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Examples of underdetermined Scattering Problems

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Abstract :

The determination of the shape of a scatterer by non-destructive methods such as scattering experiments raises the question of indetermination or ambiguity problems for the scatterer's shape, depending on the given scattering data. These ambiguity problems are discussed by means of numerical constructions of scattering equivalent problems, i.e. we determine scatterers that produce the same scattering amplitude in given conditions (fixed energies, fixed incident angles and \ or directions of receivers). This construction is given in the context of a generalized scattering theory that takes into account impedance discontinuities inside the scatterer. We start with a scattering problem defined by a discontinuous curve of arbitrary shape and by boundary conditions. Then a circular curve with appropriate boundary is determined such that these two scattering problems yield the same scattering amplitude in the given conditions. We present also numerical results for two particular cases of the generalized scattering theory, calculated by means of the Nyström method. We used it to verify that the equivalence obtained in the Born approximation holds the exact scattering amplitudes.

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Examples of underdetermined Scattering Problems

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Abstract

The determination of the shape of a scatterer by non-destructive methods such as scattering experiments raises the question of indetermination or ambiguity problems for the scatterer's shape, depending on the given scattering data. These ambiguity problems are discussed by means of numerical constructions of scattering equivalent problems, i.e. we determine scatterers that produce the same scattering amplitude in given conditions (fixed energies, fixed incident angles and \ or directions of receivers). This construction is given in the context of a generalized scattering theory that takes into account impedance discontinuities inside the scatterer. We start with a scattering problem defined by a discontinuous curve of arbitrary shape and by boundary conditions. Then a circular curve with appropriate boundary is determined such that these two scattering problems yield the same scattering amplitude in the given conditions. We present also numerical results for two particular cases of the generalized scattering theory, calculated by means of the Nyström method. We used it to verify that the equivalence obtained in the Born approximation holds the exact scattering amplitudes.

1 Introduction

From previous results (Nachman [1], Colton [2]) we know that reconstruction of a finite scatterer by scalar waves is unique , if the scattering datas are known for all illumination angles and all directions of receivers, at a fixed energy. We have already shown (Dupuy and Sabatier [3]) by examples proving that if we weaken this conditions, we do not keep uniqueness of shape determination. This paper gives further examples and demonstrations of this point. In particular we present a different ambiguity case which contains the back scattering case. We use the scattering theory in the by Sabatier (see ref. [4]) and corresponding to the impedance equation, with discontinuity curves corresponding to a jump of the impedance and/or of its normal derivative and specialized to the two dimensional case as in ref [3]. Between these curves, the scatterer is assumed to be inhomogeneous, but allowing a smooth variation of parameters only,

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such that the impedance is twice differentiable. We recall briefly this model and we insist on the part that takes into account the discontinuities only. In §4 of ref [3], We showed that this restriction is relevant. Here we present a numerical resolution of two particular cases of impedance equation without diffuse scattering and we calculate the exact scattering amplitude by using the Nyström method, with a particular decomposition used by R. Kress (see ref. [5]), in order to isolate the logarithmic singularities of the two-dimensional case. Finally, we present a construction method of equivalent scattering problems, and we show and discuss numerical examples.

2 The scattering problem

Figure 1: Scattering problem with discontinuities : Shape of discontinuities

We start from the impedance equation

$$(\alpha^{-2} \operatorname{div} \alpha^2 \operatorname{grad} + k^2 - V(x))\varphi(k, x) = 0 \quad (2.1)$$

where $k, x \in \mathbb{R}^2$, $\alpha > 0$ is defined inside domains $\Omega_1, \Omega_2, \dots, \Omega_{N+1}$, such that $\Omega_i \cap \Omega_j = \emptyset$ for any $i \neq j$, $\mathbb{R}^2 = \sum_{i=1}^{N+1} \overline{\Omega}_i$, and $\delta\Omega_i = S_i$ is the external boundary of Ω_i , and the internal boundary of Ω_{i+1} . The domains are ordered from Ω_1 to Ω_{N+1} and are all finite except Ω_{N+1} , which extends to infinity in all directions (see fig 1). In addition we assume that each S_i is $\mathcal{C}^2$, and α is $\mathcal{C}^2$ inside $\mathbb{R}^2 \setminus S$, where $S = \cup_{i=1}^N S_i$, with $\alpha(x)$ and $\frac{\partial \alpha(x)}{\partial \nu_x}$ going to finite limits at any point $x_i \in S_i$ as $x \rightarrow x_i$ inside the domain Ω_i or Ω_{i+1} , ν is a vector normal to S_i and pointing outwards, i.e, in Ω_{i+1} , towards S_{i+1} . At any point $x_p \in S_p$ labelling the + and - sides of ν_p as external and internal parts, we can characterize the jump of α and its derivative throughout S_p by the following “singular data”:

Transmission and reflection factors

$$\frac{1}{t_p} = \frac{1}{2} \left[\frac{\alpha_p^+}{\alpha_p^-} + \frac{\alpha_p^-}{\alpha_p^+} \right] \quad r_p = \frac{1}{2} \left[\frac{\alpha_p^+}{\alpha_p^-} - \frac{\alpha_p^-}{\alpha_p^+} \right]$$

slope factor

$$\frac{\bar{s}_p}{t_p} = \frac{1}{2} \nu \cdot \left[\frac{\operatorname{grad} \alpha_p^-}{\alpha_p^+} - \frac{\operatorname{grad} \alpha_p^+}{\alpha_p^-} \right]$$

The impedance scattering problem was studied by Sabatier [4] in the three dimensional case. Here we write the results in the two dimensional case. (see [3]). First we have the following result.

