BV and BFV Formulation of a Gauge Theory of Quadratic Lie Algebras in 2-d and a Construction of $W_3$ Topological Gravity

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Abstract

The recently proposed generalized field method for solving the master equation of Batalin and Vilkovisky is applied to a gauge theory of quadratic Lie algebras in 2-dimensions. The charge corresponding to BRST symmetry derived from this solution in terms of the phase space variables by using the Noether procedure, and the one found due to the BFV-method are compared and found to coincide. $W_3$ algebra, formulated in terms of a continuous variable is employed in the mentioned gauge theory to construct a $W_3$ topological gravity. Moreover, its gauge fixing is briefly discussed.

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1 Introduction

Gauge theories of Lie groups offered an understanding of elementary particles physics. Going beyond Lie groups can be obtained by deforming their algebra. The simplest deformation is to let the commutator of two generators possess some quadratic terms in the generators, in addition to the linear ones. Obviously this is not a Lie algebra, nevertheless, it is known as “quadratic Lie algebra”. Yang-Mills theory is not suitable to formulate a gauge theory of quadratic Lie algebras. A gauge theory Lagrangian of quadratic Lie algebras is proposed in [1], by utilizing the formulation given in [2]. Its gauge algebra is closed on mass shell.

The most powerful methods of quantization of the gauge systems possessing a gauge algebra closing on mass shell and/or irreducible gauge generators are Batalin-Vilkovisky (BV)[3] and Batalin-Fradkin-Vilkovisky (BFV)[4] schemes. The former is a Lagrangian and the latter is a Hamiltonain formulation.

The first step in the BV-method is to find the proper solution of the master equation. Usually this is a hard task. Fortunately, for a vast class of first order systems a general solution is known[5]. We apply this method to the gauge theory of quadratic Lie algebras and find the proper solution of the master equation. By using this solution and the Noether procedure we find a charge corresponding to BRST symmetry, in terms of the related phase space variables. On the other hand we can find a BRST charge of this system following from the BFV-method. A priori there is not any argument to conlude that these two charges are the same\(^1\). Here, we show that these two charges coincide for the gauge theory of quadratic Lie algebras.

The gauge theory Lagrangian which we deal with does not depend on the space-time metric and does not lead to local excitations. Hence it is a topological quantum field theory (for a review see [7]).

\(^{1}\)The relation between BV- and BFV- method is studied in [6]. They give the solution of the master equation if the BFV-BRST charge is known.

\(^{2}\)Gauging Lie algebras which have infinite generators are studied in [10].
form. They also pointed out that a topological $W_3$ gravity based on their formulation is expected to be possible. Here, we give a formulation of $W_3$ topological gravity in this spirit. But, we employ the continuous variables from the beginning to write $W_3$ algebra commutators, instead of introducing them as a bookkeeping device. Obviously our construction is different from the topological $W_3$ gravity formulation based on $sl(3)$ given in [11].

2 BV and BFV Formulation of a Gauge Theory of Quadratic Lie Algebras in 2-d

The simplest deformation of a Lie algebra generated by $T_a$ is

$$[T_a, T_b] = f_{ab}^c T_c + V_{ab}^d T_d + k_{ab},$$

(1)

which is known as quadratic Lie algebra. Here, $f$, $V$, and $k$ are constants and their symmetry properties are

$$f_{ab}^c = -f_{ba}^c, \quad V_{ab}^{cd} = -V_{ba}^{cd}, \quad V_{ab}^{cd} = V_{ba}^{dc}, \quad k_{ab} = -k_{ba}.$$  (2)

Commutators should satisfy the Jacobi identities, hence the constants $f$, $V$, and $k$ should be chosen such that

$$f_{[ab}^d f_{cd]}^e = 0,$$

$$f_{[ab}^d V_{cd]}^f + V_{[ab}^d f_{cd]}^e + V_{[ab}^d f_{cd]}^f = 0,$$  (3)

$$V_{[ab}^{de} V_{cd]}^f = 0,$$

$$f_{[ab}^d k_{cd]}^e = 0,$$

$$V_{[ab}^{de} k_{cd]}^f = 0,$$

are obeyed. Here, $[ ]$ denote antisymmetrization in the indices which are within them.

