Two dimensional Sen connections in general relativity

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Abstract

The two dimensional version of the Sen connection for spinors and tensors on spacelike 2-surfaces is constructed. A complex metric $\gamma_{AB}$ on the spin spaces is found which characterizes both the algebraic and extrinsic geometrical properties of the 2-surface $\mathcal{S}$. The curvature of the two dimensional Sen operator $\Delta_\epsilon$ is the pull back to $\mathcal{S}$ of the anti-self-dual part of the spacetime curvature while its ‘torsion’ is a boost gauge invariant expression of the extrinsic curvatures of $\mathcal{S}$. The difference of the 2 dimensional Sen and the induced spin connections is the anti-self-dual part of the ‘torsion’. The irreducible parts of $\Delta_\epsilon$ are shown to be the familiar 2-surface twistor and the Weyl–Sen–Witten operators. Two Sen–Witten type identities are derived, the first is an identity between the 2 dimensional Sen and the Weyl–Sen–Witten operators and the integrand of Penrose’s charge integral, while the second contains the ‘torsion’ as well. For spinor fields satisfying the 2-surface twistor equation the first reduces to Tod’s formula for the kinematical twistor.

Introduction

It is well known that one cannot associate gauge-independent energy-momentum (and angular momentum) density with the gravitational ‘field’; i.e. any such expression is pseudotensorial or, in the tetrad formalism of gravity, depends on the tetrad field too. For asymptotically flat spacetimes, however, one can define the total energy-momentum [1-3]. One of the most important results of the last decade in the classical relativity theory is the better understanding of the energy-momentum of localized gravitating systems, especially the proof of the positivity of the ADM and Bondi–Sachs masses [4-8]. These results can naturally be recovered from the spinorial Sparling equation if on the hypersurfaces extending either to spatial or null infinity the 3 dimensional Sen connection is used [9,10]. This Sen connection seems therefore to be the ‘natural’ connection on these hypersurfaces in the energy-momentum problems of gravity.

These successes inspired several relativists to search for expressions of the gravitational energy-momentum at the quasi-local level; i.e. to associate these physical quantities with closed spacelike 2-surfaces [11-25]. The definition of the quasi-local energy-momentum, however, is far from being so obvious as for example the Bondi–Sachs energy-momentum. There are several inequivalent proposals for it [26], and it is not clear how they are related to each other. The usual formalism to carry out the calculations in the spinorial constructions [18-25] is the elegant GHP formalism [27-29]. Since however its form is not covariant, the geometric content of the expressions is not always obvious and a lot of experience is needed to ‘see’ the geometric content.
The covariance of a formalism may help and suggest why and what to calculate. Furthermore, since the 3 dimensional Sen connection seems to be 'natural' in the energy-momentum problems of gravity, one might conjecture that the two dimensional version of the Sen connection has also significance in general relativity [30]. Thus in the present paper, which is intended to be the first of a four paper series, we would like to develop such a covariant spinor formalism that might be the 2 dimensional version of the usual Sen connection and in which the various constructions can be compared. This formalism may help to find the 'most natural' quasi-local energy-momentum expression; or at least yields a better understanding of the energy-momentum problems of general relativity by giving new insight into the geometry of the spacelike 2-surfaces.

In the first two sections we review the geometry of spacelike 2 dimensional submanifolds, introduce the 2 dimensional version of the Sen operator and calculate its curvature and torsion. Then, in section 3, we discuss the algebra of surface spinors and find a complex metric $\gamma_{AB}$ on the space of spinors. $\gamma_{AB}$ will have fundamental importance in what follows. In section 4 the spinor form of the 2 dimensional Sen operator will be discussed, and, in section 5, we extend the 2 dimensional covariant differentiation to spinors. We will see that the 2 dimensional Sen connection is not simply a copy of the 3 dimensional one in one less dimensions. An important difference between the one and two co-dimensional submanifolds is that while in the one co-dimensional case the normal is uniquely determined, in the two co-dimensional case there is a 1 parameter family of unit normals, and as a consequence of this 'boost gauge freedom' the 2 dimensional Sen (and spin) connection has always a 'normal' piece as well.

In section 6 the irreducible chiral parts of the 2 dimensional Sen operator will be determined. They are precisely the right and left handed parts of the 2-surface twistor and the 2 dimensional Weyl-Sen-Witten operators, where the chirality is defined by the $\gamma^{AB}$ spinor. In section 7 we discuss the 2 dimensional counterparts of the 3 dimensional Sen-Witten type identities. These are identities between the 2 dimensional Weyl-Sen-Witten and 2 dimensional twistor operators and the integrand of various quasi-local charge integrals of the curvature and the torsion of the 2 dimensional Sen operator. The Tod formula for the kinematical twistor is a special consequence of them.

In the present paper we work out only the general formalism. This formalism will be applied only in the forthcoming papers for the quasi-local energy-momentum, the quasi-local characterization of pp-wave spacetimes and the gravitational radiative modes. Throughout the present paper the abstract index formalism [28] will be used unless otherwise stated.

1. Two dimensional spacelike submanifolds

Let $(M, g)$ be a four dimensional Lorentzian geometry with signature $-2$ and $\Sigma$ be a two dimensional spacelike submanifold. Let $t^a$ and $v^a$ be timelike and spacelike unit normals to $\Sigma$ being orthogonal to each other: $t^at_a = 1$, $v^av_a = -1$ and $t^av_a = 0$. (If $\Sigma$ is orientable and an open neighbourhood of $\Sigma$ in $M$ is space and time orientable, which will be assumed in the present paper, then $t^a$ and $v^a$ are globally well defined.) $t^a$ and $v^a$ are, of course, not unique, there is the gauge freedom

\[ t^a = t^a \cosh u + v^a \sinh u \]
\[ v^a = t^a \sinh u + v^a \cosh u. \]  (1.1)
(1.1) will be called a boost gauge transformation. The projection to the tangent spaces of \( S \) is given by

\[
\Pi^a_b := \delta^a_b - t^a t_b + v^a v_b. \tag{1.2}
\]

A tensor \( T^{a_1 \ldots a_r}_{b_1 \ldots b_r} \) is called a surface tensor if \( T^{a_1 \ldots a_r}_{b_1 \ldots b_r} = \Pi^{a_1}_{e_1} \ldots \Pi^{a_r}_{e_r} T^{e_1 \ldots e_r}_{f_1 \ldots f_r} \). Specially \( q_{ab} := \Pi^a_b \Pi^b_a g_{ij} = g_{ab} - t_a t_b + v_a v_b \) is the induced (negative definite) metric and (in the non-abstract index formalism) \( dS = \frac{1}{2} t^a v^b \varepsilon_{abcd} dx^c \wedge dx^d \) is the induced volume element on \( S \). Obviously, \( \Pi^a_b, q_{ab}, dS \) are all boost gauge invariant.