Theorem 1 *The function $\psi := \alpha\varphi$ is a solution of the chain of Schrödinger equations:*

$$(\Delta + k^2 - V - \alpha^{-1}\Delta\alpha)\psi(k, x) = 0 \quad x \in \mathbb{R}^2 \setminus S \quad (2.2)$$

$$(2.3)$$

coupled by the continuity conditions through S of ψ/α and $\alpha \frac{\partial \psi}{\partial \nu} - \psi \frac{\partial \alpha}{\partial \nu}$.

Because of this Theorem (1), there always exist two equivalent formulations of the same physical problem and it is useful to go back and forth from one to the other. In the impedance formulation of the scattering problem, $\varphi(k, x)$ is sought such that:

$$(\alpha^{-2} \operatorname{div} \alpha^2 \operatorname{grad} + k^2 - V(x))\varphi(k, x) = 0 \quad x \in \mathbb{R}^2 \quad (2.4)$$

$\varphi_s := \alpha(x)\varphi(k, x) - \exp[ik \cdot x]$ is Sommerfeld outgoing, i.e. in twodimensional cases :

$$\left(\frac{x}{|x|} \cdot \operatorname{grad} \varphi_s(x) \right) - ik\varphi_s(x) = o\left(\frac{1}{|x|^{\frac{1}{2}}} \right) \quad |x| \rightarrow \infty \quad (2.5)$$

In the Schrödinger chain formulation, $\psi(k, x)$ is sought such that:

$$(\Delta + k^2 - V - \alpha^{-1}\Delta\alpha)\psi(k, x) = 0 \quad x \in \mathbb{R}^2 \setminus S \quad (2.6)$$

$$\begin{aligned} \alpha^+(x) \frac{\partial \psi^+(x)}{\partial \nu} - \psi^+(x) \frac{\partial \alpha^+(x)}{\partial \nu} &= \alpha^-(x) \frac{\partial \psi^-(x)}{\partial \nu} - \psi^-(x) \frac{\partial \alpha^-(x)}{\partial \nu} \\ \psi^+(x)/\alpha^+(x) &= \psi^-(x)/\alpha^-(x) \quad x \in S \end{aligned} \quad (2.7)$$

$$\psi(k, x) - \exp[ik \cdot x] \text{ is Sommerfeld outgoing} \quad (2.8)$$

We recall that this Schrödinger chain formulation was used to derive a generalized Lipmann-Schwinger equation and a scattering amplitude in two parts (see ref [4]). One part is due to the discontinuities (related to the relation (2.7)), and noted A_0 . The other one due to diffuse scattering in the presence of discontinuities (it is related to the potential $V + \alpha\Delta\alpha$, and noted A_1).

In this paper we consider only the part due to the discontinuities. Using the same notation of the previous paper, we note ψ_{in} , the solution of the scattering problem with only discontinuities, i.e. the solution of the following system:

$$\begin{aligned} (\Delta + k^2)\psi_{in}(k, x) &= 0 \quad x \in \mathbb{R}^2 \setminus S \\ \psi_{in}/\alpha \text{ and } \alpha \frac{\partial \psi_{in}}{\partial \nu} - \psi_{in} \frac{\partial \alpha}{\partial \nu} &\text{ continuous } / S \\ \psi_{in}(x, y) - \exp[ik \cdot x] &\text{ is Sommerfeld outgoing} \end{aligned} \quad (2.9)$$

So as to solve this system we write down $u(x) := \psi_{in}(k, x) - e^{ikx}$, with

$$u(x) = \sum_{j=1}^N \int_{S_j} ds(z) \Phi(z, x) \phi_j(z) + \sum_{j=1}^N \int_{S_j} ds(z) \frac{\partial \Phi}{\partial \nu_z}(z, x) \psi_j(z) \quad (2.10)$$

where $\Phi(x, y)$ is the Helmholtz Green function, i.e. :

$$\Phi(x, y) = \frac{i}{4} H_0^{(1)}(|k||x - y|) \quad x, y \in \mathbb{R}^2, x \neq y \quad (2.11)$$

As in the three-dimensional case $u(x)$ satisfies (2.9) and the Sommerfeld condition. We determine ϕ_j and ψ_j by the continuity conditions (2.9), obtaining:

$$\begin{cases} \psi &= 2\beta e^{i\vec{k} \cdot x} + \beta \mathbf{S} \phi + \beta \mathbf{K} \psi \\ \phi + \gamma \psi &= 2\beta \frac{\partial e^{i\vec{k} \cdot x}}{\partial \nu} - 2\beta' e^{i\vec{k} \cdot x} + \beta \mathbf{T} \psi - \beta' \mathbf{K} \psi + \beta \mathbf{K}' \phi - \beta' \mathbf{S} \phi \end{cases} \quad (2.12)$$

with

$$\begin{aligned} \psi_j &= \psi \quad \text{for } x \in S_j \\ \beta(x) &= \frac{\alpha^+(x) - \alpha^-(x)}{\alpha^+(x) + \alpha^-(x)} ; \beta'(x) = \frac{\alpha'^+(x) - \alpha'^-(x)}{\alpha^+(x) + \alpha^-(x)} ; \gamma(x) = \frac{\alpha'^+(x) + \alpha'^-(x)}{\alpha^+(x) + \alpha^-(x)} \end{aligned}$$

and the surface operators $\mathbf{S}, \mathbf{K}, \mathbf{K}', \mathbf{T}$ are defined by:

$$(\mathbf{S}_{ij} f)(x) = 2 \int_{S_j} ds(z) \Phi(z, x) f(z) \quad x \in S_i \quad (2.13)$$

$$(\mathbf{K}_{ij} f)(x) = 2 \int_{S_j} ds(z) \frac{\partial \Phi}{\partial \nu_z}(z, x) f(z) \quad x \in S_i \quad (2.14)$$

$$(\mathbf{K}'_{ij} f)(x) = 2 \int_{S_j} ds(z) \frac{\partial \Phi}{\partial \nu_x}(z, x) f(z) \quad x \in S_i \quad (2.15)$$

$$(\mathbf{T}_{ij} f)(x) = 2 \frac{\partial}{\partial \nu_x} \int_{S_j} ds(z) \frac{\partial \Phi}{\partial \nu_z}(z, x) f(z) \quad x \in S_i \quad (2.16)$$