Gauge theory of this algebra in 2-d space-time is given by the Lagrange density[1]

$$\mathcal{L} = -\frac{1}{2} \epsilon^{\mu\nu} \left\{ \Phi_a (\partial_\mu h^a_\nu - \partial_\nu h^a_\mu + f_{bc}^{\quad a} h^b_\mu h^c_\nu + V_{bc}^{\quad d} \Phi_d h^b_\mu h^c_\nu) + k_{ab} h^a_\mu h^b_\nu \right\},$$  (4)

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which is invariant under the gauge transformations

$$\delta h^a_\mu = \partial_\mu \lambda^a + f^a_{bc} h^b_\mu \lambda^c + 2 V^{ad}_{bc} \Phi^d_\mu \lambda^c,$$

$$\delta \Phi_a = f^a_{bc} \Phi^b_c \lambda^c + V^{ad}_{bc} \Phi^d_c \lambda^b + k_{ab} \lambda^b,$$  \hspace{1cm} (5, 6)

whose algebra is closed on mass shell.

To find the proper solution of the (BV-) master equation whose classical limit is given by (4), we can apply the generalized field method of [5], because, the gauge transformations (5)-(6) can be written as

$$\delta h^a_\mu = \epsilon_{\mu\nu} \frac{\partial^2 L}{\partial \Phi^a_\nu \partial h^b_\nu} \lambda^b,$$

$$\delta \Phi_a = \epsilon_{\mu\nu} \frac{\partial^2 L}{\partial h^a_\mu \partial h^b_\nu} \lambda^b.$$  \hspace{1cm} (7, 8)

In the method given in [5] one introduces the generalized fields

$$\tilde{h} = h_{(1,0)} + \eta_{(0,1)} - \Phi^*_{(2,-1)},$$

$$\tilde{\Phi} = - h^*_{(1,-1)} - \eta^*_{(2,-2)} + \Phi_{(0,0)},$$  \hspace{1cm} (9, 10)

where the first number in the parenthesis is the order of differential forms and the second is the ghost number. $\eta^*$ are the ghost fields, and the star denotes the antifields as well as the Hodge map. As one can observe the total degree of the generalized fields $\tilde{h}$, $\tilde{\Phi}$, which is the sum of ghost number and order of differential form are 1 and 0. Now by replacing the fields $h$, $\Phi$ with the generalized ones $\tilde{h}$, $\tilde{\Phi}$ one can obtain the proper solution of the master equation:

$$S = - \int d^2 \tilde{x} \frac{1}{2} \{ \tilde{\Phi}_a (d \tilde{h}^a + f^a_{bc} \tilde{h}^b \tilde{h}^c + V^{ad}_{bc} \tilde{\Phi}^d \tilde{h}^b \tilde{h}^c) + k_{ab} \tilde{h}^a \tilde{h}^b \}. \hspace{1cm} (11)$$

It is the solution of the master equation, because $S$ is invariant under the gauge transformation obtained as the generalization of (7)-(8). In the multiplication of two identical generalized fields the field which appears first in the generalized field, will be the first term in the multiplication. The proper solution of the master equation $S$, in components is
\[
S = \int d^2 x \left\{ \mathcal{L} + h^a_b ( \partial_a \eta^c + f_{ba} e^b \eta^c + 2 V_{bc}^d \Phi_d h^b \eta^c ) \\
\quad + \Phi^{ab} ( f_{ba}^c \Phi^c e^b + V_{ba}^d \Phi^d \eta^b + k_{ab} \eta^b ) \\
\quad + \eta^a ( \frac{1}{2} f_{ac}^b \eta^b \eta^c + V_{bc}^d \Phi^d \eta^b \eta^c ) \\
\quad - \frac{1}{2} \epsilon_{ab} V_{bc}^d h^a \eta^b h^d \eta^c \right\}. \tag{12}
\]

One can explicitly check that indeed (12) is the proper solution of the master equation, by using the symmetries and the identities which \( f, V, \) and \( K \) satisfy (2)-(3).

The proper solution of the master equation \( S \), is the full Lagrangian which can be used in the related path integrals after gauge fixing. Instead of discussing gauge fixing on general grounds, we prefer to do it after constructing \( W_3 \) topological gravity.

Let us deal with Hamiltonian formalism of the theory. Because of being first order the non-vanishing Poisson brackets can be read easily from the Lagrange density (4) as

\[ \{ \Phi^a, h^b \} = -\delta^b_a. \]

Now, one can see that \( h^a \) behave as Lagrange multipliers and derivations with respect to them lead to the constraints

\[ \Psi_a = \partial_a \Phi^a + f_{ac}^b h^c h^1 + V_{ab}^d \Phi^d h^1 + k_{ab} h^1. \tag{13} \]

Canonical Hamiltonian vanishes, so that there is no other constraint. By making use of the symmetry properties (2), and the identities (3) of \( f, V, \) and \( k \), one finds that \( \Psi \) satisfy

\[ \{ \Psi_a, \Psi_b \} = (f_{ab}^c + 2V_{ab}^d \Phi_d) \Psi_c, \tag{14} \]

so that they are first class.