For any surface vector field \( X^a \) let us define

\[
\delta_a X^b := \Pi^b_a \nabla_a X^f. \tag{1.3}
\]

Since \( \delta_a q_{bc} = 0 \) and \( (\delta_a \delta_b - \delta_b \delta_a) \phi = 0 \) for any function \( \phi \), \( \delta_a \) is the unique torsion free metric Levi-Civita covariant differentiation. This is also boost gauge invariant.

To characterize the extrinsic geometry of \( S \) in \( M \) certain boost gauge dependent quantities have to be introduced:

\[
\tau_{ab} := \Pi^c_a \Pi^d_b \nabla_c X^f, \\
\nu_{ab} := \Pi^c_a \Pi^d_b \nabla_c v^f. \tag{1.4}
\]

are the (symmetric) extrinsic curvatures and let

\[
A_a := \Pi^b_a \nabla_f t_e v^e. \tag{1.5}
\]

Under a boost gauge transformation they transform as

\[
\tau'_{ab} = \tau_{ab} \cosh u + \nu_{ab} \sinh u, \\
\nu'_{ab} = \tau_{ab} \sinh u + \nu_{ab} \cosh u, \\
A'_a = A_a - \delta_a u. \tag{1.6}
\]

If the curvature tensors are defined by \( R^a_{bcd} X^b := - (\nabla_c \nabla_d - \nabla_d \nabla_c) X^a \) and \( ^b R^a_{bcd} X^b := - (\delta_a \delta_d - \delta_d \delta_a) X^a \) for surface vectors, respectively, then

\[
R_{ef \delta} \Pi^e_a \Pi^f_b \Pi^\delta_c = ^b R_{abcd} + \tau_{ac} \tau_{bd} - \tau_{ad} \tau_{bc} - \nu_{ac} \nu_{bd} + \nu_{ad} \nu_{bc}, \\
t^a R_{ef \delta} \Pi^e_a \Pi^f_b = \delta_e \tau_{db} - \delta_d \tau_{eb} + A_c \nu_{eb} - A_d \nu_{eb}, \\
v^a R_{ef \delta} \Pi^e_a \Pi^f_b = \delta_e \nu_{db} - \delta_d \nu_{eb} + A_c \tau_{eb} - A_d \tau_{eb}, \\
t^a v^b R_{ef \delta} \Pi^e_a \Pi^f_b = \tau_{ec} \nu_{fb} - \tau_{fd} \nu_{ec} + \delta_e A_d - \delta_d A_e. \tag{1.7}
\]

The remaining six ‘irreducible’ parts of the curvature, \( t^a t^f R_{ae \delta f} \Pi^e_a \Pi^\delta_c, \ldots, t^a v^b t^c v^d R_{abcd} \), can also be expressed by the extrinsic curvatures, the vector potential \( A_a \) and additional boost gauge dependent quantities. However, they are rather complicated and we do not need them. Since \( S \) has dimension 2, its curvature tensor can be characterized completely by its curvature scalar:

\[
^b R_{abcd} = \frac{1}{2} ^b R(q_{ae} q_{bd} - q_{ad} q_{be}).
\]
2. The two dimensional Sen operator $\Delta_a$

Let us define the 2 dimensional version of the Sen operator: $\Delta_a := \Pi_a^c \nabla_c$. Obviously $\Delta_a$ is a differential operator acting on any tensor field and annihilates the metric: $\Delta_a g_{bc} = 0$. Let

$$Q^\epsilon_{ab} := -\Pi^\epsilon_a \Delta_b \Pi^b_c = \tau^\epsilon_{ab} - \nu^\epsilon_{ab}. \quad (2.1)$$

This is by definition boost gauge invariant, and for any surface vector $X^a$ one has

$$\delta_a X_b = \Delta_a X_b + Q^\epsilon_{ab} X^\epsilon. \quad (2.2)$$

The action of the commutator of the $\Delta_a$’s on arbitrary functions and vector fields are

$$(\Delta_a \Delta_b - \Delta_b \Delta_a) \phi = -2Q^\epsilon_{[ab]} \Delta_c \phi \quad (2.3)$$

$$(\Delta_a \Delta_b - \Delta_b \Delta_a) X^\epsilon = -R^\epsilon_{f [ab]} \Pi^f_a X^f - 2Q^\epsilon_{[ab]} \Delta_f X^f. \quad (2.4)$$

The curvature and the torsion of $\Delta_a$ are therefore $F^\epsilon_{f ab} := R^\epsilon_{f [ab]} \Pi^f_a$ and $T^\epsilon_{ab} := 2Q^\epsilon_{[ab]}$, respectively. By (2.1) the torsion is built up only from extrinsic quantities, while the curvature can be expressed by the intrinsic geometry of $\Sigma$, the extrinsic curvatures and the vector potential $A_a$ (eq.(1.7)). The curvature can be reexpressed by the intrinsic curvature, the boost gauge independent $Q^\epsilon_{ab}$ and its $\Delta_a$-derivatives and the field strength $\delta_c A_d - \delta_d A_c$ of the $A_c$ field:

$$F_{abcd} = \delta R_{abcd} + (\nu_a t_b - \nu_b t_a) \left( \delta_c A_d - \delta_d A_c \right) +$$

$$+ 2 \Delta_a Q_{[a|b]} - 2 \Delta_c Q_{[d|a]} - 4Q_{[c|\eta]} Q^\epsilon_{[\epsilon|\eta]} +$$

$$+ g^{ef} \left( Q_{ecb} Q_{fda} - Q_{edb} Q_{fca} - Q_{ace} Q_{bdf} + Q_{ad} Q_{bc} \right) \quad (2.5)$$

It might be worth noting that $\Delta_a$ is a covariant differentiation in the sense of [31] on the pull back to $\Sigma$ of the tensor bundle over $M$; and its curvature, defined by $-F^a_{b[cd]X^b V^c Z^d} := \Delta V \Delta Z X^a - \Delta Z \Delta V X^a - \Delta [V, Z] X^a$ for any $X^a$ and surface vector fields $V^a$, $Z^a$, is precisely what we called curvature. For connections on principle fibre bundles not isomorphic to a (not necessarily nontrivial) reduced subbundle of the linear frame bundle of the base manifold the torsion is not defined.

