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Normal Ordering and the Leading Logarithm Sum

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The $\overline{MS}$ and normal ordered renormalization prescriptions are compared. The $\overline{MS}$ prescription is shown to be superior when one is interested in summing the leading logarithm radiative corrections since the normal ordered prescription does not account for the leading logarithm corrections in any simple manner.

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1. Introduction

It is well known that the mass-independent minimal subtraction renormalization prescription is a convenient one to use when computing the effects of a heavy particle on low energy physics in an effective field theory formalism which involves two different renormalization scales. This is because when one makes use of this prescription to compute the one-loop order contributions to the β function and the anomalous dimensions, the solution to the resulting renormalization group equation includes a sum of the leading logarithm radiative corrections that result from a change in the renormalization scale [1].

Although the usual prescription for renormalization of an effective field theory is minimal subtraction, it has been suggested [2] that the normal ordered renormalization prescription can also be used for this purpose. Specifically, let us briefly consider a computation [2,3] of the leading logarithm QCD corrections to the W mass from the $W^2 b\bar{b}$ operator, where W is the charged weak boson field and b is the bottom quark field. Naively, one might have thought that no computation of Feynman diagrams was necessary since we could have worked with normal ordered interactions, in which case the one-loop order contribution to the W mass would vanish. However, as we show in this letter, complications arise when computing with normal ordered interactions: although the one-loop order correction to the W mass vanishes, the two-loop and higher order corrections do not, and the solution to the renormalization group equation (using β functions and anomalous dimensions computed using the normal ordered prescription) does not include a sum of the leading logarithm corrections to the W mass, contrary to what happens when using the minimal subtraction prescription.

2. Comparison of Prescriptions

We compare the modified minimal subtraction ($\overline{\text{MS}}$) and normal ordered (NO) renormalization prescriptions by examining a simple example in $\lambda\phi^4$ field

theory using dimensional regularization. Consider the tree-level Lagrangian density,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + s_\phi\phi, \quad (1)$$

where s_ϕ is the source for the field ϕ . At higher orders in a loop expansion this Lagrangian gives rise to divergences in the Green's functions. To retain a theory with finite Green's functions we include counterterms in the Lagrangian to cancel these divergences order-by-order. This is most easily done by using dimensional regularization where the spacetime dimension of the Lagrangian is analytically continued to dimension $d = 4 - \epsilon$. The divergences then appear in the form of poles proportional to $1/\epsilon$. In d dimensions the field ϕ has mass dimension $d/2 - 1$. In order to keep the coupling constant λ dimensionless we introduce a mass scale μ and replace $\lambda\phi^4$ in equation (1) by $\lambda\mu^\epsilon\phi^4$ which is of dimension d .

Using the minimal subtraction renormalization prescription we can cancel the divergences in the Green's functions by including counterterms in the Lagrangian to cancel the $1/\epsilon$ poles. Using $\overline{\text{MS}}$ we also include counterterms that cancel the finite constant $\ln(4\pi)$ and Euler's constant γ . For convenience we include all three constants in the parameter Δ where $\Delta \equiv 1/\epsilon - \gamma/2 + (1/2)\ln(4\pi)$. The Feynman diagrams of Figure 1 contribute to the $\overline{\text{MS}}$ 1PI Green's functions at the one-loop order.

Let us consider the same problem, this time using the NO prescription. Our rule for this prescription follows that for the $\overline{\text{MS}}$ prescription, but in addition we also include counterterms in the Lagrangian that order-by-order in a loop expansion cancel all operator mixing resulting from the contraction of two fields emanating from the same vertex at some renormalization scale μ . More specifically, when using the NO prescription we require that the contribution to the 1PI two-point function from the Feynman diagram of Figure 1a vanish. We shall see that the NO renormalization prescription essentially reorders the perturbation expansion derived when using the $\overline{\text{MS}}$ renormalization prescription, and results in new renormalized constants, differing from those derived using the $\overline{\text{MS}}$ prescription.