The following result is central:

Theorem 2 *We assume that $\beta, \beta', \gamma, \psi$ are in $C^1(S)$ and ϕ in $\mathcal{C}(S)$.*

If $N(1 - \frac{\mathbf{B}}{1+\beta^2}) = 0$ and $N(1 - \beta \mathbf{K}) = 0$ then the system (2.12) has a unique solution:

$$\psi = (\mathbf{1} - \beta \mathbf{K})^{-1} \beta (2e^{i\vec{k} \cdot x} + \mathbf{S} \phi) \quad (2.17)$$

$$\phi = [(1 + \beta^2)\mathbf{1} - \mathbf{B}]^{-1} \mathbf{A} e^{i\vec{k} \cdot x} \quad (2.18)$$

with:

$$\mathbf{A} e^{i\vec{k} \cdot x} = 2\beta \frac{\partial e^{i\vec{k} \cdot x}}{\partial \nu} - 2[\beta' + (\gamma \mathbf{1} + \beta' \mathbf{K} - \beta \mathbf{T})(1 - \beta \mathbf{K})^{-1} \beta] e^{i\vec{k} \cdot x} \quad (2.19)$$

$$\mathbf{B} = -(\gamma \mathbf{1} + \beta' \mathbf{K})(1 - \beta \mathbf{K})^{-1} \beta \mathbf{S} + \beta \mathbf{K}' - \beta' \mathbf{S} + \beta \mathbf{C} \quad (2.20)$$

$$\mathbf{C} = \mathbf{T}(\mathbf{1} - \beta \mathbf{K})^{-1} \beta \mathbf{S} + \beta \mathbf{1} \quad (2.21)$$

Proof: see ref. [3]

Q.E.D.

Now we define the scattering amplitude, A_0 related to ψ_{in} and due to the discontinuities only. To show how it comes, we apply Green's theorem to $\psi_{in}(k, y)$ and $\Phi(x, y)$ inside the domain defined by $|y| \leq R$ and $y \in \Omega_{N+1}$, before we let $R \rightarrow \infty$, and we obtain:

$$\psi_{in}(k, x) = e^{ik \cdot x} - \frac{1}{(8\pi|k|)^{\frac{1}{2}}} \frac{e^{i[|k||x| + \frac{\pi}{4}]}}{|x|^{\frac{1}{2}}} A_0(|k|\hat{x}, k) + o\left(\frac{1}{|x|^{\frac{1}{2}}}\right) \quad (2.22)$$

with

$$A_0(\vec{k}', \vec{k}) = \int_{S_N^+} e^{-i\vec{k}' \cdot y} \left[\frac{\partial \psi_{in}(\vec{k}, y)}{\partial \nu_y} + i\vec{k}' \cdot \nu(y) \psi_{in}(\vec{k}, y) \right] ds(y) + o\left(\frac{1}{|x|^{\frac{1}{2}}}\right) \quad (2.23)$$

where $\vec{k}' := k\hat{x}$

There is another expression of A_0 in terms of ϕ and ψ . To get, one puts the asymptotic expansions of $\Phi(x, y)$ and $\frac{\partial \Phi(x, y)}{\partial \nu_x}$ inside the definition (2.10) of u , obtaining:

$$\begin{aligned} \phi_{in}(\vec{k}, x) &= e^{i\vec{k} \cdot x} + \sum_{j=1}^N \frac{1}{(8\pi k)^{\frac{1}{2}}} \frac{e^{i[k|x| + \frac{\pi}{4}]}}{|x|^{\frac{1}{2}}} \int_{S_j} ds(y) e^{-i\vec{k}\hat{x} \cdot y} \phi(y) \\ &\quad - \sum_{j=1}^N \frac{i}{(8\pi k)^{\frac{1}{2}}} \frac{e^{i[k|x| + \frac{\pi}{4}]}}{|x|^{\frac{1}{2}}} \int_{S_j} ds(y) (\nu_y \cdot k\hat{x}) e^{-i\vec{k}\hat{x} \cdot y} \psi(y) \end{aligned} \quad (2.24)$$

Comparing (2.22) one finally gets:

$$A_0(\vec{k}', \vec{k}) = i \sum_{j=1}^N \int_{S_j} ds(y) (\nu_y \cdot \vec{k}') e^{-i\vec{k}' \cdot y} \psi(y) - \sum_{j=1}^N \int_{S_j} ds(y) e^{-i\vec{k}' \cdot y} \phi(y) \quad (2.25)$$

where $\vec{k}' = k\hat{x}$

It is possible to trace back in the calculations the first and second order terms with respect to the potential's size and to r_p , s_p , t_p . In fact, with the assumptions of theorem 2, we can derive the following expansion up to second order of ψ and ϕ .

$$\begin{cases} \phi &= 2\beta \frac{\partial e^{i\vec{k} \cdot x}}{\partial \nu} - 2\beta' e^{i\vec{k} \cdot x} + 2(\beta K' \beta - \beta' S \beta) \frac{\partial e^{i\vec{k} \cdot x}}{\partial \nu} \\ &+ 2(\beta T \beta - \beta K' \beta' - \beta' K \beta + \beta' S \beta') e^{i\vec{k} \cdot x} + O(\|\beta\|^3) \\ \psi &= 2\beta e^{i\vec{k} \cdot x} + 2(\beta K \beta - \beta S \beta') e^{i\vec{k} \cdot x} + 2\beta S \beta \frac{\partial e^{i\vec{k} \cdot x}}{\partial \nu} + O(\|\beta\|^3) \end{cases} \quad (2.26)$$

We get the scattering amplitude at the first (Born approximation) and second order (quadratic approximation), by using the relation (2.25) with the expansions of ψ and ϕ .