Here the principle fibre bundle on which the 2 dimensional Sen connection is defined is the pull back to $\Sigma$ of the linear frame bundle $L(M)$. Thus the torsion of the Sen connection cannot be defined in the strict sense of [31]. In fact, while the curvature $F^a_{b[cd]}$ is a $gl(4, \mathbb{R})$ valued 2-form on $\Sigma$, the ‘torsion’ defined in (2.3) is not an $\mathbb{R}^4$ valued 2-form on $\Sigma$. However, in the calculations and formulae, e.g. in (2.3,4), $2Q^\epsilon_{[a|\eta]}$ behaves as a true torsion. This is the reason why we will call $2Q^\epsilon_{[a|\eta]}$ the torsion further on.

3. The algebra of 2-surface spinors

If $t^{A_1}$ and $v^{A_1}$ are the spinor form of the normals to $\Sigma$ then

$$2t^{A_1} t_{B_2} = \delta^A_B, \quad 2v^{A_1} v_{B_2} = -\delta^A_B \quad (3.1)$$
and let us define

$$\gamma^A_{\ B} := 2t^{AR}v_{BR}. \quad (3.2)$$

It is easy to show that $\gamma^A_{\ B}$ is boost gauge independent, invariant with respect to the conformal rescaling of the metric and

$$\gamma^R_{\ R} = 0, \quad \gamma^A_{\ R}\gamma^R_{\ B} = \delta^A_{\ B}. \quad (3.3)$$

Thus $\gamma^A_{\ B}$ is nondegenerate and $\gamma : \mathbb{S}^4 \rightarrow \mathbb{S}^4 : \lambda^A \mapsto \gamma^A_{\ R}\lambda^R$ is an isomorphism of the spinor space $\mathbb{S}^4$ onto itself. Its eigenvalues are $\pm 1$, and hence $\gamma^A_{\ B}$ plays a role similar to the $\gamma_5$ matrix. Its eigenspinors may be called left handed and right handed with respect to $\gamma^A_{\ B}$. The spinor

$$\pi^{\pm A}_{\ B} := \frac{1}{2}(\delta^A_{\ B} \pm \gamma^A_{\ B}) \quad (3.4)$$

is the projection of the spin space to the subspace of right/left handed spinors: $\gamma^A_{\ R}\pi^{\pm R}_{\ B} = \pm \pi^{\pm A}_{\ B}$. The right/left handed spinors are pure spinors in the sense of [28]. (I am greatful to prof. H. Urbanik for this remark.) Right handed covariant spinors should be defined by $\pi^R_{\ AB}$: $\mu_R = \mu_S\pi^S_{\ R}$. This definition ensures that $\mu_S$ is a right/left handed covariant spinor iff $\mu^R$ is a right/left handed contravariant spinor. The spinor form of the induced volume form on $\mathbb{S}$ and on the 2-surface element orthogonal to $\mathbb{S}$ are given by

$$t^a\varepsilon_{abcd} = \frac{i}{2}\left(\gamma_{CD}\varepsilon_{C'D'} - \gamma_{C'D}\varepsilon_{CD}\right),$$

$$t_a\varepsilon_{ab} - t_b\varepsilon_{ab} = -\frac{1}{2}\left(\gamma_{CD}\varepsilon_{C'D'} + \gamma_{C'D}\varepsilon_{CD}\right), \quad (3.5)$$

respectively. Thus geometrically $\gamma_{AB}$ is the anti-self-dual part of the 2-surface element orthogonal to $\mathbb{S}$. $\gamma_{AB}$ can also be considered as a complex metric on $\mathbb{S}^4$. The null spinors of the metric $\gamma_{AB}$ are just the eigenspinors of $\gamma^A_{\ B}$ and hence ($\lambda^A, \gamma^A_{\ R}\lambda^R$) is an independent system iff $\lambda^A$ is not a null spinor. The group leaving invariant the complex metric $\gamma_{AB}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{C}^\ast$, where $\mathbb{C}^\ast := \mathbb{C} - \{0\}$; and the group leaving invariant both the symplectic and complex metrics is $\mathbb{C}^\ast$. As a consequence of the existence of the extra structure $\gamma_{AB}$ on $\mathbb{S}^4$ the decomposition $\phi_{AB} = \frac{1}{\sqrt{2}}\epsilon_{AB}\phi_R + \phi_{(AB)}$ of a spinor $\phi_{AB}$ is not irreducible any more. Its symmetric part can be decomposed further as $\phi_{(AB)} = -\frac{1}{2}\gamma_{AB}\phi_{RS} + \phi_{(AB)} + \frac{1}{2}\gamma_{AB}\gamma_{RS}\phi_{RS}$, the sum of the $\gamma$-trace and the trace-free symmetric part of $\phi_{AB}$. The elements of the spinor space $\mathbb{S}^4$ will be called spacelike 2 dimensional spinors if a spinor $\gamma^A_{\ B}$ satisfying (3.3) is given on $\mathbb{S}^4$. (These ‘surface’ spinors should not be confused with the one component reduced spinors [28] of the 2 dimensional geometry ($\mathbb{S}, q_{ab}$), which may also be called surface spinors.)

Let $(\alpha_4, \iota_4)$ be a normalized spinor dyad such that

$$t^a = \frac{1}{\sqrt{2}}\left(o^A\delta^4 + i^4T^4\right), \quad \nu^a = \frac{1}{\sqrt{2}}\left(o^A\delta^4 - i^4T^4\right). \quad (3.6)$$

Then $\gamma^A_{\ B} = o^A\iota_B + i^4o_B$ and hence

$$\gamma^A_{\ B}o^B = -o^A, \quad \gamma^A_{\ B}i^B = i^A. \quad (3.7)$$
Thus $\sigma^A$ and $\nu^A$ are null spinors, $\sigma^A$ is left handed and $\nu^A$ is right handed. Conversely, if $\sigma_A$, $\nu_A$ are left and right handed spinors, respectively, satisfying $\sigma_A \nu^A = 1$ then they form a GHP spinor dyad adapted to $\S$. The linear isomorphism $\gamma : \mathbb{S}^A \to \mathbb{S}^A$ can obviously be extended to the whole tensor algebra over $\mathbb{S}^A$. Its action on the vectors of the complex null tetrad is

$$
\gamma(l^a) = \gamma(\sigma^A)\overline{\gamma}(\tilde{\sigma}^A) = \bar{l}^a
$$

$$
\gamma(n^a) = \gamma(\nu^A)\overline{\gamma}(\tilde{\nu}^A) = \bar{n}^a
$$

$$
\gamma(m^a) = \gamma(\sigma^A)\overline{\gamma}(\tilde{\nu}^A) = -\bar{m}^a.
$$

Therefore a vector $X^a$ is tangent to $\S$ if $\gamma(X^a) = -X^a$ and $X^a$ is orthogonal to $\S$ if $\gamma(X^a) = X^a$. The spinor form of the projection $\Pi^a_2$ is therefore

$$
\Pi^a_2 = \left(\delta^A_B \delta^B_A - \gamma^A_B \gamma^B_A\right).
$$

and hence $q_{AB} = \frac{1}{2}(\varepsilon_{AB} \varepsilon_{CD} - \gamma_{AB} \gamma_{CD})$. The vectors $l^a$, $n^a$, $m^a$ and $\bar{m}^a$ can also be characterized as the cokernel of the projections $\pi^{A \bar{B}} \pi^{A \bar{B}}$, $\pi^{A \bar{B}} \pi^{A \bar{B}}$, $\pi^{A \bar{B}} \pi^{A \bar{B}}$ and $\pi^{A \bar{B}} \pi^{A \bar{B}}$, respectively. $\pi^{A \bar{B}} := \pi^{A \bar{B}} \pi^{A \bar{B}}$ define a complex structure on the tangent spaces of $\S$ and they are the projections to the subspace of $(1,0)$ and $(0,1)$ type vectors, respectively [32].

