equiv \sqrt{p_\mu p^\mu}$.

For infinitesimal transformation it is more convenient to parametrize a and b as $a = 1 - (i/2)d\alpha^3$ and $b = -(i/2)d\alpha^1 + (1/2)d\alpha^2$, and the invariance of the Lagrangian (1) under the transformation (10) leads to the conservation law of S^r ($r=1, 2, 3$) given by

$$\left. \begin{aligned} S^1 &= m(\xi i\sigma_2 \xi + c.c.)/2, \\ S^2 &= -im(\xi i\sigma_2 \xi - c.c.)/2, \\ S^3 &= m(\xi^* \Sigma \xi). \end{aligned} \right\} \quad (12)$$

In order to clarify the physical meaning of the transformation (10) and the new conservation law for S^r , let us compare the motion of our system in the rest frame of its center of mass defined by $\gamma^\mu = (0, 0, 0, m)$ (C. M. frame) with that of a non-relativistic rigid sphere.

If we fix the center of mass of the sphere at the origin, the motion of the non-relativistic rigid sphere can be described by specifying the orientation of a triad $\{a_i^r\}$ ($r, i = 1, 2, 3$) fixed to the sphere the origin of which is placed at its center of mass (body frame), where r designates three unit vectors forming the triad, and i refers to their components in the laboratory frame. Under the rotation of the laboratory frame a_i^r transforms as $a_i^r \rightarrow R_{ij} a_j^r$, where R_{ij} is an orthogonal transformation. On the other hand, the choice of the body frame is left completely arbitrary so long as it is fixed to the body, and we can go freely from one to another. We shall call this the rotation of the body frame. Under this a_i^r transformas as $a_i^r \rightarrow A^{rs} a_i^s$, where A^{rs} is also orthogonal.

It is usual to introduce Euler angles (φ, θ, ψ) to describe a_i^r . Then in terms of them the laboratory frame and the body frame component of the angular velocity with which the sphere is rotating, ω_i and ω^r , are given by

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^{*}) Substituting $\xi \rightarrow a\xi - b\Sigma' i\sigma_2 \xi^*$ and $\dot{\xi} \rightarrow a\dot{\xi} - b(\dot{\Sigma}' i\sigma_2 \xi^* + \Sigma' i\sigma_2 \dot{\xi}^*)$ together with their complex conjugates, L is transformed as

$$L \rightarrow L + p(ab^* \xi i\sigma_2 \Sigma' \dot{\Sigma} \xi - c.c.).$$

Substituting explicit forms for Σ and Σ' , we find

$$\xi i\sigma_2 \Sigma' \dot{\Sigma} \xi = \xi i\sigma_2 \{(\mathbf{p}\boldsymbol{\sigma} + p^0)(\dot{\mathbf{p}}\boldsymbol{\sigma} - \dot{p}^0) - p_\mu \dot{p}^\mu\} \xi / p^2.$$

Here terms proportional to σ_i in the curled bracket vanish since for Grassmann spinor $\xi i\sigma_2 \sigma_i \xi = 0$. Therefore the first term reduces to $p_\mu \dot{p}^\mu$, which cancels with the second term. So $\xi i\sigma_2 \Sigma' \dot{\Sigma} \xi = 0$, and L is invariant under the transformation (10).

$$\left. \begin{aligned} \omega_1 &= -\sin \psi \dot{\theta} + \sin \theta \cos \psi \dot{\varphi}, \\ \omega_2 &= \cos \psi \dot{\theta} + \sin \theta \sin \psi \dot{\varphi}, \\ \omega_3 &= \dot{\psi} + \cos \theta \dot{\varphi}, \end{aligned} \right\} \quad (13)$$

and

$$\left. \begin{aligned} \omega^1 &= \sin \varphi \dot{\theta} - \sin \theta \cos \varphi \dot{\psi}, \\ \omega^2 &= \cos \varphi \dot{\theta} + \sin \theta \sin \varphi \dot{\psi}, \\ \omega^3 &= \dot{\varphi} + \cos \theta \dot{\psi}. \end{aligned} \right\} \quad (14)$$