Prior to computing the renormalized Lagrangians using $\overline{\text{MS}}$ and NO, we establish our notation with a brief review of the renormalization group. The renormalization group equation for the connected Green's functions is

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + m^2 \gamma_m \frac{\partial}{\partial m^2} + \int d^4x \beta_{s_\phi}(x) \frac{\delta}{\delta s_\phi(x)} \right) W[s_\phi] = 0, \quad (2)$$

where W is the generating function for connected Green's functions, γ_m is the anomalous dimension of m^2 , β_λ is the beta function for the coupling constant λ , $\beta_{s_\phi(x)} \equiv s_\phi(x) \gamma_\phi$, and γ_ϕ is the anomalous dimension of the field ϕ . We can derive the renormalization group equation for the 1PI Green's functions by performing a Legendre transform with respect to the source s_ϕ ,

$$\Gamma[\Phi] = W[s_\phi] - \int d^4x s_\phi \Phi(x), \quad (3)$$

where $\Gamma[\Phi]$ is the generating function for the 1PI functions and $\Phi(x) = \delta W / \delta s_\phi(x)$. Then the renormalization group equation for the 1PI Green's functions is

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + m^2 \gamma_m \frac{\partial}{\partial m^2} - \int d^4x \Phi(x) \gamma_\phi \frac{\delta}{\delta \Phi(x)} \right) \Gamma[\Phi] = 0. \quad (4)$$

By taking functional derivatives of equation (4) with respect to $\Phi(x)$, and subsequently setting Φ to zero, we find the renormalization group equation for the 1PI n -point function $\Gamma^{(n)}$,

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + m^2 \gamma_m \frac{\partial}{\partial m^2} - n \gamma_\phi \right) \Gamma^{(n)}(\mu) = 0. \quad (5)$$

From this equation we can compute γ_ϕ , γ_m , and β_λ .

Once we have evaluated the anomalous dimensions and the n -point 1PI functions at scale μ , we can solve the renormalization group equation (5) to determine an expression for the Green's functions at other energy scales. In order to evaluate $\Gamma^{(n)}$ at some fixed scale μ' it is convenient to introduce the variable t defined through the relation $\mu = e^{-t} \mu'$. From equation (5) we have the differential equation for $\Gamma^{(n)}(\mu' e^{-t})$,

$$\left(-\frac{\partial}{\partial t} + \beta_\lambda \frac{\partial}{\partial \lambda} + m^2 \gamma_m \frac{\partial}{\partial m^2} - n \gamma_\phi \right) \Gamma^{(n)}(\mu' e^{-t}) = 0. \quad (6)$$

The solution to equation (6) is found in two steps. First we solve the homogeneous equation,

$$\left(-\frac{\partial}{\partial t} + \beta_\lambda \frac{\partial}{\partial \lambda} + m^2 \gamma_m \frac{\partial}{\partial m^2} \right) \Gamma^{(n)}(\lambda, m, \mu' e^{-t}) = 0. \quad (7)$$

The dependence of the function $\Gamma^{(n)}$ on t , and hence on μ , can be expressed solely through its dependence on the functions $\bar{\lambda}(t, \lambda)$ and $\bar{m}(t, \lambda, m)$, defined by the relations

$$\begin{aligned} \int_\lambda^{\bar{\lambda}(t, \lambda)} \frac{dx}{\beta_\lambda(x)} &= t \\ \bar{m}^2(t, \lambda, m) &= m^2 \exp \left[\int_\lambda^{\bar{\lambda}(t, \lambda)} dx \frac{\gamma_m(x)}{\beta_\lambda(x)} \right]. \end{aligned} \quad (8)$$

The solution of the homogeneous equation (7) is

$$\Gamma^{(n)}[\lambda(\mu), m(\mu), \mu] = \Gamma^{(n)}[\bar{\lambda}, \bar{m}, \mu'] \equiv \Gamma^{(n)}[\lambda(\mu'), m(\mu'), \mu']. \quad (9)$$