For any spinor $\lambda^A$ let us define

$$
L^a := (\pi^{A \bar{B}} \lambda^B) (\tilde{\pi}^{A \bar{B}} \tilde{\lambda}^B),
$$

$$
N^a := (\pi^{A \bar{B}} \lambda^B) (\tilde{\pi}^{A \bar{B}} \tilde{\lambda}^B),
$$

$$
M^a := (\pi^{A \bar{B}} \lambda^B) (\tilde{\pi}^{A \bar{B}} \tilde{\lambda}^B).
$$

It is easy to see that $L^a$ and $N^a$ are future directed null vectors orthogonal to $\S$, $M^a$ and $\bar{M}^a$ are complex null vectors tangent to $\S$ and $L^a N_a = -M^a \bar{M}_a = \frac{1}{\pi} |\gamma_{AB} \lambda^A \lambda^B|^2$ and $\varepsilon_{abcd} L^a N^b M^c \bar{M}^d = -\frac{1}{16} |\gamma_{AB} \lambda^A \lambda^B|^4$. $\{L^a, N^a, M^a, \bar{M}^a\}$ is therefore a future directed right handed complex null tetrad adapted to $\S$ unless $\lambda^A$ is a null spinor. If $\lambda^A$ becomes null then both $M^a$, $\bar{M}^a$ and either $L^a$ or $N^a$ will be zero. The corresponding orthogonal vector basis is

$$
T^a = \frac{1}{\sqrt{2}} (\lambda^A R^A + \gamma^A R^A \gamma^R \tilde{R}^A),
$$

$$
Z^a = -\frac{1}{2\sqrt{2}} (\lambda^A R^A \tilde{R}^A + \gamma^A R^A \tilde{R}^A),
$$

$$
X^a = \frac{1}{\sqrt{2}} (\lambda^A R^A - \gamma^A R^A \gamma^R \tilde{R}^A),
$$

$$
Y^a = \frac{i}{2\sqrt{2}} (\gamma^A R^A \tilde{R}^A - \lambda^A \gamma^R \tilde{R}^A).
$$

The length of these vectors is $\frac{1}{2} |\gamma_{AB} \lambda^A \lambda^B|$. If $\lambda^A$ becomes null then $T^a$ and $Z^a$ will be parallel null vectors. Thus a single non-null spinor field on $\S$ is able to define an orthogonal vector basis in the Lorentzian tangent spaces. Similar statement holds for any nonzero spinor field on a spacelike hypersurface [33].

The spinor $\gamma^A_B$ can be defined by intrinsic quantities too: if $x^A A^1$ and $y^A A^1$ are orthonormal vectors tangent to $\S$ then $J^A_B := 2x^A B^1 y^B R^R = i\gamma^A_B$. $J^A_B$, and hence $\gamma^A_B$ also, is rotation gauge invariant. The fact that $\gamma^A_B$ is globally well defined is a consequence of its definition (3.2) and the globality of $l^a$ and $n^a$. If we defined $\gamma^A_B$ by the intrinsic properties of $\S$ then it would not by definition be globally well defined. But, using its rotation gauge invariance, it would be easy to show that it is, in fact, globally well defined.
4. The spinor form of $\Delta_s$

The action of the commutator of the $\Delta_s$'s on a spinor field:

\[
\left(\Delta_e \Delta_d - \Delta_d \Delta_e\right)\xi^A = -R^A_{\cdots} \Pi^f \Pi^d_{\cdots} \xi^B - 2Q\Lambda A \Delta_e \xi^A, \tag{4.1}
\]

where $R^A_{\cdots} \Pi^f \Pi^d_{\cdots}$ is the anti-self-dual part of the spacetime curvature, and hence the curvature of the operator $\Delta_e$ is the pull back to $8$ of the anti-self-dual part of the spacetime curvature. In terms of the Weyl and Ricci spinors and the $\Lambda$ scalar it is given by

\[
F_{ABD} = R_{ABf} \Pi^{f}_{CC} \Pi^{d}_{DD} = \frac{1}{4} \varepsilon^{C,D} \left( \psi_{ABCD} - \psi_{ABEF} \gamma^E \gamma^F_D + \gamma_{CD} \phi_{ABEF} \tilde{\gamma}^{E/F}_D + \gamma_{AC} \gamma_{BD} - \gamma_{AD} \gamma_{BC} \right) - \frac{1}{4} \varepsilon^{C,D} \left( \tilde{\gamma}^{C,D} \psi_{ABEF} \gamma^{EF} + \phi_{ABCD} - \phi_{ABEF} \gamma^{E/F}_D - 2\Lambda \delta_{AB} \gamma^{C,D} \right). \tag{4.2}
\]

Its contraction with the induced volume form:

\[
F_{AB} e^f \varepsilon_{f} e^{cd} = -i \left( \psi_{ABCD} \gamma^{CD} - \phi_{ABCD} \gamma^{C,D} + 2\Lambda \delta_{AB} \right). \tag{4.3}
\]

To find the spinor form of the torsion and the expression of the curvature in terms of extrinsic and intrinsic geometrical quantities, let us define

\[
Q^{E}_a F := \frac{1}{2} \Delta_s \gamma^E R^a_{RF}. \tag{4.4}
\]

Then obviously $Q^{R}_a R = 0$ and by the definition (2.1) of $Q^{E}_a b$ and the spinor form (3.9) of the projection one has

\[
Q^{E}_a b = \frac{1}{2} \left( \delta_{b}^{E} \gamma^a_{RF} + \delta_{a}^{E} \gamma^b_{RF} + \gamma^E_{aR} \gamma^R_{bF} + \gamma^E_{bR} \gamma^R_{aF} \right). \tag{4.5}
\]