Introducing a two-component spinor u by

$$u = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i(\varphi+\psi)/2} \\ \sin \frac{\theta}{2} e^{-i(\varphi-\psi)/2} \end{pmatrix}, \quad (15)$$

they are expressed as

$$\omega_i = i(u^* \sigma_i \dot{u} - \dot{u}^* \sigma_i u) \quad (16)$$

and

$$\left. \begin{aligned} \omega^1 &= \dot{u} \sigma_2 u + c.c., \\ \omega^2 &= -i(\dot{u} \sigma_2 u - c.c.), \\ \omega^3 &= i(u^* \dot{u} - \dot{u}^* u). \end{aligned} \right\} \quad (17)$$

With these variables the angular momentum $\mathbf{m}$ is expressed as

$$\mathbf{m} = I \boldsymbol{\omega}, \quad (18)$$

where I is the moment of inertia, and the Lagrangian is given by

$$L = \mathbf{m}^2 / 2I \quad (19)$$

Starting from this Lagrangian, we can go over to the Hamiltonian formalism without difficulty, and we find Poisson bracket relations [4]

$$\left. \begin{aligned} (m_i, m_j)_{PB} &= m_k, (i, j, k = 1, 2 \text{ or } 3 \text{ and are cyclic}), \\ (m^r, m^s)_{PB} &= -m^t, (r, s, t = 1, 2 \text{ or } 3 \text{ and are cyclic}), \\ (m_i, m^r)_{PB} &= 0. \end{aligned} \right\} \quad (20)$$

and

$$(H, m_i)_{PB} = (H, m^r)_{PB} = 0, \quad (21)$$

where the Hamiltonian H is given by L , and $(\ )_{PB}$ means the Poisson bracket.

From Eqs.(20) we see that $\mathbf{m}^2$, m^3 and m_3 form a complete set of constant of motion in an involution. Therefore the motion of the systems can be determined by giving their values. This implies that, quantum mechanically, the wave function of the system is specified by three quantum numbers giving respectively the eigenvalue of $\mathbf{m}^2$, m^3 and m_3 .

Eq.(21) shows that m_i and m^r are constant in time. Therefore if we take the direction of a_i^3 to that of $\mathbf{m}$, we find from Eqs.(14) $\dot{\theta} = \dot{\psi} = 0$ and $\dot{\varphi} = \omega$, where ω is a constant. Thus $\varphi = \varphi_0 + \omega t$, $\theta = \theta_0$ and $\psi = \psi_0$, where φ_0 , θ_0 and ψ_0 are constant in time, and u can be written as

$$u = u_0 e^{-i\omega t/2}, \quad (22)$$

where u_0 is an expression obtained by replacing (φ, θ, ψ) by $(\varphi_0, \theta_0, \psi_0)$ in Eq.(15), which is therefore constant in time. Furthermore substituting Eq.(22) we have from Eqs.(16) and (18)

$$m_i = I\omega(u^* \sigma_i u). \quad (23)$$

On the other hand $S_{\mu\nu}$ given by Eq.(9) reduces, in C. M. frame, to $S_i(\equiv S_{jk}) = m(\xi^* \sigma_i \xi)$ and $S_{i0} = 0$. Therefore in C. M. frame $S_{\mu\nu}$ agrees with m_i except for unessential constant factors if we replace ξ by u , and we find that ξ corresponds exactly to u .

Let us now try to clarify the physical meaning of the transformation (10) and the conservation law for S^r based upon this correspondence. We note that for u two kinds of transformations are possible. One is the rotation of the laboratory frame under which u is transformed as

$$u \rightarrow (1 + \frac{i}{2} \sigma_i d\alpha_i) u, \quad (24)$$

and the other is the rotation of the body frame under which u is transformed as

$$u \rightarrow au - b i \sigma_2 u^* \quad (25)$$

with $a = 1 - (i/2)d\alpha^3$ and $b = -(i/2)d\alpha^1 + (1/2)d\alpha^2$, where $d\alpha_i$ and $d\alpha^r$ denote now respectively the infinitesimal angle of rotation of the laboratory and the body frame around their i -th and r -th axis, which are given in terms of Euler angles by

$$\left. \begin{aligned} d\alpha_1 &= -\sin \psi d\theta + \sin \theta \cos \psi d\varphi, \\ d\alpha_2 &= \cos \psi d\theta + \sin \theta \sin \psi d\varphi, \\ d\alpha_3 &= d\psi + \cos \theta d\varphi, \end{aligned} \right\} \quad (26)$$

and

$$\left. \begin{aligned} d\alpha^1 &= \sin \varphi d\theta - \sin \theta \cos \varphi d\psi, \\ d\alpha^2 &= \cos \varphi d\theta + \sin \theta \sin \varphi d\psi, \\ d\alpha^3 &= d\varphi + \cos \theta d\psi. \end{aligned} \right\} \quad (27)$$