The solution of the inhomogeneous equation (6) is then

$$\Gamma^{(n)}[\lambda(\mu')] = \Gamma^{(n)}[\lambda(\mu)] \exp \left[n \int_{\lambda(\mu)}^{\lambda(\mu')} \gamma_\phi(x) \frac{dx}{\beta_\lambda(x)} \right]. \quad (10)$$

Using the rules for renormalizing the two theories outlined above we can compute the Lagrangians renormalized at the one-loop order using the $\overline{\text{MS}}$ and NO prescriptions. The fourier-transformed Lagrangian renormalized using $\overline{\text{MS}}$ is

$$\begin{aligned} \mathcal{L}^{\overline{\text{MS}}} &= \frac{1}{2} p^2 \phi_{\overline{\text{MS}}}^2 - \frac{1}{2} m_{\overline{\text{MS}}}^2 \phi_{\overline{\text{MS}}}^2 \left(1 + \frac{\lambda_{\overline{\text{MS}}}}{16\pi^2} \Delta \right) \\ &\quad - \frac{1}{4!} \lambda_{\overline{\text{MS}}} \mu^\epsilon \phi_{\overline{\text{MS}}}^4 \left(1 + \frac{3\lambda_{\overline{\text{MS}}}}{16\pi^2} \Delta \right) + s_\phi \phi_{\overline{\text{MS}}}, \end{aligned} \quad (11)$$

and that renormalized using NO is

$$\begin{aligned} \mathcal{L}^{\text{NO}} &= \frac{1}{2} p^2 \phi_{\text{NO}}^2 - \frac{1}{2} m_{\text{NO}}^2 \phi_{\text{NO}}^2 \left[1 + \frac{\lambda_{\text{NO}}}{16\pi^2} \left(\Delta + 1/2 + \ln(\mu/m_{\text{NO}}) \right) \right] \\ &\quad - \frac{1}{4!} \lambda_{\text{NO}} \mu^\epsilon \phi_{\text{NO}}^4 \left(1 + \frac{3\lambda_{\text{NO}}}{16\pi^2} \Delta \right) + s_\phi \phi_{\text{NO}}. \end{aligned} \quad (12)$$

To determine the anomalous dimensions we compute the 1PI n -point functions $\Gamma^{(n)}$ using the Lagrangians of equations (11) and (12):

$$\begin{aligned}\Gamma^{(2)\overline{MS}}(\mu) &= p^2 - m_{\overline{MS}}^2 \left[1 - \frac{\lambda_{\overline{MS}}}{16\pi^2} \left(1/2 + \ln(\mu/m_{\overline{MS}}) \right) \right] \\ \Gamma^{(4)\overline{MS}}(\mu) &= -\lambda_{\overline{MS}} + \frac{3\lambda_{\overline{MS}}^2}{16\pi^2} \ln(\mu/m_{\overline{MS}}) \\ \Gamma^{(2)NO}(\mu) &= p^2 - m_{NO}^2 \\ \Gamma^{(4)NO}(\mu) &= -\lambda_{NO} + \frac{3\lambda_{NO}^2}{16\pi^2} \ln(\mu/m_{NO}),\end{aligned}\tag{13}$$

where the Feynman diagrams of Figure 1 contribute, and where we have taken the limit $\epsilon \rightarrow 0$. Using equations (5) and (13) we find the anomalous dimensions and β functions

$$\begin{aligned}\gamma_{\phi}^{\overline{MS}} &= 0 + \mathcal{O}(\lambda^2) \\ \gamma_m^{\overline{MS}} &= \frac{\lambda_{\overline{MS}}}{16\pi^2} + \mathcal{O}(\lambda^2) \\ \beta_{\lambda}^{\overline{MS}} &= \frac{3\lambda_{\overline{MS}}^2}{16\pi^2} + \mathcal{O}(\lambda^3) \\ \gamma_{\phi}^{NO} &= 0 + \mathcal{O}(\lambda^2) \\ \gamma_m^{NO} &= 0 + \mathcal{O}(\lambda^2) \\ \beta_{\lambda}^{NO} &= \frac{3\lambda_{NO}^2}{16\pi^2} + \mathcal{O}(\lambda^3).\end{aligned}\tag{14}$$