Since $Q^{E}_a b = Q^{E}_{a b}$ (cf. eq. (2.1)), $Q^{E}_a F$ has the ‘hidden’ symmetry

\[
Q_{EA E' F'} =\frac{1}{2} \left( \delta_{a}^{E} \gamma^R_{E'R'} + \gamma^{E'}_{aR} \gamma^R_{E'F'} \right) Q_{AF E'R} + \frac{1}{2} \left( \delta_{a}^{E} \gamma^E_{R F'} - \gamma^{E'}_{aF} \gamma^E_{R F'} \right) Q_{A E F'R}. \tag{4.6}
\]

The spinor form of the torsion is therefore

\[
T_{EE' A A' B B'} = - \left( \varepsilon_{A' B'} \gamma^E_{EE'} + \varepsilon_{A B} \gamma^E_{EE'} \right), \tag{4.7}
\]

and the expression (2.5) for the curvature is

\[
F^A_{B C D} = \frac{1}{2} \left( R^A_{B C D} \right) + \frac{1}{2} \left( \delta_{E}^{A} \gamma^B_{D} \gamma^D_{E} \gamma^C_{A} - \delta_{E}^{A} \gamma^B_{C} \right) + \Delta \gamma D \Delta Q^A - \Delta C \Delta Q^A \Delta D \Delta B + \frac{1}{2} \left( \gamma^B_{D} \gamma^D_{B} Q^A + \gamma^B_{C} \gamma^C_{B} Q^A \right) - \frac{1}{2} \left( \gamma^B_{D} \gamma^D_{B} Q^A + \gamma^B_{C} \gamma^C_{B} Q^A \right) + \frac{1}{2} \left( \gamma^B_{D} \gamma^D_{B} Q^A + \gamma^B_{C} \gamma^C_{B} Q^A \right) + \frac{1}{2} \left( \gamma^B_{D} \gamma^D_{B} Q^A + \gamma^B_{C} \gamma^C_{B} Q^A \right). \tag{4.8}
\]
where $^8R_{ABCD}$ is the curvature tensor of $^8$. As one may expect, the torsion contains all the information on the divergences and shears of the null geodesics orthogonal to $^8$. In fact,

$$
\sigma^A \sigma^B \sigma^C \sigma^D \sigma_{ABCD} = \sigma \quad \tau^A \tau^B \tau^C \tau^D \tau_{ABCD} = -\rho \quad (4.9)
$$

are the familiar GHP spin coefficients and the remaining contractions are zero. As a consequence of the 'hidden' symmetry (4.6) $\rho$ and $\rho'$ are, of course, real.

5. The induced 2-surface spin connection

To motivate how to define the action of $\delta_\sigma$ on spinors let us consider the $\delta_\sigma$-derivative of the complex null surface vector $M^\alpha$ defined by eq.(3.10):

$$
\delta_\sigma M^\alpha = \Delta_\sigma M^\alpha - \Delta_\sigma \Pi^\alpha\beta M^\beta = \frac{1}{4} \left( \Delta_\sigma (\beta^B - \gamma^B R^R) + \frac{1}{2} \Delta_\sigma \gamma^B R^R (\lambda^R - \gamma^R S^S) \right) \left( \lambda^B R^R + \gamma^B R^R \gamma^R \right) + 
\frac{1}{4} \left( \Delta_\sigma (\lambda^B - \gamma^B R^R) \right) \left( \Delta_\sigma (\lambda^B R^R + \gamma^B R^R) - \frac{1}{2} \Delta_\sigma \gamma^B R^R (\lambda^R + \gamma^R S^S) \right).
$$

Thus it seems natural to define the action of $\delta_\sigma$ on spinors by

$$
\delta_\sigma (\lambda^B \pm \gamma^B R^R) := \Delta_\sigma (\lambda^B \pm \gamma^B R^R) \mp \frac{1}{2} \Delta_\sigma \gamma^B R^R (\lambda^R \pm \gamma^R S^S); (5.1)
$$

i.e. by

$$
\delta_\sigma \lambda^B := \Delta_\sigma \lambda^B - Q^B \gamma^R \lambda^R. (5.2)
$$

$\delta_\sigma$ annihilates both the symplectic and complex metrics:

$$
\delta_\sigma \varepsilon_{RS} = 0, \quad \delta_\sigma \gamma_{RS} = 0. (5.3)
$$

Therefore the 2-surface spinor curvature, defined by

$$
^8R_{ABCD} \xi^B := -(\delta_\sigma \delta_\sigma - \delta_\sigma \delta_\sigma) \xi^A, (5.4)
$$

has the algebraic symmetries $^8R_{ABCD} = ^8R_{BCAD}$ and $\gamma_{AD} ^8R_{Bcd} = \frac{1}{2} \varepsilon_{AB}^{EF} \gamma_{EF} R_{EFcd}$ and hence

$$
^8R_{ABcd} = \frac{1}{2} \gamma_{AB}^{EF} \varepsilon_{EF} R_{EFcd}. (5.5)
$$

Calculating the action of the commutator $[\delta_\sigma \delta_\sigma - \delta_\sigma \delta_\sigma]$ on the surface vector $\Pi^A_{BB'} \lambda^B \bar{\lambda}^{B'}$ we obtain the spinor form of the curvature tensor of $^8$ by the spin curvature:

$$
^8R_{ABBB'}_{cd} = \frac{i}{2} (\varepsilon_{A'B'} \gamma_{AB} - \varepsilon_{AB}^{EF} \gamma_{EF}^{A'B'}) \frac{i}{2} \left( -\gamma_{EF}^{B'} R_{EFcd} + \gamma_{EF}^{B'} R_{EFcd} \right). (5.6)
$$

This is the product of the surface volume form and the imaginary part of $\gamma_{AB}^{EF} R_{EFcd}$, and hence, contracting with the volume form and using the expression of the curvature tensor $^8R_{abcd}$ by the
induced metric and the curvature scalar $^8R$, the imaginary part of $\gamma^{AB} R_{ABcd}$ can be expressed by the curvature scalar $^8R$ and the volume form. To determine the full spinor curvature first let us rewrite the commutator $(\delta_e \delta_d - \delta_d \delta_e) \xi^A$ by (5.2) and (4.1) to obtain

$$-^8 R_{ABcd} = -F_{ABcd} + \Delta_{DD} Q_{ACCD} - \Delta_{CC} Q_{ACDB} + Q_{ACCD} Q_{DDCA} - Q_{ACDB} Q_{DDBC} + Q_{ABCD} \left( \delta^{R}_{E} \delta^{R}_{DDC} + \delta^{R}_{C} \delta^{R}_{DDC} - \delta^{R}_{D} \delta^{R}_{CCD} - \delta^{R}_{D} \delta^{R}_{CCD} \right).$$

