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On the Precision of the Computation of the QCD Corrections to Electroweak Vacuum Polarizations

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We demonstrate that the dispersive computation of the threshold enhancements to heavy quark vacuum polarizations is unstable. Because of the slow convergence of the dispersion relations the result critically depends on the intermediate energy region where the non-relativistic approximation, intrinsic to threshold calculations, is invalid. We discuss other ambiguities precluding a reliable calculation of the threshold contribution to the vacuum polarizations. In the absence of a solution prudence should force one to assign an error to the radiative corrections not far below the level of the perturbative $O(\alpha_s)$ contributions. This may preclude the extraction of the Higgs mass from precision measurements.

Measurements of the electroweak parameters are sufficiently precise to require a quantitative evaluation of the QCD corrections [1]. Diagrams where a gluon is exchanged between the b,t-quarks inside vacuum polarization loops cannot be neglected. Whereas the perturbative part of this $O(\alpha_s)$ gluon correction to the quark loops can be reliably calculated using either dispersive or straightforward diagrammatic techniques [2-4], evaluation of the threshold contribution of heavy quark bound states seems less straightforward. We demonstrate in this paper that the use of dispersion relations to account for threshold effects is, in this case, unfortunately intrinsically unstable. The calculational results critically depend on the behavior of the vacuum polarizations functions at an intermediate energy where the asymptotic (perturbative) limit is not valid, but which is sufficiently high to invalidate the threshold resummation. Alternatively, the result depends critically on the computation of the threshold effects in an energy region where the non-relativistic approximation, central to all estimates, is no longer valid.

Although we offer no solution, we find the problem sufficiently important to expose the unstable nature of the computation in this paper. Conflicting results in the literature can be readily understood. More importantly, the ambiguity associated with the computation of the QCD corrections to the vacuum polarizations is large enough to make any determination of the Higgs mass via high precision measurements of electroweak parameters questionable in the absence of a resolution of the problem described in what follows.

Computation of QCD corrections to the vacuum polarization can be performed via once-subtracted dispersion relations. As usual, the subtraction is not unique. Two subtraction prescriptions appear in the literature by Sirlin and Kniehl(SK) [5],

$$\Pi^{VA}(s, m_1, m_2) = \frac{1}{\pi} \left[\int_{(m_1+m_2)^2}^{\Lambda^2} ds' \frac{Im\Pi^{VA}(s', m_1, m_2)}{s' - s - i\epsilon} + \int_{(m_1+m_2)^2}^{\Lambda^2} ds' Im\chi^{VA}(s', m_1, m_2) \right] \quad (1)$$

and Gaemers et al.(G) [2]

$$\Pi^{V,A}(s, m_1, m_2) = \frac{1}{\pi} \left[\int_{(m_1+m_2)^2}^{\Lambda^2} ds' \frac{Im\Pi^{V,A}(s', m_1, m_2)}{s' - s - i\epsilon} - \frac{1}{2} \sum_{i=1}^2 \int_{4m_i^2}^{\Lambda^2} ds' \frac{Im\Pi^V(s', m_i, m_i)}{s' - s - i\epsilon} \right], \quad (2)$$

where $\Pi^{V,A}$ and $\lambda^{V,A}$ are the $g_{\mu\nu}$ and $q_\mu q_\nu$ coefficients of the vacuum polarization tensors for the vector and axial currents, *i.e.*

$$\Pi_{\mu\nu}^{V,A}(q, m_1, m_2) = \Pi^{V,A}(s, m_1, m_2) g_{\mu\nu} + \lambda^{V,A}(s, m_1, m_2) q_\mu q_\nu \quad (3)$$

Perturbatively, the functions Π and λ can be expanded in the coupling constant:

$$\Pi(s) = \Pi^0(s) + \frac{\alpha_s}{\pi} \Pi^1(s) + \dots \quad (4)$$

Both subtractions yield identical results for the perturbative gluon correction to quark loops in the gauge bosons self energies. This result also agrees with direct computation of the 2-loop diagrams for a constant value of α_s . This was first demonstrated for the case of a massless bottom quark. Recently the equivalence between the SK dispersion and the dimensional regularization evaluation of $\Delta\rho$ was established to $O(\alpha_s)$ [6] for a finite m_b . We explicitly demonstrated that both subtraction prescriptions yield the same result for a finite m_b for Δr and $\Delta\rho$.

Although the choice of subtraction is ambiguous, arguments can be advanced supporting the scheme proposed by Kniehl and Sirlin [7]. They demonstrated that the gluon corrections to the process $H^0 \rightarrow l^+ l^-$ are different for the two dispersion relations at the $O(\alpha_s)$ level. The result of a direct computation of the diagrams using dimensional regularization agrees with the first approach, Eq. (1). In the following we will therefore use the SK subtraction.