The result can then be reduced to the simplest form by using the standard equivalence (voir ref.[4]): in the Born approximation it is:

$$A_0^B(k', k) = -2 \sum_{j=0}^N \int_{S_j} ds(z) \beta(z) \frac{\partial}{\partial \nu_z} [e^{i(k-k') \cdot z}] + 2 \sum_{j=0}^N \int_{S_j} ds(z) \beta'(z) e^{i(k-k') \cdot z} \quad (2.27)$$

3 Numerical resolution of two particular cases

Now we present a numerical solution for ψ_{in} in two special cases. In the first one we only have $\frac{\partial \alpha}{\partial \nu}$ discontinuous and in the second one the relative discontinuity $\frac{\alpha^+(x)}{\alpha^-(x)}$ does not depend on the position of x on the surface. We recall that the general case where ϕ and ψ are defined by (2.18) and (2.17), is much more difficult to solve because the operator $\mathbf{T}$ (in particular $\mathbf{T}_{ii}$) is not compact, and this must be circumvented.

So as to solve the integral equations we use the Nyström method and a particular decomposition of their kernels to circumvent the logarithmic singularity. This is done as an application of the result of R. Kress (See [5] and [6]), where it is shown a numerical treatment of exterior boundary value problems for the Helmholtz equation expressed by means of integral equations.

First, we give results of unicity and existence of ψ_{in} for these particular cases. After that we give the decomposition of kernels to be used in the Nyström method. Finally we give the systems to be solve numerically and a convergence result.

3.1 The two particular cases

The first case is $\alpha \in \mathcal{C}(\mathbb{R}^2)$. In this case we have $\beta = 0$ and from the equation (2.18) and (2.17) we obtain the following system.

$$\begin{aligned} \psi &= 0 \\ (1 + \beta' \mathbf{S})\phi &= -2\beta' e^{i\vec{k} \cdot x} \end{aligned} \quad (3.28)$$

Theorem 3 *If $\beta' \in \mathcal{C}(S)$ and S is a bounded surface belonging to the class $\mathcal{C}^2$, then $(1 + \beta' \mathbf{S})^{-1}$ exists as a bounded operator on $\mathcal{C}(S)$. The solution of the system (3.28) is:*

$$\begin{cases} \psi = 0 \\ \phi = -2(1 + \beta' \mathbf{S})^{-1} \beta' e^{i\vec{k} \cdot x} \end{cases} \quad (3.29)$$

Proof : Using the result on the homogen chain in ref. [1] we know that if there exists a solution, this solution is unique. The existence is holds in general because $\beta' \mathbf{S}$ is a compact operator on $\mathcal{C}(S)$ (See [4])

Q.E.D.

For the second case the relative discontinuity $\alpha^+(x)/\alpha^-(x)$ does not depend on the x position on the surface. We introduce the function $\sigma(x)$ defined by:

- $\sigma_{N+1}(x) = 1$, $x \in \Omega_{N+1}$
- $\sigma_i(x) = \sigma_{i+1}(x) \alpha^-(x) / \alpha^+(x)$, $x \in \Omega_i$, $i = 1, \dots, N$

So as to find ψ_{in} , we pose $v := \frac{u}{\sigma}$ which solves the following system.

$$(\Delta_x + k^2)v(x) = 0 \quad x \in \mathbb{R}^2 \setminus S \quad (3.30)$$

$$v(x) \text{ et } \sigma^2 \left[\frac{v}{\sigma} \frac{\partial \alpha}{\partial \nu} - \frac{\partial v}{\partial \nu} \right] (x) \text{ continuous through } S \quad (3.31)$$

$$v(x) \text{ is Sommerfeld outgoing (2.5)} \quad (3.32)$$

We seek $v(x)$ by using simple and double-layer potentials, i.e. we write

$$v(x) = \sum_{j=1}^N \int_{S_j} ds(z) \Phi(z, x) \phi_j(z) + \sum_{j=1}^N \int_{S_j} ds(z) \frac{\partial \Phi}{\partial \nu_z}(z, x) \psi_j(z) \quad (3.33)$$

which satisfies (3.30) and (2.5). We obtain the following system by imposing then the continuity conditions (3.31).

$$\psi = 0 \quad (3.34)$$

$$(1 - \tilde{\beta} \mathbf{K}' + \tilde{\beta}' \mathbf{S}) \phi = -2\tilde{\beta}' e^{i\vec{k} \cdot x} + 2\tilde{\beta} \frac{\partial e^{i\vec{k} \cdot x}}{\partial \nu_x} \quad (3.35)$$

with:

$$\tilde{\beta} = \frac{\sigma_+^2 - \sigma_-^2}{\sigma_+^2 + \sigma_-^2} \quad \text{et} \quad \tilde{\beta}' = \frac{\frac{\sigma_+^2}{\alpha_+} \frac{\partial \alpha_+}{\partial \nu} - \frac{\sigma_-^2}{\alpha_-} \frac{\partial \alpha_-}{\partial \nu}}{\sigma_+^2 + \sigma_-^2}$$

Notice that $\tilde{\beta}$ does not depend on the position on the curve but $\tilde{\beta}'$ depends on this one in general. As in the previous case, we have a result of existence and uniqueness.