Using the anomalous dimensions of equation (14) we can find an expression for the evolution of the mass and coupling parameters from equation (8):

$$\begin{aligned}m_{\overline{MS}}^2(\mu') &= m_{\overline{MS}}^2(\mu) \left(\frac{\lambda_{\overline{MS}}(\mu')}{\lambda_{\overline{MS}}(\mu)} \right)^{1/3} \\ \lambda_{\overline{MS}}(\mu') &= \lambda_{\overline{MS}}(\mu) \left[1 - \frac{3\lambda_{\overline{MS}}(\mu)}{16\pi^2} \ln(\mu'/\mu) \right]^{-1} \\ m_{NO}^2(\mu') &= m_{NO}^2(\mu) \\ \lambda_{NO}(\mu') &= \lambda_{NO}(\mu) \left[1 - \frac{3\lambda_{NO}(\mu)}{16\pi^2} \ln(\mu'/\mu) \right]^{-1}.\end{aligned}\tag{15}$$

Let us compare the results derived by using the two different renormalization prescriptions. By comparing the two Lagrangians of equations (11)

and (12) we note that at one-loop order the fields in the two theories are not renormalized and are equivalent,

$$\phi_{\overline{MS}} = \phi_{NO} \equiv \phi;\tag{16}$$

and the coupling constants (renormalized at one-loop order) are equivalent,

$$\lambda_{\overline{MS}} = \lambda_{NO} \equiv \lambda.\tag{17}$$

However, the mass parameters in the two theories differ; at scale μ they are related by

$$m_{\overline{MS}}^2(\mu) = m_{NO}^2(\mu) \left[1 + \frac{\lambda(\mu)}{16\pi^2} \ln \left(\frac{\mu}{m_{NO}} \right) + \mathcal{O}(\lambda^2) + \dots \right],\tag{18}$$

where the ellipses denote the subleading logarithm term $m_{NO}^2 \lambda_{NO}/(32\pi^2)$. As we run the two theories from the scale μ to the scale μ' , the parameters evolve according to equation (15), and the relationship between the mass parameters at scale μ' [derived from a comparison of $\mathcal{L}^{\overline{MS}}(\mu')$ and $\mathcal{L}^{NO}(\mu')$] becomes

$$m_{\overline{MS}}^2(\mu') = m_{NO}^2(\mu') \left[1 + \frac{\lambda(\mu')}{16\pi^2} \ln \left(\frac{\mu'}{m_{NO}(\mu')} \right) + \mathcal{O}(\lambda^2) \right].\tag{19}$$

Let us compare the two sides of equation (19). On the left hand side we have

$$m_{\overline{MS}}^2(\mu') = m_{\overline{MS}}^2(\mu) \left(\frac{\lambda(\mu')}{\lambda(\mu)} \right)^{1/3}.\tag{20}$$

Expanding the right hand side of equation (20) in λ we find

$$m_{\overline{MS}}^2(\mu') = m_{\overline{MS}}^2(\mu) \left[1 + \frac{\lambda(\mu)}{16\pi^2} \ln \left(\frac{\mu'}{\mu} \right) + 2 \frac{\lambda^2(\mu)}{(16\pi^2)^2} \ln^2 \left(\frac{\mu'}{\mu} \right) + \dots \right],\tag{21}$$

where the ellipses denote a sum of the leading logarithm corrections to $m_{\overline{MS}}^2$.