(5.7)

Comparing its right hand side with (4.8) we have

$$^8 R_{ABcd} = - \frac{1}{2} \left( \delta_c A_d - \delta_d A_c \right) - \frac{^8 R}{4} \left( \xi_{CD} \gamma_{CD} - \xi_{CD} \gamma_{CD} \right).$$

(5.8)

Thus the imaginary part of $\gamma^{AB} R_{ABcd}$ is, in fact, related to the curvature tensor $^8R_{abed}$, i.e. to the intrinsic geometry of $\mathfrak{g}$. However, its real part is an extrinsic geometrical quantity: the ‘field strength’ of the vector potential $A_e$. Thus the spinor curvature is not the anti-self-dual part of the intrinsic curvature of $\mathfrak{g}$. With this extension of $\delta_e$ from surface tensors to spinors we have extended $\delta_e$ to arbitrary tensors: For any vector field $X^a$ orthogonal to $^8 \delta_e X^a = \Pi^a_{\delta} \delta^a B^b \nabla_j X^b$. If $(o^R, \tilde{r}^R)$ is a spinor dyad normalized by (3.6) then twice the real part of the spinor connection coefficient, $B_e := -\sigma^A \Delta_e A$, is the $SO(1,1)$-gauge potential: $B_e + \tilde{B}_e = A_e$, while the only independent part of the $SO(2)$-Ricci rotation coefficients is $B_e - \tilde{B}_e = \tilde{\eta}^a \delta_e m_a$. The GHP coefficients representing the connection are related to $B_e$ by $\beta = -\tilde{B}_e \eta^a$ and $\beta' = B_e \tilde{\eta}^a$, respectively. The curvature associated with the action of $\delta_e$ on vectors orthogonal to $\mathfrak{g}$ is $\frac{1}{2} (\xi_{AB} \gamma_{AB} + \xi_{AB} \gamma_{AB}) (\delta_e A_d - \delta_d A_e)$. This result is in accordance with the following geometrical picture [31]: The surface $\mathfrak{g}$ defines a principle fibre bundle over $\mathfrak{g}$ with structure group $SO(2) \circ SO(1,1)$ such that $SO(2)$ acts naturally on the tangent bundle and $SO(1,1)$ on the normal bundle of $\mathfrak{g}$. The principle $C^*\text{-bundle}$ $(B, \mathfrak{g}, C^*)$ of the normalized spinor dyads satisfying (3.6) over $\mathfrak{g}$ is its double covering bundle. Since this principle $C^* \approx U(1) \circ (0, \infty)$-bundle over $\mathfrak{g}$ is the pull back to $\mathfrak{g}$ of the sum of the principle $U(1)$- and $(0, \infty)$-bundles along the diagonal map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$: $p \mapsto (p, p)$, the curvature of any connection on $(B, \mathfrak{g}, C^*)$ must be the sum of the two curvatures corresponding to the $U(1)$ and $(0, \infty)$ subgroups, respectively. In fact, the real part of the spinor curvature is connected with the $(0, \infty)$, while its imaginary part with the $U(1)$ subgroup.

Finally contracting (5.8) with $\gamma^{AB}$ and integrating for $\mathfrak{g}$:

$$\int_\mathfrak{g} \gamma^{AB} R_{ABcd} dx^c \wedge dx^d = 2 \int_\mathfrak{g} dA + i \int_\mathfrak{g} \frac{^8 R}{2} \mu_b \xi_{abed} dx^c \wedge dx^d = 2 \int_\mathfrak{g} dA + i \int_\mathfrak{g} Rd\mathfrak{g}.$$

(5.9)

Thus for closed $\mathfrak{g}$ the real part of $^8 R_{ABcd}$ does not contribute to the total, integral curvature; and by the Gauss–Bonnet theorem the total curvature of $\mathfrak{g}$ is $8 \pi i (1 - G)$ where $G$ is the genus of $\mathfrak{g}$. This is a simple, direct verification of the fact [28] that the integral for a closed $\mathfrak{g}$ of the imaginary part of the complex Gauss curvature of $\mathfrak{g}$, given by $K = -\frac{i}{2} \mu_b \xi_{abed} \delta_e A_d + \frac{1}{4} ^8 R$ in the present formalism, is zero.
6. The irreducible parts of $\Delta_{\lambda}$

There are essentially two irreducible parts of the first Sen–derivative of a spinor field: the contraction $\Delta_{R^{[B}}\lambda_{R]}$ and

$$T_{R^{[RS}}K^{\lambda_{K}} := \Delta_{R^{[R}}\lambda_{R]} + \frac{1}{2} \gamma_{R^{[S}}^{EF} \Delta_{R^{E}R^{F]} \lambda_{F}},$$

(6.1)

the trace-free symmetric part of $\Delta_{R^{[R}}\lambda_{R]}$. The `$\gamma$-trace' $\gamma_{R^{[S}}^{RS} \Delta_{R^{R}}\lambda_{R]}$ is not independent since, because of the argumentation following eq. (3.8), $\gamma_{R^{[S}}^{RS} \Delta_{R^{R}}\lambda_{R]} = \tilde{\tau}_{R^{S]} R} \Delta_{S^{R}} \lambda_{R}^{R}$, and hence, using (3.9),

$$\Delta_{R^{[R}}\lambda_{R]} = \Pi_{R^{[R}} \varepsilon_{A} \Delta_{A^{R}}^{R^{K}} \lambda_{K} + T_{R^{[RS}}K^{\lambda_{K}}.$$  

(6.2)

$\Delta_{R^{[R}}\lambda_{R]} \equiv 0$ is the 2 dimensional Weyl–Sen–Witten operators. Recalling that under the conformal rescaling $\varepsilon_{AB} \rightarrow \varepsilon_{AB} := \Omega \varepsilon_{AB}$ the spacetime covariant differentiation is known to transform [28] as $\nabla_{R^{R}} \psi_{A}^{\lambda_{B}} := \nabla_{R^{R}} \psi_{A}^{\lambda_{B}} := \nabla_{R^{R}} \psi_{A}^{\lambda_{B}} + C^{A} R_{E} E \psi_{A}^{\lambda_{B}} + \ldots - C^{F} R_{R} \psi_{A}^{\lambda_{B}} - \ldots$, where $C^{E} R_{R} := \delta_{R} T_{R^{E}}^{R}$ and $T_{R} := \nabla_{R} \ln \Omega$, it is easy to deduce the behaviour of $\Delta_{R^{R}}$, $Q_{R^{R}B}$, etc. under conformal rescalings. In particular if $\lambda_{R}$ has conformal weight $w \in \mathbb{R}$ (i.e. $\lambda_{R} \rightarrow \lambda_{R} := \Omega^{w} \lambda_{R}$ under the conformal rescaling) then the conformal behaviour of $T_{R^{[RS}}K^{\lambda_{K}}$ and the 2 dimensional Weyl–Sen–Witten operators:

$$\tilde{\Delta}_{R^{[R}}\lambda_{R]} = \Omega^{w-1} \Delta_{R^{[R}}\lambda_{R]} - \Omega^{w-1} \lambda_{R}^{R_{1}} \left[ (1 + w) \delta_{R}^{K} \delta_{R}^{K} + (1 - w) \gamma_{R}^{R} \delta_{R}^{K} \right] \Delta_{R^{[R}}\lambda_{R]} + \frac{1}{2} \gamma_{R^{[S}}^{R} \gamma_{R^{R]}^{R} \lambda_{R}^{R}} \Omega \left[ (1 + w) \delta_{R}^{K} \delta_{R}^{K} + (1 - w) \gamma_{R}^{R} \delta_{R}^{K} \right] \Delta_{R^{[R}}\lambda_{R]}$$

(6.3)

Thus, in contrast to the four dimensional Weyl neutrino operator $\nabla_{R^{R}}$, the Weyl–Sen–Witten operator $\Delta_{R^{R}}$ does not have definite conformal weight, while $T_{R^{[RS}}K^{\lambda_{K}}$ has zero conformal weight if it acts on spinor fields of unit conformal weight. However if $\lambda_{R}$ has unit conformal weight then under the conformal rescaling the spinor

$$\tau_{A} := -i \Delta_{A^{R}}^{R} \lambda_{A}$$

(6.5)

transforms like the secondary part of a twistor; i.e. $Z^{a} := (\lambda^{A}, \tau_{A^{i}})$ is a local twistor [1] on $\Sigma$. One can calculate the covariant derivative of a local twistor defined on $\Sigma$ in the direction tangential to $\Sigma$; i.e. to define the 2 dimensional Sen derivative of any local twistor $Z^{a} = (\lambda^{A}, \tau_{A^{i}})$ defined on $\Sigma$:

$$\Delta_{R}Z^{a} := \Pi_{R} \nabla_{R} Z^{a} = \left( \Delta_{BB^{i}}^{B} \lambda^{A} + i \Pi_{BB}^{A^{i}} \tau_{A^{i}} \right), \Delta_{BB^{i}}^{A^{i}} \tau_{A^{i}} + i \lambda^{A} \frac{1}{2} \left( \frac{1}{6} R_{g_{ab}} - R_{ab} \right) \Pi_{BB}^{EE}$$

(6.6)

Its primary part is $T_{BB^{i}}^{A^{i}} \lambda_{K}^{K} + i \Pi_{BB}^{A^{i}} \left( \tau_{K^{i}}^{K} + i \Delta_{K^{i}}^{K} \lambda_{K}^{K} \right)$; and hence the primary part of the Sen derivative of a twistor satisfying (6.5) is just the $T_{R^{[RS}}K^{\lambda_{K}}$-derivative of the primary spinor part of the twistor. Thus $T_{R^{[RS}}K^{\lambda_{K}}$ is precisely the 2-surface twistor operator. (Borrowing the idea how the twistor covariant differentiation, $\nabla_{R} Z^{a}$, is defined [1] one can define the induced 2 dimensional covariant derivative $\delta_{R} Z^{a}$ of $Z^{a}$ too. This, however, will not be used in the present paper.)
It will be useful to introduce the following chiral differential operators:

\[ \Delta_{R}^{\pm} \psi := \mp R \Delta_{R} \psi, \]

\[ \mathcal{T}_{RS}^{\pm K} \lambda := \mp R \mathcal{T}_{RS} \lambda. \]

\[ \mathcal{T}_{RS}^{\pm K} \lambda \]

may be called the right/left handed parts of the twistor operator \( \mathcal{T}_{RS}^{K} \) and \( \Delta_{R}^{\pm} \) the right/left handed part of the Weyl–Sen–Witten operator. Any further application of the symmetry operations and the projections on \( \mathcal{T}_{A}^{\pm AB} \) and \( \Delta_{R}^{\pm} \) yields zero or is an identity; i.e. \( \mathcal{T}_{A}^{\pm AB} \), \( \Delta_{R}^{\pm} \) form the complete irreducible decomposition of \( \Delta_{A}^{AB} \).

Finally determine the GHP form of these irreducible parts. Let \((\alpha, \mu)\) be a spinor dyad normalized by \(\alpha \mu = 1\) and \((3,6)\), and define \(\lambda^{0} = \lambda_{1}\) and \(\lambda^{1} = -\lambda_{0}\) by \(\lambda^{A} =: \lambda^{0} \sigma^{A} + \lambda^{1} \rho^{A}\). They are scalars of weight \((-1,0)\) and \((1,0)\), respectively \([27-29]\). Then the GHP form of the irreducible chiral operators are

\[ -\tau_{R}^{\pm} \Delta_{R}^{\pm} \lambda_{R} = R m \nabla_{\sigma} \lambda_{R} = \partial \lambda^{0} - \rho \lambda^{1}, \]

\[ -\sigma_{R}^{\pm} \Delta_{R}^{\pm} \lambda_{R} = -\sigma_{R} \nabla_{\sigma} \lambda_{R} = \partial \lambda^{1} - \rho \lambda^{0}, \]

and all the remaining contractions are zero. Thus the GHP form of \( \mathcal{T}_{RS}^{K} \lambda_{K} = 0 \) is the familiar 2-surface twistor equation \([1,18-20]\). The GHP form \((6.9-12)\) of the irreducible chiral parts of the 2 dimensional Sen operator define differential operators on the Whitney sum of certain vector bundles \(E(p,q)\) of scalars of weight \((p,q), p - q \in \mathbb{Z}\).

\[ -\Delta^{-} : E_{-}^{\infty}(p-1,q) \otimes E_{-}^{\infty}(p+1,q) \rightarrow E_{-}^{\infty}(p,q-1) : (\lambda^{0}, \lambda^{1}) \rightarrow (\partial \lambda^{0} - \rho \lambda^{1}), \]

\[ -\Delta^{+} : E_{+}^{\infty}(p-1,q) \otimes E_{+}^{\infty}(p+1,q) \rightarrow E_{+}^{\infty}(p,q+1) : (\lambda^{0}, \lambda^{1}) \rightarrow (\partial \lambda^{1} - \rho \lambda^{0}), \]

\[ -\tau^{-} : E_{-}^{\infty}(p-1,q) \otimes E_{-}^{\infty}(p+1,q) \rightarrow E_{-}^{\infty}(p+2,q-1) : (\lambda^{0}, \lambda^{1}) \rightarrow (\partial \lambda^{1} - \sigma \lambda^{0}), \]

\[ -\tau^{+} : E_{+}^{\infty}(p-1,q) \otimes E_{+}^{\infty}(p+1,q) \rightarrow E_{+}^{\infty}(p+2,q+1) : (\lambda^{0}, \lambda^{1}) \rightarrow (\partial \lambda^{0} - \sigma \lambda^{1}). \]

Here \(E^{\infty}(p,q)\) is the space of the smooth cross sections of \(E(p,q)\) (see e.g. \([34]\)). For \(p = q = 0\) these operators reduce to the GHP form of the irreducible chiral parts of the 2 dimensional Weyl–Sen–Witten and twistor operators acting on the space of the smooth covariant spinor fields \(C^{\infty}(\mathbb{R}, S_{A}) \cong E_{-}^{\infty}(-1,0) \otimes E_{+}^{\infty}(1,0)\).