The dispersive approach is, in principle, most suitable to incorporate the $t\bar{t}$ threshold enhancement into the vacuum polarization functions since such effects are usually expressed in terms of contributions to the absorptive parts. They are calculated below threshold by summing over the $t\bar{t}$ resonances,

$$\begin{aligned} \frac{1}{s} Im\Pi^V(s, m_t, m_t) &= N_c(1 - 4C_F \frac{\alpha_s}{\pi}) \sum_n \frac{[R_n(0)]^2}{M_n^2} \delta(s - M_n^2) \\ - Im\lambda^A(s, m_t, m_t) &= N_c(1 - 3C_F \frac{\alpha_s}{\pi}) \sum_n \frac{[R_n(0)]^2}{M_n^2} \delta(s - M_n^2) \end{aligned} \quad (5)$$

where R_n is the wave function of the S -wave n^{th} resonance $t\bar{t}$ state with mass M_n .

Above threshold there are a variety of equivalent ways to introduce the threshold effects.

They may be estimated using the Green's function by solving the Schrödinger equation [8].

The Green function is readily related to the vacuum polarization functions:

$$\begin{aligned} \frac{1}{s} Im\Pi^V(s, m_t, m_t) &= N_c(1 - 4C_F \frac{\alpha_s}{\pi}) \frac{1}{2m_t^2} ImG(0, 0; \sqrt{s} - 2m_t + i\Gamma_t) \\ - Im\lambda^A(s, m_t, m_t) &= N_c(1 - 3C_F \frac{\alpha_s}{\pi}) \frac{1}{2m_t^2} ImG(0, 0; \sqrt{s} - 2m_t + i\Gamma_t). \end{aligned} \quad (6)$$

$G(0, 0; \sqrt{s} - 2m_t + i\Gamma_t)$ is the Green's function evaluated at the origin and Γ_t is the top width. Although the Green's function must, in general, be evaluated numerically, for a Coulomb potential the Schrödinger equation is solvable and we obtain the well-known resummation factor

$$\begin{aligned} \frac{1}{s} Im\Pi^V(s, m_t, m_t) &= N_c(1 - 4C_F \frac{\alpha_s}{\pi}) \frac{\frac{4}{3} \frac{\pi\alpha_s}{\beta}}{1 - \exp(-\frac{4}{3} \frac{\pi\alpha_s}{\beta})} \frac{1}{s} Im\Pi_0^V(s, m_t, m_t) \\ - Im\lambda^A(s, m_t, m_t) &= N_c(1 - 3C_F \frac{\alpha_s}{\pi}) \frac{\frac{4}{3} \frac{\pi\alpha_s}{\beta}}{1 - \exp(-\frac{4}{3} \frac{\pi\alpha_s}{\beta})} \frac{1}{s} Im\Pi_0^V(s, m_t, m_t) \end{aligned} \quad (7)$$

These equations give the correct behavior of the imaginary parts of the vacuum polarization functions close to threshold. The factor $(1 - C_F \alpha_s/\pi)$ incorporates the next term in the threshold expansion $(\alpha_s/\beta)^n$, where β is the relative velocity between the $t\bar{t}$ pair. We show in figure 1 the solutions of the Schrödinger equation for the potential J of Ref. [9] for three different values of Λ_{QCD} . Also shown are the perturbative results for three values of the argument of α_s ($\alpha_s(m_t/4)$, $\alpha_s(m_t)$, $\alpha_s(4m_t)$). We here should draw attention to the fact that the threshold resummation is valid only near threshold, *i.e.*

$$\beta = \left(1 - \frac{4m_t^2}{s}\right)^{1/2} \ll 1. \quad (8)$$

At this point the calculation is, in principle, straightforward. One calculates the threshold behavior of the quarks in the loop at low β and subsequently "forces" the result to smoothly join the perturbative answer [4]. The latter indeed represents the correct answer far above

threshold. The value where this junction occurs we denote by β_{cut} . The contribution above threshold to the real part of the polarization functions is then calculated by

$$\Pi(s') = \frac{1}{\pi} \int_{4m_t^2}^{s_{\text{cut}}} ds \frac{\text{Im}\Pi_{\text{thres}}(s) - \text{Im}\Pi_{\text{pert}}(s)}{(s - s')} \quad (9)$$