Theorem 4 *If $\tilde{\beta}, \tilde{\beta}' \in \mathcal{C}(S)$ and S is a bounded surface belonging to the class $\mathcal{C}^2$, then $(1 - \tilde{\beta} \mathbf{K}' + \tilde{\beta}' \mathbf{S})^{-1}$ exists and is bounded operator on $\mathcal{C}(S)$. The solution of the system (3.35) is :*

$$\begin{cases} \psi = 0 \\ \phi = -2(1 - \tilde{\beta} \mathbf{K}' + \tilde{\beta}' \mathbf{S})^{-1} \left(\tilde{\beta}' e^{i\vec{k} \cdot x} + 2\tilde{\beta} \frac{\partial e^{i\vec{k} \cdot x}}{\partial \nu_x} \right) \end{cases} \quad (3.36)$$

Proof: The proof is similar to the previous one. Recall that $\tilde{\beta} \mathbf{K}'$ and $\tilde{\beta}' \mathbf{S}$ are compact operators.

Q.E.D.

3.2 Numerical resolution

For the numerical resolution, we assume that the curve S is star shaped and belongs to the class $\mathcal{C}^2$. Its equation is of the form $R = f(\theta)$ where f is a 2π periodic function. Then for two points x, y of S we can write

$$x := \begin{pmatrix} f(\theta_x) \cos \theta_x \\ f(\theta_x) \sin \theta_x \end{pmatrix} \quad \text{and} \quad y := \begin{pmatrix} f(\theta_y) \cos \theta_y \\ f(\theta_y) \sin \theta_y \end{pmatrix}$$

Now, for solving the integral equations (3.28) and (3.35) we isolate the logarithmic singularities by using similar notations of Kress's article we obtain the following parametric form of $\mathbf{S}$ and $\mathbf{K}'$ (See [5]).

$$\mathbf{S} \phi(x) = \int_0^{2\pi} M(x, y) \phi(y) d\theta_y \quad (3.37)$$

$$-\tilde{\beta} \mathbf{K}' \phi(x) = \tilde{\beta} \int_0^{2\pi} L'(x, y) \phi d\theta_y \quad (3.38)$$

with

$$M(x, y) = \frac{i}{2} H_0^{(1)}(kr(x, y)) \sqrt{f^2(\theta_y) + f'^2(\theta_y)} \quad (3.39)$$

$$L'(x, y) = \frac{ik}{2} \left[f^2(\theta_x) - f(\theta_x)f(\theta_y) \cos(\theta_y - \theta_x) + f'(\theta_x)f(\theta_y) \sin(\theta_y - \theta_x) \right] \\ \times \sqrt{\frac{f^2(\theta_y) + f'^2(\theta_y)}{f^2(\theta_x) + f'^2(\theta_x)}} \frac{H_1^{(1)}(kr(x, y))}{r(x, y)} \quad (3.40)$$

and

$$r(x, y) = \left[f^2(\theta_x) + f^2(\theta_y) - 2f(\theta_x)f(\theta_y) \cos(\theta_x - \theta_y) \right]^{\frac{1}{2}} \quad (3.41)$$

The logarithmic singularities of $\mathbf{S}$ and $\mathbf{K}'$ are isolated if we write :

$$M(x, y) = M_1(x, y) \log 4 \left[\sin^2 \frac{(\theta_x - \theta_y)}{2} \right] + M_2(x, y) \quad (3.42)$$

$$L'(x, y) = L'_1(x, y) \log 4 \left[\sin^2 \frac{(\theta_x - \theta_y)}{2} \right] + L'_2(x, y) \quad (3.43)$$

with

$$M_1(x, y) = -\frac{1}{2\pi} J_0(kr(x, y)) \sqrt{f^2(\theta_y) + f'^2(\theta_y)} \quad (3.44)$$

$$M_2(x, y) = M(x, y) - M_1(x, y) \log 4 \left[\sin^2 \frac{(\theta_x - \theta_y)}{2} \right] \quad (3.45)$$

$$L'_1(x, y) = -\frac{k}{2\pi} \left[f^2(\theta_x) - f(\theta_x)f(\theta_y) \cos(\theta_y - \theta_x) + f'(\theta_x)f(\theta_y) \sin(\theta_y - \theta_x) \right] \\ \times \sqrt{\frac{f^2(\theta_y) + f'^2(\theta_y)}{f^2(\theta_x) + f'^2(\theta_x)}} \frac{J_1(kr(x, y))}{r(x, y)} \quad (3.46)$$

$$L'_2(x, y) = L'(x, y) - L'_1(x, y) \log 4 \left[\sin^2 \frac{(\theta_x - \theta_y)}{2} \right] \quad (3.47)$$

In particular for $y = x$, using the series expansion of H_0 (See [6]), we have

$$M_2(x, x) = \left\{ \frac{i}{2} - \frac{\gamma}{\pi} - \frac{1}{2\pi} \log \left[\frac{k^2}{4} (f^2(\theta_x) + f'^2(\theta_x)) \right] \right\} \sqrt{f^2(\theta_x) + f'^2(\theta_x)} \\ L'_2(x, x) = L'(x, x) = \frac{1}{2\pi} \frac{f^2(\theta_x) + 2f'(\theta_x)f''(\theta_x) - f(\theta_x)f'''(\theta_x)}{f^2(\theta_x) + f'^2(\theta_x)}$$