Expanding the right hand side of equation (19) in λ we have

$$\begin{aligned}m_{NO}^2(\mu') &= m_{NO}^2(\mu) \left[1 + \frac{\lambda(\mu)}{16\pi^2} \ln \left(\frac{\mu'}{m_{NO}(\mu')} \right) + \mathcal{O}(\lambda^2) \right] \\ &= m_{NO}^2(\mu) \left[1 + \frac{\lambda(\mu)}{16\pi^2} \ln \left(\frac{\mu'}{m_{NO}} \right) \left(1 + \frac{3\lambda(\mu)}{16\pi^2} \ln(\mu'/\mu) + \dots \right) + \mathcal{O}(\lambda^2) \right].\end{aligned}\tag{22}$$

Whereas the expansion for the parameter $m_{\overline{MS}}^2(\mu')$ in equation (21) includes a sum of the leading logarithm corrections, there is no such sum in the expression for $m_{NO}^2(\mu')$ of the NO Lagrangian. To $\mathcal{O}(\lambda)$ we see that both equations (21) and (22) include the lowest order leading logarithm correction to equation (18), $\lambda(\mu) \ln(\mu'/\mu)/(16\pi^2)$. However, at higher order in λ the NO prescription fails to incorporate the leading logarithm corrections to the mass parameter in any simple manner. To derive the leading logarithm corrections to m_{NO}^2 of $\mathcal{O}(\lambda^2)$ we would need to compute the two-loop order corrections to m_{NO}^2 . [This is nonzero due to the contribution to $\Gamma^{(2)NO}$ from the “setting sun” diagram of Figure 2.]

3. Conclusion

In this letter we have compared the $\overline{MS}$ and NO renormalization prescriptions. Using each prescription we have computed the one-loop order corrections, and by solving the resulting renormalization group equations we have shown how the parameters evolve as we change the renormalization scale μ . As is well known, when renormalizing the theory using $\overline{MS}$ the solution of the renormalization group equation includes a sum of the leading logarithm radiative corrections. We have shown that this does not happen when one makes use of the NO renormalization prescription.

Let us comment on the NO prescription. The NO prescription is a mass-dependent renormalization prescription. Three complications arise. First, the anomalous dimensions depend explicitly on λ as well as on the dimensional parameters m_{NO} and μ . Because it depends on three variables, the renormalization group equation can be difficult to solve.

Secondly, the solution to the NO prescription renormalization group equation fails to include a sum of the leading logarithm radiative corrections. We can see this by looking at the expression for γ_m^{NO} in equation (14). At one-loop order $\gamma_m^{NO} = 0$, which implies that the leading logarithm corrections to m_{NO} vanish even though we know they are nonzero due to the contribution from the “setting sun” diagram and similar diagrams of higher order in λ . Since there is no term of $\mathcal{O}(\lambda)$, the second order leading logarithm contribution of $\mathcal{O}(\lambda^2)$ is

included with the subleading logarithm contribution at $\mathcal{O}(\lambda^2)$ in γ_m^{NO} . Instead of the good ordering we have when using $\overline{MS}$ [where $\gamma_m^{\overline{MS}} = A_1\lambda + A_2\lambda^2 + \dots$, where A_1 accounts for the leading logarithm contributions, A_2 accounts for the subleading logarithm contributions, ...], we have a situation in the NO case where the ordering is confused: γ_m^{NO} accounts for both leading and subleading logarithmic contributions at the same order, λ^2 . When using the NO prescription to derive leading logarithm results of $\mathcal{O}(\lambda^n)$ it may be necessary to compute the contributions to the Green’s functions from n -loop Feynman diagrams. [We have seen above that to compute $\mathcal{O}(\lambda^2)$ leading logarithm corrections to m_{NO} it is necessary to include corrections from the two-loop Feynman diagrams.]

Finally, in the NO Lagrangian we have terms of $\mathcal{O}[\ln(\mu/m_{NO})]$. The concern is that if we consider the theory at some high renormalization scale μ , with $\mu \gg m_{NO}$, these logarithmic terms in the NO Lagrangian become divergent and we can no longer derive reliable results when working in perturbation theory. Alternatively in the $\overline{MS}$ Lagrangian when we consider the theory at some high renormalization scale, the parameters of the theory evolve according to equation (15): the divergent logarithm terms are absorbed into the evolution of the parameters, and because the leading logarithms are summed the parameters in the $\overline{MS}$ Lagrangian remain finite.

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References

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Fig. 1. Diagrams contributing to the 1PI Green's functions at the one-loop order.

Fig. 2. "Setting sun" diagram.