where $s_{\text{cut}} = s(\beta_{\text{cut}})$. One must realize that, no matter what approach one follows, an equation like (9) holds. E.g., Ynduráin introduces it explicitly in Eq. (11) of Ref. [10]. In the paper by Sirlin and Kniehl [7], they simulate the threshold behavior by introducing a running α_s with an argument which is singular at threshold

$$\text{Im}\Pi_{\text{thres}}(s) = \text{Im}\Pi^0(s) + \frac{\alpha_s(xp^2)}{\pi} \text{Im}\Pi^1(s) \quad (10)$$

where $p = \sqrt{\frac{s}{4} - m_t^2}$ is the three momentum of the $t\bar{t}$ pair and x is a number of order 1. $\text{Im}\Pi^0$ are defined by Eq. (4). The threshold contribution is then obtained by integrating the function (10) w.r.t to the matching point $xp^2 = m_t^2$. In this way they fit the threshold function by the running of α_s at low β and, at high energy, they match the perturbative expression.

It is clear that these procedures are far from unique. We will see that the results, unfortunately, depend critically on these arbitrary procedures. We illustrate this further by using the elegant technique proposed by Voloshin [11]. Voloshin has obtained an interpolating formula between the threshold behavior and the asymptotic perturbative result for the production of $\tau^+\tau^-$ from e^+e^- [11]. This formula correctly accounts for the limiting behavior at both ends ($\beta \rightarrow 0, 1$) and, smoothly interpolates between them. It is straightforward to rewrite it for a heavy quark threshold. Using the Coulombic resummation (7) his result can be written as

$$\text{Im}\Pi(s) = F_t(s) \text{Im}\Pi^0(s) (1 + X(s)) \quad (11)$$

where $F_t = \frac{4\pi\alpha_s}{1 - \exp(-\frac{4\pi\alpha_s}{3\beta})}$ is the Coulomb threshold factor, $\text{Im}\Pi^0$ is the zero order, one loop, vacuum polarization function and X can be written as

$$X(\beta) = \frac{\alpha_s}{\pi\beta} \left\{ (1 + \beta^2) \left[\ln \frac{1+\beta}{1-\beta} \ln \frac{1+\beta}{1-\beta} + 2L\iota_2\left(\frac{1+\beta}{1+\beta}\right) - 2L\iota_2\left(\frac{1+\beta}{2}\right) - 4L\iota_2(\beta) + L\iota_2(\beta^2) \right] + \left[\frac{11}{8}(1 + \beta^2) - 3\beta + \frac{1}{2}\frac{\beta^4}{1-\beta^2} \right] \ln \frac{1+\beta}{1-\beta} + 6\beta \ln \frac{1+\beta}{2} - 4\beta \ln \beta + \frac{3}{4}\beta \frac{5-3\beta^2}{1-\beta^2} \right\}. \quad (12)$$

Eq. (11) incorporates the radiative corrections with correct $\beta \rightarrow 0, 1$ limits. In the above equation, we will parametrize the scale of α_s to be

$$\mu^2 = xp^2 + \beta (\mu_{\text{pert}}^2 - xp^2) \quad (13)$$

where μ_{pert} is the perturbative scale, and x is a number of order 1. This way we reproduce indeed the threshold correction at low β and the perturbative result at high β . In figure 2 we plot the perturbative imaginary vacuum polarization function for $\mu_{\text{pert}}^2 = m_t^2$, the threshold function as given by Eq. (10) and the Voloshin result of Eq. (11) for $x = 1/4, 1, 4$. As expected, the matching point is a function of x , a parameter undetermined to this order. The crucial point is that the dependence of the result on the value of β_{cut} is highly unstable. Numerically β_{cut} is not small. In the paper of Ynduráin, he writes $\beta_{\text{cut}} = \pi/\sqrt{2}C_F\alpha_s$ which is of the order of 0.3. In the work by Sirlin and Kniehl, they use $x = 4$ which gives $\beta_{\text{cut}} = \sqrt{(1 - 4/5)} = 0.45$. As will be illustrated next, the threshold contribution to the electroweak parameters obtained using Eq. (9) not only depend strongly on β_{cut} , one might question the use of the non-relativistic approximation at such large values of the velocity.

We next calculate the threshold contribution of the $t\bar{t}$ pair to the electroweak parameter $\Delta\rho$ and $\Delta\gamma$. In figures 3–4 we plot the contribution to these parameters as a function of the “matching point” for the approach of Sirlin and Kniehl. We repeat the calculation using the Voloshin interpolation formula. We present the results for three values of x , $x = 1/4, 1, 4$. In Tables I and II we show the order zero and first order contributions to these parameters as well as the range of threshold contributions obtained using the two methods. As can be seen from the figures the actual value of the contribution depends strongly on the value of β_{cut} or, alternatively on the value of x .