The Nyström method relies on approximating the integrals by quadrature formulas. To do it, let us set

$$M_1(x, y)\phi(y) \approx \sum_{k=0}^{2n-1} M_1(x, y_k) \phi_k^{(n)}(y_k) L_k^{(n)}(y) \quad (3.48)$$

where $\phi_k^{(n)}(y_k)$ are the values of ϕ at the $2n$ points $y_k, y_k := k\frac{\pi}{n}, k = 0, \dots, 2n-1$, and the $L_k^{(n)}(y)$ are the Lagrange basis for the trigonometric interpolation defined by

$$L_k^{(n)}(y) = \frac{1}{2\pi} \left\{ 1 + \sum_{l=0}^{n-1} \cos l(\theta_y - y_k) + \cos n(\theta_y - y_k) \right\} \quad (3.49)$$

Hence we obtain:

$$\int_0^{2\pi} d\theta_y M_1(x, y) \log 4 \left[\sin^2 \frac{(\theta_x - \theta_y)}{2} \right] \phi(y) \approx \sum_{k=0}^{2n-1} M_1(x, y_k) \phi_k^{(n)}(y_k) R_k^{(n)}(x)$$

with $R_k^{(n)}(x) := \int_0^{2\pi} d\theta_y \log 4 \left[\sin^2 \frac{(\theta_x - \theta_y)}{2} \right] L_k^{(n)}(y)$.

It gives after a few calculations (see ref. [5]):

$$R_k^{(n)}(x) = -\frac{2\pi}{n} \sum_{l=0}^{n-1} \frac{1}{l} \cos l(\theta_x - y_k) - \frac{\pi}{n^2} \cos n(\theta_x - y_k) \quad (3.50)$$

The regular parts of the integrals are calculated by trapezoidal method. We use the same type of approximation in the second integral equation and we obtain the following approximate equations:

$$\begin{aligned} \phi^{(n)}(x) + \beta'(x) \sum_{k=0}^{2n-1} \left[R_k^{(n)}(x) M_1(x, y_k) + \frac{\pi}{n} M_2(x, y_k) \right] \phi_k^{(n)}(y_k) \\ = -2\beta'(x) e^{ikf(\theta_x) \cos(\theta_x - \theta_{\vec{k}})} \end{aligned} \quad (3.51)$$

$$\begin{aligned} \phi^{(n)}(x) + \tilde{\beta}'(x) \sum_{k=0}^{2n-1} \left[R_k^{(n)}(x) M_1(x, y_k) + \frac{\pi}{n} M_2(x, y_k) \right] \phi_k^{(n)}(y_k) \\ + \tilde{\beta} \sum_{k=0}^{2n-1} \left[R_k^{(n)}(x) L'_1(x, y_k) + \frac{\pi}{n} L'_2(x, y_k) \right] \phi_k^{(n)}(y_k) \\ = -2\tilde{\beta}' e^{ikf(\theta_x) \cos(\theta_x - \theta_{\vec{k}})} + 2i\tilde{\beta}(\nu_{x_j}, \vec{k}) e^{ikf(\theta_x) \cos(\theta_x - \theta_{\vec{k}})} \end{aligned} \quad (3.52)$$

Finally we must solve finite linear equations obtained by writing the equation (3.51) and (3.52) at quadrature points $\phi_j^{(n)}(x_j)$, $x_j := j\frac{\pi}{n}$, $j = 0, \dots, 2n-1$. These systems are:

$$\begin{aligned} \phi_j^{(n)}(x_j) + \beta'(x_j) \sum_{k=0}^{2n-1} \left[R_k^{(n)}(x) M_1(x_j, y_k) + \frac{\pi}{n} M_2(x_j, y_k) \right] \phi_k^{(n)}(y_k) \\ = -2\beta'(x_j) e^{ikf(x_j) \cos(x_j - \theta_{\vec{k}})} \end{aligned} \quad (3.53)$$

$$\begin{aligned} \phi_j^{(n)}(x_j) + \sum_{k=0}^{2n-1} \tilde{\beta}'(x_j) \left[R_k^{(n)}(x) M_1(x_j, y_k) + \frac{\pi}{n} M_2(x_j, y_k) \right] \phi_k^{(n)}(y_k) \\ + \tilde{\beta} \sum_{k=0}^{2n-1} \left[R_k^{(n)}(x_j) L'_1(x_j, y_k) + \frac{\pi}{n} L'_2(x_j, y_k) \right] \phi_k^{(n)}(y_k) \\ = -2\tilde{\beta}'(x_j) e^{ikf(x_j) \cos(x_j - \theta_{\vec{k}})} + 2i\tilde{\beta}(\nu_{x_j}, \vec{k}) e^{ikf(x_j) \cos(x_j - \theta_{\vec{k}})} \end{aligned} \quad (3.54)$$

where $y_k := k\frac{\pi}{n}$, $k = 0, \dots, 2n-1$.

After solving these linear systems, we obtain the respective approximate solutions $\phi^{(n)}$ at any point $0 \leq x \leq 2\pi$, by replacing the values $\phi_k^{(n)}(y_k)$ at $y_k = k\frac{\pi}{n}$, $k = 0, \dots, 2n-1$ in the equations (3.51) and (3.54).

Finally, this method we have the following result

Theorem 5 *As the equation (3.28) has a unique solution and the kernels $M_1(x, y)$ and $M_2(x, y)$ are continuous, and the functions $\beta'(x)$ and $e^{i\vec{k} \cdot x}$ are continuous too, then:*

- the approximating linear system (3.53) has a unique solution for all sufficiently large n .
- For $n \rightarrow \infty$, the approximation solutions $\phi^{(n)}$ converge uniformly to the solution ϕ of the integrale equation (3.28).

In addition, if $\beta'(x)$ and $e^{i\vec{k}\cdot x}$ and S are analytic then the errors $\|\phi^{(n)} - \phi\|_\infty$ decrease exponentially; i.e.

$$\|\phi^{(n)} - \phi\|_\infty = O(e^{-ns})$$

where $s > 0$.