### 7. The spinor identities

Using \((5.2)\), the commutator \((4.1)\) and \((4.2-3)\) one has the following identity for any two spinor fields \(\lambda^{A}\) and \(\mu^{A}\):

\[ \frac{1}{2} \gamma^{A'B'}(\Delta_{A'B'} \lambda^{A}) (\Delta_{B'B} \mu^{B}) = \delta_{AA'} (\gamma^{A'B'} \lambda_{B} \Delta_{B'B} \mu^{B}) - \]

\[ -\frac{i}{2} \lambda^{A} \mu^{B} R_{AB} \epsilon^{ef} \epsilon_{ef}^{cd}. \]

\[ \frac{1}{2} \gamma^{A'B'}(\Delta_{A'B'} \lambda^{A}) (\Delta_{B'B} \mu^{B}) = \delta_{AA'} (\gamma^{A'B'} \lambda_{B} \Delta_{B'B} \mu^{B}) - \]

\[ -\frac{i}{2} \lambda^{A} \mu^{B} R_{AB} \epsilon^{ef} \epsilon_{ef}^{cd}. \]
The second derivatives of the spinor fields appear only in the form of a total divergence again. This is the two dimensional version of the Sen identity: apart from the total divergence, the right hand side of these identities has surprisingly symmetrical structure. If we integrate the identities (7.1) and (7.2) for the spacelike 2-submanifold \( \Sigma \) which is orientable and closed then the total divergences disappear. Expressing the \( \Delta_a \) operators by the two dimensional Weyl–Sen–Witten and twistor operators we obtain identities between the integral of the Weyl–Sen–Witten and twistor derivatives and the charge integrals of the curvature and the torsion of \( \Delta_a \):

\[
\frac{3}{2} \varepsilon^{AB} (\Delta_A \mu^A)(\Delta_B \lambda^B) = \delta_{RR} \left( \mu^R \Delta_S^R \lambda^S + \mu^S \Delta_S^R \lambda^R \right) + \varepsilon^{AB} (\Delta_A (\lambda_B))(\Delta_B (\lambda_A)) - i \frac{1}{2} \mu^A \gamma_A K \lambda^B R_{KBCd} d^d \varepsilon_{ef} \varepsilon^{cd} - \frac{1}{2} \left( 2 \varepsilon^{RA} \mu^B \lambda^S - \mu^R \Delta^R (\lambda_B) - \mu^B \Delta^R (\lambda_A) - \varepsilon^{RA} \mu_S \Delta^R (\lambda_B) \right) \varepsilon^{AB} T_{RR} A^A B^B.
\]

(7.2)

The second derivatives of the spinor fields appear only in the form of a total divergence again [30]. However, in contrast to the identity (7.1), eq. (7.2) contains the torsion \( T_{\alpha \beta} \) of the operator \( \Delta_a \) too. If we integrate the identities (7.1) and (7.2) for the spacelike 2-submanifold \( \Sigma \) which is orientable and closed then the total divergences disappear. Expressing the \( \Delta_a \) operators by the two dimensional Weyl–Sen–Witten and twistor operators we obtain identities between the integral of the Weyl–Sen–Witten and twistor derivatives and the charge integrals of the curvature and the torsion of \( \Delta_a \):

\[
\oint_{\Sigma} \varepsilon^{RS} \left( \Delta_R^R \lambda^R \Delta_S^S \mu_S + T_{RS} \left( \lambda^R \Delta_S^S \mu_S + T_{RS} \lambda^R \Delta_S^S \mu_S \right) \right) d\Sigma = - \frac{1}{2} \oint_{\Sigma} \lambda^A \mu^B R_{ABcd} d^c \lambda^d 
\]

and

\[
\oint_{\Sigma} \varepsilon^{RS} \left( \Delta_R^R \mu^R \Delta_S^S \lambda_S - T_{RS} \left( \mu^R \Delta_S^S \lambda_S + \mu^S \Delta_S^R \lambda_S \right) \right) d\Sigma = - \frac{1}{2} \oint_{\Sigma} \mu^A \gamma_A K \lambda^B R_{KBCd} d^c \lambda^d - \oint_{\Sigma} \mu^A \left( \delta^A_R \mu^B \lambda^S + \varepsilon_{KR} \lambda^R \right) d^d \lambda^d + \oint_{\Sigma} \mu^A \left( \delta^A_R \mu^B \lambda^S + \varepsilon_{KR} \lambda^R \right) d^d \lambda^d.
\]

(7.4)

The left hand side of these identities has surprisingly symmetrical structure. If \( \pi_{\alpha A} := -i \Delta_a \mu^A \) and \( \pi_{\beta B} := -i \Delta_a \lambda_B \) and their spinor components are defined by \( \pi_{\alpha A} := \bar{e}_{A} \pi_{\alpha} - i \lambda_{A} \pi_{\alpha} \) then

\[
\pi^{AB} \delta_{\alpha A} \mu^A \lambda_B = - \left( \pi_{\alpha 0} \pi_{\beta 1} + \pi_{\beta 1} \pi_{\alpha 0} \right)
\]

\[
\varepsilon^{AB} \delta_{\alpha A} \mu^A \lambda_B = - \left( \pi_{\alpha 0} \pi_{\beta 1} - \pi_{\beta 1} \pi_{\alpha 0} \right).
\]

(7.5)

The right hand side of (7.5) is the well known expression related to Penrose’s kinematical twistor (eq. 9.9.29 of [1]), while (7.6) to the infinity twistor (eq. 9.9.27 of [1]). In fact, if \( \lambda_B \) is a solution of the 2-surface twistor equation then identity (7.3) reduces to Tod’s expression of the kinematical twistor [1, 18, 20]; while (7.4) is an (as far as we know) new identity.
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