The transformation (25) is nothing but the transformation (10) in C. M. frame where $\Sigma' = 1$. Therefore (10) corresponds to the rotation of the body frame of the non-relativistic rigid sphere and the parameters $d\alpha^r$ get the meaning of the infinitesimal angle of rotation corresponding to those given by Eq.(27). Then it is expected that S^r given by Eqs.(12) should correspond to the body frame component of the spin angular momentum. To

check this point let us calculate the Dirac bracket between $S_{\mu\nu}$ and S^r in C. M. frame [5].*) Then we find

$$\left. \begin{aligned} (S_i, S_j)_{DB} &= S_k, \\ (S^r, S^s)_{DB} &= -S^t, \\ (S_i, S^r)_{DB} &= 0, \end{aligned} \right\} \quad (28)$$

which are exactly the same as Eqs.(20). Furthermore, substituting Eq.(22) we have $m^r = (0, 0, I\omega(u^*u))$, which agrees with S^r in C.M. frame if we replace ξ by u in S^r , just as m_i is obtained from $S_{\mu\nu}$ by the same substitution. These results show that our expectation is in fact realized.

The fact that the Lagrangian of the superparticle allows a transformation that corresponds to the rotation of the body frame of the non-relativistic rigid sphere with the conservation law that corresponds to the conservation law of the body frame component of its angular momentum seems to suggest that the superparticle has an *extended* structure similar to the non-relativistic rigid sphere the rotational motion of which is described by the *Grassmann* spinor ξ , although it may not necessarily be an intuitive one.

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*) From Eqs.(2) we find

$$\eta^\alpha \equiv \pi^\alpha - im\xi_\rho^* \Sigma^{\rho\alpha} = 0, \quad \eta^{\dot{\alpha}} \equiv \pi^{\dot{\alpha}} - im\Sigma^{\dot{\alpha}\rho} \xi_\rho = 0. \quad (A)$$

Then $(\eta^\alpha, \eta^\beta)_{PB} = 0$, $(\eta^{\dot{\alpha}}, \eta^{\dot{\beta}})_{PB} = 0$, $(\eta^\alpha, \eta^{\dot{\beta}})_{PB} = -2im\Sigma^{\beta\alpha}$ and $(\eta^{\dot{\alpha}}, \eta^\beta)_{PB} = -2im\Sigma^{\dot{\alpha}\beta}$, where in case q_i and p_i are Grassmann numbers the Poisson bracket is defined by

$$(FG)_{PB} \equiv (\partial F / \partial q_i)(\partial / \partial p_i)G + (\partial F / \partial p_i)(\partial / \partial q_i)G,$$

with $(\partial F / \partial q_i)$ and $(\partial / \partial p_i)G$ denoting respectively the right and the left derivatives. Therefore C defined by $C^{ab} = (\eta^a, \eta^b)_{PB}$ where $a = (\alpha, \dot{\alpha})$ and $b = (\beta, \dot{\beta})$ is written as

$$C = \begin{pmatrix} 0 & -2im\Sigma^{T\alpha\dot{\beta}} \\ -2im\Sigma^{\dot{\alpha}\beta} & 0 \end{pmatrix} \quad (B)$$

C has non-vanishing determinant, and C^{-1} is given by

$$C^{-1} = \begin{pmatrix} 0 & \frac{i}{2m}\Sigma'_{\alpha\dot{\beta}} \\ \frac{i}{2m}\Sigma'^T_{\dot{\alpha}\beta} & 0 \end{pmatrix} \quad (C)$$

Therefore (A) is the second class constraint, and the Dirac bracket is given by

$$(FG)_{DB} = (FG)_{PB} - (F\eta^a)_{PB}C_{ab}^{-1}(\eta^b G)_{PB}.$$

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