Proof: this result is built on the idea of collectively compact operators. For more details see ([3]) chapters 10, 11, 12, and in particular the theorems 10.8, 10.9, 11.5, 12.13 .

Q.E.D.

Of course, we have a similar convergence result for the second case.

4 Ambiguities with $\theta_{\vec{k}'} = \theta_{\vec{k}} + \delta$ at fixed energy

Now we present the numerical construction of equivalent scattering problems. We determine scatterers which give the same scattering amplitude for two fixed energies, all illumination angles but the observation angles fixed in such a way that there is always a difference of a fixed value δ between the observation angle and the illumination angle. The scattering problems are treated with one discontinuity curve, without diffuse scattering (i.e. we consider the scattering amplitude A_0) and only on the Born approximation.

For the construction we assume that S_1 and S_2 , the curves of two scattering prob-

Figure 2: Ambiguities with $\theta_{\vec{k}'} = \theta_{\vec{k}} + \delta$ at fixed energie

lems, are star-shaped (see figure 2) with respect to the same center. Their equations are $R_1 = f_1(\theta)$ and $R_2 = f_2(\theta)$. Their singular datas are noted $\beta_1(\theta)$, $\beta'_1(\theta)$, $\beta_2(\theta)$ and $\beta'_2(\theta)$. All these functions are 2π periodic. To summarize, P_1 is characterised by $\{R_1(\theta), \beta_1(\theta), \beta'_1(\theta)\}$ and P_2 by $\{R_2(\theta), \beta_2(\theta), \beta'_2(\theta)\}$. For the sake of simplicity, we choose a circle for the discontinuity curve of P_2 .

Our goal is to construct the singular datas $\beta_2(\theta)$ and $\beta'_2(\theta)$ supposing $\{R_1(\theta), \beta_1(\theta), \beta'_1(\theta)\}$ and R_2 given, such that we obtain the same amplitude for a given value of k , all illumination angles, $\theta_{\vec{k}}$, and the observation angles, $\theta_{\vec{k}'}$, satisfying the condition $\theta_{\vec{k}'} = \theta_{\vec{k}} + \delta$.

So as to obtain it, we start from the Born approximation relation (2.27) which gives for $\theta_{\vec{k}'} = \theta_{\vec{k}} + \delta$ the following expression.

$$\begin{aligned} A_{0,k,\delta}^B(\theta_{\vec{k}}) &= 2 \int_0^{2\pi} d\theta \, l(\theta) e^{-2ikf(\theta) \sin \frac{\delta}{2} \sin(\theta - \theta_{\vec{k}})} \\ &+ 4ik \sin \frac{\delta}{2} \int_0^{2\pi} d\theta \, g(\theta) \sin(\theta - \theta_{\vec{k}} - \sin \frac{\delta}{2}) e^{-2ikf(\theta) \sin \frac{\delta}{2} \sin(\theta - \theta_{\vec{k}} - \sin \frac{\delta}{2})} \\ &- 4ik \sin \frac{\delta}{2} \int_0^{2\pi} d\theta \, h(\theta) \cos(\theta - \theta_{\vec{k}} - \sin \frac{\delta}{2}) e^{-2ikf(\theta) \sin \frac{\delta}{2} \sin(\theta - \theta_{\vec{k}} - \sin \frac{\delta}{2})} \end{aligned}$$

with:

$$\begin{cases} l(\theta) &= \beta'(\theta) \sqrt{f^2(\theta) + f'^2(\theta)} \\ g(\theta) &= \beta(\theta) f(\theta) \\ h(\theta) &= \beta(\theta) f'(\theta) \end{cases} \quad (4.55)$$

One can notice that $A_{0,k,\delta}^B(\theta_{\vec{k}})$ for fixed values of k and δ , is a 2π periodic function of the illumination angle $\theta_{\vec{k}}$. Therefore we calculate the Fourier coefficients of $A_{0,k,\delta}^B(\theta_{\vec{k}})$, A_m and afterwards the coefficients $B_m^{(1),(2)}$ and $C_m^{(1),(2)}$ defined by the following expressions.

$$\begin{cases} B_m^{(1),(2)} &= A_m^{(1),(2)} + A_{-m}^{(1),(2)} \\ C_m^{(1),(2)} &= A_m^{(1),(2)} - A_{-m}^{(1),(2)} \end{cases} \quad \text{avec } m \in \mathbb{N}^{+*}$$

The two problems P_1 and P_2 are equivalent if:

$$\begin{cases} A_0^{(1)} &= A_0^{(2)} \\ B_m^{(1)} &= B_m^{(2)} \\ C_m^{(1)} &= C_m^{(2)} \end{cases} \quad m \in \mathbb{N}^*$$

After a few calculations we find the following infinit system :

$$\mu_0 \Re e \beta_0^{(2)} + \nu_0 \Re e \beta_0^{(2)} = A_0^{(1)} \quad (4.56)$$

$$\mu_m \Re e \beta_m^{(2)} + \nu_m \Re e \beta_m^{(2)} = B_m^{(1)} \quad (4.57)$$

$$\mu_m \Im m \beta_m^{(2)} + \nu_m \Im m \beta_m^{(2)} = C_m^{(1)} \quad (4.58)$$

with

$$\begin{cases} \mu_m &= 2\pi R_2 J_m(2kR_2) \\ \nu_m &= 2\pi k \sin \frac{\delta}{2} R_2 [J_{m+1}(2kR_2) - J_{m-1}(2kR_2)] \end{cases}$$

$$A_0^{(1)} = 2 \int_0^{2\pi} d\theta \, \left(l(\theta) J_0(2kf_1(\theta)) + 2k \sin \frac{\delta}{2} g(\theta) J_1(2kf_1(\theta)) \right)$$