To perform the BFV analysis, let us enlarge the phase space by introducing the ghost variables \( \eta^a \), and their canonical conjugate \( \mathcal{P}_a \) which satisfy

\[ \{ \mathcal{P}_a, \eta^b \} = \delta^b_a, \]

where \( \{ A, B \} = (\partial A/\partial \mathcal{P}_a) \partial B/\partial \eta^a + (\partial A/\partial \eta_a) \partial B/\partial \mathcal{P}_a \). \( \eta \), and \( \mathcal{P} \) are anti-commuting fields, and their ghost numbers are 1 and \(-1\), respectively.

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The BFV-charge $\Omega$, defined to possess ghost number 1, to be fermionic and to satisfy
\[ \{\Omega, \Omega\}_+ = 0, \]
by using (2)-(3) can be found as
\[ \Omega = \eta^a \Psi_a - \frac{1}{2} \eta^a \eta^b \mathcal{P}_c(f_{ab}^c + 2 V^{cd}_{ab} \Phi_d). \]  \hfill (15)

To understand the relation between the BV and the BFV approaches, we would like to compare $\Omega$ of the latter with the Noether charge of the former corresponding to the BRST symmetry given by $\delta_t S/\delta \Phi$. In the definition of the latter charge one uses the related equations of motion, but in the former case the charge is not aware of the equations of motion. Hence, a priori their relation is not clear.

The Noether charge, in terms of the phase space variables ($\hat{P}$, $\hat{h}$), is given by
\[ \Omega_N = \hat{P}_A \frac{\delta_t S}{\delta \Phi_A} - K, \]  \hfill (16)
where the generalized momentum is
\[ \hat{P} = \frac{\delta_r S}{\delta \dot{h}}, \]  \hfill (17)
and $K$ is defined to satisfy
\[ \frac{\partial K}{\partial \hat{P}_A} - \frac{\partial (\delta_t S/\delta \Phi_B)}{\partial \hat{P}_A} \dot{P}_B = 0. \]

From (17) one finds that the canonical momenta are
\[ P_{\Phi_a} = -\Phi_a, \ \ P_{\eta^a} = 0, \ \ P_{\eta^a} = h_0^{a}. \]
In terms of these canonical variables one can show that $K$ is
\[ K = V_{ab}^{cd} (-P_{\Phi_a} \mathcal{P}_b \eta^c \eta^d + P_{\eta^a} P_{\Phi_a} h_0^c h_0^d), \]
which yields
\[ \Omega_N = \Omega. \]
Thus we can conclude that BV and BFV approaches lead to the same BRST charge.

One replaces the generalized Poisson brackets \{,\}, \{,\}_+, with the commutator [ , ], or anticommutator [ , ]_+, to achieve operator quantization. Here, this will lead to some complications in the definition of BRST operator due to operator ordering problems. Nevertheless, let us suppose that a suitable operator ordering exists such that \( \Omega_{op} \), the operator resembling the BRST charge \( \Omega \), is nilpotent. Then, solution of its cohomology will give the physical states of the system under study. Unfortunately, there is not any general procedure to solve this problem.

Recently a general formulation of nonlinear Lie algebras, which includes also the one which we studied is given[12]. Also for this formulation the method of [5] is suitable for finding the proper solution of the BV master equation.

3 \( W_3 \) Topological Gravity

\( W_3 \) algebra can be given in terms of operator product expansion or equivalently in terms of its modes. Here we will follow yet another, but an equivalent description. The algebra is defined in terms of the generators

\[
[G_A(z), G_B(w)] = \int dw f_{AB}^C(z, w, v)G_C(v) + \int dvdr V_{AB}^{CD}(z, w, v, r)G_C(v)G_D(r),
\]

where \( f_{AB}^C \), and \( V_{AB}^{CD} \) are given as

\[
f_{11}^1(z, w, v) = \partial_z \delta(z - w) \delta(z - v) - \partial_w \delta(w - z) \delta(w - v) \tag{19}
\]

\[
f_{12}^2(z, w, v) = 3 \partial_z \delta(z - w) \delta(z - v) + 2 \delta(z - w) \partial_z \delta(z - v) - \delta(w - v) \delta(w - r) \partial_v \delta(z - w). \tag{20}
\]

\[
V_{22}^{11}(z, w, v, r) = \delta(z - v) \delta(z - r) \partial_z \delta(z - w) - \delta(w - v) \delta(w - r) \partial_w \delta(z - w). \tag{21}
\]