We conclude that

- there is a wide range of results which depend on an arbitrary parameter which is not controlled to this precision of perturbation theory,
- the threshold enhancement is potentially large. As the calculation turns out to be unstable it is impossible to make any arguments in favor of small or large results. The uncalculable threshold contribution is possibly as large as the total Higgs contribution to Δr , which varies from -2.78×10^{-3} to 1.1×10^{-2} depending on the value of the Higgs mass (0.1–1 TeV),
- large values of the threshold enhancement do correspond to large values of β_{cut} and one must conclude that the result, in such a case, is suspect. The essential problem is however that the dispersion relations converge slowly and the results therefore critically depend on the threshold enhancement in a region where the low β approximation is no longer valid. The values for β_{cut} are high for all examples.

The discussion has so far ignored other challenges in performing a reliable calculation

- threshold corrections have only been calculated in the abelian limit,
- the calculation is unreliable far below threshold (we followed the method of Ref. [5]),
- the subtraction is not unique. The two subtraction schemes discussed here give different results which are roughly opposite in sign. The origin of the problem is the large mass splitting between top and bottom. We have been unsuccessful in tracing the discrepancy between both dispersion relations.

Obviously the problem discussed here is urgent. In the absence of a solution prudence will force one to essentially assign errors to the vacuum polarizations not far below the level of the leading QCD correction. This precludes extracting any information on the Higgs mass and degrades significantly the indirect information on the top mass, especially in the case of a heavy top.

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TABLES

TABLE I. Perturbative and threshold contributions to Δr . The $O(\alpha_s)$ was computed for $\Lambda_{QCD} = 400$ MeV and the scales $\mu_{\text{pert}}^2 = m_t^2/4, m_t^2$ (central value), $4m_t^2$. The contributions below threshold are given for the J potential for $\Lambda_{QCD} = 100\text{--}500$ MeV. The maximum contribution above threshold is quoted for $x = 4$ in Eq. (10).

m_t	$O(\alpha)$	$O(\alpha_s) \times 10^3$	below threshold $\times 10^3$	above threshold $\times 10^3$
120	-0.045	2.10 +0.24 -0.18	0.15 - 0.34	0 - 0.94
160	-0.059	3.04 +0.33 -0.25	0.28 - 0.58	0 - 1.5
200	-0.076	4.32 +0.46 -0.34	0.38 - 0.12	0 - 2.1

TABLE II. Same as previous table for $\Delta\rho$.

m_t	$O(\alpha)$	$O(\alpha_s) \times 10^4$	below threshold $\times 10^4$	above threshold $\times 10^4$
120	0.0043	-4.56 +0.39 -0.52	-(0.43 - 0.96)	-(0 - 2.8)
160	0.0077	-7.96 +0.82 -0.66	-(0.81 - 1.7)	-(0 - 4.0)
200	0.012	-11.8 +0.93 -1.2	-(1.1 - 2.4)	-(0 - 6.3)

FIGURES

FIG. 1. Contributions to the imaginary part of the vacuum polarization function from $O(\alpha)+O(\alpha\alpha_s)$ radiative corrections (dotted lines) and from the threshold enhancement given by the solution of the Schrödinger equation using the Igi-Ono potentials (full lines). Dotted lines are the perturbative functions for $\mu_{\text{pert}}^2 = m_t^2/4, m_t^2, 4m_t^2$ (from top to bottom) and $\Lambda_{QCD\text{pert}}^4 = 400 \text{ MeV}$. Continuous lines are the solutions of the Igi-Ono potentials for $\Lambda_{QCD}^4 = 100, 300, 500 \text{ MeV}$ (from bottom to top).

FIG. 2. Imaginary part of the vacuum polarization function for $m_t = 160 \text{ GeV}$. Full line represents the perturbative result for $\mu_{\text{pert}}^2 = m_t^2$ and $\Lambda_{QCD\text{pert}}^4 = 400 \text{ MeV}$. Dotted lines are the interpolating formula by Voloshin with $x = 1/4, 1, 4$ (top to bottom). Dashed lines are the perturbative result with running α_s as used by Sirlin and Kniehl with the same values of x .

FIG. 3. Threshold contribution to $\Delta\rho$ for $m_t = 160 \text{ GeV}$ as a function of β_{cut} for two approaches discussed in the text. Dashed lines correspond to the prediction of Sirlin and Kniehl with $x = 1/4, 1, 4$. Dotted lines are the prediction using the Voloshin interpolating formula with the same values of x .

FIG. 4. Same as in figure 3 for Δr .

Fig. 1

Fig. 2

Fig. 4

Fig. 3