$$\begin{aligned} B_m^{(1)} &= \int_0^{2\pi} d\theta \, l^{(1)}(\theta) J_m(2kf_1(\theta)) \cos m\theta \\ &+ k \sin \frac{\delta}{2} \int_0^{2\pi} d\theta \, g^{(1)}(\theta) [J_{m+1}(2kf_1(\theta)) - J_{m-1}(2kf_1(\theta))] \cos m\theta \\ &- k \sin \frac{\delta}{2} \int_0^{2\pi} d\theta \, h^{(1)}(\theta) [J_{m+1}(2kf_1(\theta)) + J_{m-1}(2kf_1(\theta))] \sin m\theta \end{aligned}$$

$$\begin{aligned}
C_m^{(1)} &= - \int_0^{2\pi} d\theta \, l^{(1)}(\theta) J_m(2kf_1(\theta)) \sin m\theta \\
&\quad - k \sin \frac{\delta}{2} \int_0^{2\pi} d\theta \, g^{(1)}(\theta) [J_{m+1}(2kf_1(\theta)) - J_{m-1}(2kf_1(\theta))] \sin m\theta \\
&\quad - k \sin \frac{\delta}{2} \int_0^{2\pi} d\theta \, h^{(1)}(\theta) [J_{m+1}(2kf_1(\theta)) + J_{m-1}(2kf_1(\theta))] \cos m\theta
\end{aligned}$$

Remark for m fixed that there are two equations for four unknowns. To get four equations we choose two fixed energies. One could as well choose one energy but two different radii and obtain three equivalent problems.

Numerical examples are dispatched in figures. For each figure, we represent the characteristics of the original scattering situation P_1 , i.e. $\{R_1(\theta), \beta_1(\theta), \beta_1'(\theta)\}$, the chosen circular discontinuous curve of P_2 , and its singular data $\beta^{(2)}$ and $\beta'^{(2)}$ determined so that P_1 and P_2 are equivalent. We show the Born approximation of the scattering amplitude for the two chosen energies k_1 and k_2 , all illumination angles and the observation angles such as there is a fixed difference of δ between the observation angle and the illumination angle. Further we represent the total scattering amplitude of P_1 and P_2 for the same conditions (energies, illumination and observation angles) so as to verify that the equivalence constructed with the Born approximation is still true for the total amplitude. These exact scattering amplitudes are calculated by means of the Nystöm method.

For these examples we can see that there is yet an ambiguity for the exact scattering although the second chosen curve is fixed (a circle).

The figure (7) represents an example of the evolution of the amplitude according to the energy in the range of values between k_1 and k_2 . The second example of evolution (figure (8)) is according to a range of values of δ taken around δ_0 . The values k_1 , k_2 and δ_0 are the values for which the two equivalent problems have been determined. With the first example we can see that the ambiguity depends weakly on the energy. The second one shows that the construction (for this example) depends weakly on the chosen difference between the observation angle and the illumination angle. A physical situation corresponding to this type of variation is when you have a fixed seat for the illumination, a fixed seat for the observation but where the scatterer can turn around itself.

The figures (9) and (10) are examples where three equivalent problems are constructed for one given energy.

As conclusion we show also another (see previous paper [3]) realistic situation of measurement (fixed illumination angle, fixed observation angle with a scatterer turning around itself) where the uniqueness is not still assured because we have too much weakened the hypothesis of Nachman's theorem. These examples are another warning methods of reconstruction in nondestructive sensing.

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Figure 3: Scattering problems with $\delta = 1.605\pi$ with for P_1 : $R_1 = 1 + 0.2 \sin 2\theta$, $\beta_1 = 0$, $\beta'_1 = 0.03(\cos \theta + \cos 2\theta)$ and P_2 , $R_2 = 1$

Figure 4: Scattering problems with $\delta = 1.605\pi$ with for P_1 : $\beta_1 = 0.005$, $\beta'_1 = 0$, $R_1 = \frac{2.4}{[1.44\cos^2(\theta) + 4\sin^2(\theta)]}$, and for P_2 : $R_2 = 1.6$.

Figure 5: Scattering problems with $\delta = \frac{4\pi}{3}$ with for P_1 :
 $R_1 = 1 + 0.2 \sin[4(\theta + \frac{\pi}{8})]$, $\beta_1 = 0.01$, $\beta'_1 = 0.02$ and for P_2 , $R_2 = 1$

Figure 6: Scattering problems with $\delta = 1.6\frac{\pi}{2}$ with for P_1 : $R_1 = (1 + 0.2\sin[4(\theta + \frac{\pi}{4})]) (1 + 0.1\sin[8(\theta + \frac{\pi}{4})])$, $\beta_1 = 0.03$, $\beta'_1 = 0.05$ and for P_2 , $R_2 = 1.2$

Figure 7: Example of variation of the energy between the values $k_1 = 3$ and $k_2 = 3.6$ for the equivalent problems defined at the figure 3

Figure 8: Example of variation of the δ difference between the observation angle and the illumination angle around the value $\delta_0 = 1.605\pi$ rad for the equivalent problems defined at the figure 3

Figure 9: Scattering problems with $\delta = 1.605\pi$ with for $P_1 : R_1 = 1 + 0.2 \sin 2\theta$, $\beta_1 = 0$, $\beta'_1 = 0.03(\cos \theta + \cos 2\theta)$, for P_2 , $R_2 = 1$ and for P_3 , $R_3 = 1.5$

Figure 10: Scattering problems with $\delta = 1.605\pi$ with for P_1 : $R_1 = 1 + 0.2 \sin[4(\theta + \frac{\pi}{8})]$, $\beta_1 = 0.01$, $\beta'_1 = 0.03$, for P_2 , $R_2 = 1$ and for P_3 , $R_3 = 1.5$