Obviously we deal with the “classical” \( W_3 \) algebra. To obtain the mode expansion take

\[
T(z) = \sum_{n=-\infty}^{\infty} T_n z^{-n-2}, \quad W(z) = \sum_{n=-\infty}^{\infty} W_n z^{-n-3},
\]

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and perform the integrals (without summation on $A$ and $B$)

$$\int dzdw \ [G_A(z), G_B(w)]u_A(z)u_B(w),$$

where

$$u_1(z) = z^{n+1}, \quad u_2(z) = z^{n+2}. \quad (22)$$

These lead to the known description of classical $W_3$ algebra,

$$[T_m, T_n] = (n - m)T_{n+m}, \quad (23)$$
$$[T_m, W_n] = (n - 2m)W_{n+m}, \quad (24)$$
$$[W_m, W_n] = (n - m) \sum_{k=-\infty}^{\infty} T_{m-k}T_k. \quad (25)$$

Although, for the algebra given in (23)-(25) Jacobi identities are satisfied, it is not sufficient to conclude that the algebra defined in (18)-(21) also satisfies Jacobi identities. Because, one can employ different $f_{AB}^C$, and $V_{AB}^{CD}$ which lead to the same algebra in terms of the modes. This feature is a result of using continuous variables in the definition of the algebra: we should suppose that all of the functions which take value in the algebra (depend on the continuous parameters) behave such that partial integrals are always allowed. Nevertheless, after a tedious calculation one can show that

$$\int d\tau \ [f_{11}^1(z, w, r)f_{11}^1(r, v, q) + f_{11}^1(w, v, r)f_{11}^1(r, z, q)$$
$$+ f_{11}^1(v, z, r)f_{11}^1(r, w, q)] = 0,$$

$$\int d\tau \ [f_{12}^2(z, w, r)f_{12}^2(r, v, q) + f_{21}^2(w, v, r)f_{12}^2(z, r, q)$$
$$+ f_{11}^1(v, z, r)f_{21}^2(r, w, q)] = 0,$$

$$\int d\tau \ [V_{22}^{11}(z, w, v, r)f_{21}^2(p, r, q) + V_{22}^{11}(w, p, v, r)f_{21}^2(z, r, q)$$
$$+ V_{22}^{11}(p, z, v, r)f_{21}^2(w, r, q)] = 0.$$

One can also check that $f$, and $V$ possess the desired symmetry properties (2).

Now, to formulate a topological $W_3$ gravity let us introduce

$$h^A_\mu = (e_\mu, B_\mu), \quad \Phi_A = (t, w).$$

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Using these fields, and (19)-(21) in (4) yields

\[ S_0 = \int d^2 x \int dz \epsilon^{\mu \nu} (t \partial_\mu e_\nu + w \partial_\mu B_\nu + e_\mu \partial e_\nu + 3we_\mu \partial B_\nu + 2\partial we_\mu B_\nu + tB_\mu \partial B_\nu), \]

(26)

where \( \partial \) is the derivative with respect to \( z \). Thus introduce the generalized fields

\[
\hat{h} = (e_\mu, B_\mu \ominus \eta_1, \eta_2 \ominus t^*, w^*), \\
\hat{\Phi} = (\ominus e^{* \mu}, B^{* \mu} \ominus \eta_1^*, \eta_2^* \ominus t, w),
\]

(27)

to obtain the proper solution of the master equation as

\[
S_{W_3} = S_0 + \int d^2 x \int dz \{ B^{* \mu} [\partial_\mu \eta_2 + e_\mu \partial \eta_2 - 2\partial e_\mu \eta_2 - \partial B_\mu \eta_1 + 2B_\mu \partial \eta_1] \\
+ e^{* \mu} [\partial_\mu \eta_1 + e_\mu \partial \eta_1 - \partial e_\mu \eta_1 + 2t(B_\mu \partial \eta_2 - \partial B_\mu \eta_2) - e_\mu e^{w} \eta_2 \partial \eta_2] \\
+ \eta_1^* [\eta_1 \partial \eta_1 - 2t_2 \partial \eta_2 + \eta_2^* [\eta_1 \partial \eta_2 - 2\partial \eta_1 \eta_2] \\
-t^* [2t_1 \partial \eta_1 + \partial t \eta_1 + 3w \partial \eta_2 + 2w \eta_2] \\
-w^* [\partial w \eta_1 + 3w \partial \eta_1 + 2t \partial t \eta_2 + 2t \partial \eta_2] \}.
\]

(28)

Gauge transformations of this theory are in agreement with the ones given in [2]. Of course, in [2] the ghost fields \( \eta_1, \eta_2 \) and \( e^* e^* \) terms are absent.

To discuss gauge fixing conditions of this theory, we enlarge the configuration space by introducing the fields

\[
\tilde{\eta}_a(0, -1), \quad \tilde{\pi}_a(0, 0) \quad ; \quad \epsilon(\tilde{\eta}_a) = 1, \quad \epsilon(\tilde{\pi}_a) = 0,
\]

\[
\tilde{\eta}_a^*(2, 0), \quad \tilde{\pi}_a^*(2, -1) \quad ; \quad \epsilon(\tilde{\eta}_a^*) = 0, \quad \epsilon(\tilde{\pi}_a^*) = 1,
\]

where \( \epsilon \) denotes the Grassmann parity. Let us define

\[
S_\epsilon = S + \int d^2 x \int dz \tilde{\eta}_a^* \tilde{\pi}_a,
\]

which is still a solution of the master equation. Now, gauge fixing can be achieved in terms of the gauge fixing fermion

\[
\Psi_{(0, -1)}, \quad \epsilon(\Psi) = 1,
\]

after substituting the antifields in \( S_\epsilon \) by

\[
\lambda^* = \frac{\partial \Psi(\lambda)}{\partial \lambda},
\]
where $\chi$ represents the fields of the theory.

Of course, there exist different choices for gauge fixing fermion. Let us first deal with

$$\Psi = \bar{\eta}_1 e_1 + \bar{\eta}_2 B_1.$$  \hfill (29)

In this gauge, the partition function after integrating over $\pi_a$, $e_1$, $B_1$, is

$$Z = \int \prod dt dw d\phi_a d\eta_a d\bar{\eta}_a e^{\int d^2 x \sum_{a} t \partial_1 e_0 + w \partial_1 B_0 + \bar{\eta}_a \partial_1 \eta_a}.$$  \hfill (30)

To calculate the path integral let us give the mode expansion of the remaining fields:

$$e_0(z) = \sum_{n=-\infty}^{\infty} e_0^n z^n + 1; \quad \eta_1(z) = \sum_{n=-\infty}^{\infty} \eta_1^n z^n + 1$$

$$B_0(z) = \sum_{n=-\infty}^{\infty} B_0^n z^n + 2; \quad \eta_2(z) = \sum_{n=-\infty}^{\infty} \eta_2^n z^n + 2$$

$$t(z) = \sum_{n=-\infty}^{\infty} t^n z^{-n-2}; \quad \bar{\eta}_1(z) = \sum_{n=-\infty}^{\infty} \bar{\eta}_1^n z^{-n-2}$$

$$w(z) = \sum_{n=-\infty}^{\infty} w^n z^{-n-3}; \quad \bar{\eta}_2(z) = \sum_{n=-\infty}^{\infty} \bar{\eta}_2^n z^{-n-3}.$$  

Mode expansions of the ghosts $\eta_a$ and the antighosts $\bar{\eta}_a$, follow from the fact that mode expansion of the generalised fields $\bar{\Phi}$ and $\Phi$ are dictated by mode expansions of the original fields $e$, $B$, and $t$, $w$. Observe that in this gauge $\bar{\eta}_a$ behave like $e^*, B^*$.

Hence, after performing the $z$ integral one obtains

$$Z = \int \prod dt dw d\phi_a d\eta_a d\bar{\eta}_a e^{\int d^2 x \sum_{-\infty}^{\infty} [t^n \partial_1 e_0^n + w^n \partial_1 B_0^n + \bar{\eta}_a^n \partial_1 \eta_a^n]}.$$ \hfill (31)

Now, we can integrate over all of the remaining fields and find

$$Z = \sum_{[\mathcal{G}]} \text{sign}_{G}(\Delta),$$ \hfill (32)

where

$$\Delta = \lim_{N \to \infty} \frac{\det^N \partial_1 \det^N \partial_1}{|\det^N \partial_1 \det^N \partial_1|}.$$
As a matter of fact, the main point is to clarify what is meant by the set $[G]$. This should be decided due to the properties of the target manifold[7]. But this is out of the scope of this work.

One can also choose the gauge fixing fermion as

$$\Psi = \tilde{e}_1 e_0 + \tilde{e}_2 B_0,$$

which leads to temporal gauge: This choice is studied in [12] on general grounds, and it is shown that up to a field redefinition, effectively it yields a free field theory.

Before studying the topological properties, it is not possible to decide which gauge is the most interesting one. Moreover, one should quantize the related BRST charge which would follow after using (19)-(21) in (15). In the case of finding an operator ordering yielding a nilpotent BRST operator one should use it to work out the physical states. These subjects are left for future studies.

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References


