SOLUTIONS OF THE YANG-BAXTER EQUATIONS FROM
BRAIDED-LIE ALGEBRAS AND BRAIDED GROUPS

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ABSTRACT We obtain an R-matrix or matrix representation of the Artin braid group acting in a canonical way on the vector space of every (super)-Lie algebra or braided-Lie algebra. The same result applies for every (super)-Hopf algebra or braided-Hopf algebra. We recover some known representations such as those associated to racks. We also obtain new representations such as a non-trivial one on the ring \( k[x] \) of polynomials in one variable, regarded as a braided-line. Representations of the extended Artin braid group for braids in the complement of \( S^1 \) are also obtained by the same method.

1 Introduction

In this paper we apply some constructions from the theory of braided groups and braided geometry[1] to obtain a new construction for matrix solutions of the celebrated Quantum Yang-Baxter Equations (QYBE). Equivalently, we provide a new and canonical class of representations of the Artin braid group. The importance of such representations or R-matrices has been very clearly established in the last few years and is one of the primary motivations behind the celebrated quantum groups \( U_q(g) \)[2][3]. Representations lead ultimately to link invariants and families of representations to 3-manifold invariants.

By contrast to the theory of quantum groups, our construction is based on what we believe to be a more primitive object, called a braided Lie algebra[4]. The famous quantum groups \( U_q(g) \) have finite-dimensional braided-Lie algebras associated to them and one can work with them instead of the quantum group. In this case our canonical braiding reproduces the braiding associated to the quantum double[2].

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More generally however, our notion of braided Lie algebra also includes as a special case
the notion of a rack, see e.g[5]. In this case we recover the rack braiding. Super-Lie algebras,
super-racks and other much more esoteric objects are also included in the theory.

The axioms of a braided-Lie algebra are recalled in the Preliminaries. They are Lie algebra-
like objects living in a braided tensor category with braiding $\Psi$. Ordinary Lie algebras and
ordinary racks are defined with $\Psi$ given by the usual transposition. Their super-versions are
defined with $\Psi = \pm 1$ according to a $\mathbb{Z}_2$-grading. Unlike previous attempts to go beyond super-
symmetry, we need not assume that $\Psi^2 = \text{id}$.

In Section 3 we give a parallel theorem for Hopf algebras, super-Hopf algebras and more
generally, for braided-Hopf algebras[6]. The latter are Hopf algebras living in our braided tensor
category with background braiding $\Psi$. This theory is more general because not every braided-
Hopf algebra is the enveloping algebra of a braided-Lie algebra, but in the case that it is, we recover the results of Section 2. The example of a finite group and its canonical braiding
fits comfortably into either setting. We will also give some more novel examples in Section 4,
including one based on the anyonic line where $\Psi$ is given by a root of unity. Many other
important algebras of interest in the theory of $q$-deformations are not naturally Hopf algebras
but rather braided ones.

Finally, we show in the Appendix how the extended Artin braid relations for braids in the
complement of the unknot in $S^3$ can be represented equally well using the same techniques.
We assume that we are given a cocommutative representation of a braided-Hopf algebra in
the sense that it is central in the braided representation ring of the braided-Hopf algebra[7].
Representations of such extended braid relations have been used by knot theorists in [8] and
elsewhere. In short, the techniques which we use here are of quite wide applicability and this
appendix demonstrates one more instance of them.

Although we will not go as far as constructing knot and three-manifold invariants from our
canonical braiding, knot theory nevertheless enters in a fundamental way. This is because we
will be working throughout in a background braided category with braiding $\Psi$. Usually one uses
quantum groups etc to construct such a braided category and hence to obtain knot-invariants:
we proceed in exactly the reverse direction by assuming that $\Psi$ is given and doing all our proofs
by drawing braids and tangles. These techniques and the formulation of a large number of

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geometrical constructions of planes, lines, matrices, groups, differential operators etc., is the

topic of braided geometry as developed over 30-40 papers by the author in the last few years.

We refer to [9][1] for reviews and to [6][7][10][11][12][13] for some of the basic theory.

**Preliminaries**

Here we recall very briefly the definition of braided or quasitensor categories and the diagram-

matic notation for them. Firstly, a monoidal category consists of a category \( \mathcal{C} \) equipped with a

functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and functorial isomorphisms \( \Phi_{V,W,Z} : V \otimes (W \otimes Z) \to (V \otimes W) \otimes Z \) for all

objects \( V, W, Z \), and a unit object \( \mathbf{1} \) with functorial isomorphisms \( l_V : V \to \mathbf{1} \otimes V, r_V : V \to V \otimes \mathbf{1} \)

for all objects \( V \). The \( \Phi \) should obey a well-known pentagon coherence identity while the \( l \) and \( r \) obey triangle identities of compatibility with \( \Phi[14] \). We assume such a monoidal category

and suppress writing \( \Phi, l, r \) explicitly. A monoidal category also has an opposite tensor product

\( \otimes^\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) defined in the obvious way.

A braided monoidal or quasitensor category \( (\mathcal{C}, \Psi) \) is a monoidal category \( \mathcal{C} \) equipped further

with a natural transformation \( \Psi : \otimes^\circ \to \otimes \) called the braiding or quasisymmetry and subject to two hexagon coherence identities. Explicitly, this means a collection of functorial isomorphisms

\( \Psi_{V,W} : V \otimes W \to W \otimes V \) for any two objects and such that

\[
\Psi_{V,W} \otimes Z = \Psi_{V,Z} \circ \Psi_{V,W}, \quad \Psi_{V} \otimes W,Z = \Psi_{V,Z} \circ \Psi_{W,Z}.
\]  

(1)

One can deduce also that \( \Psi_{\mathbf{1},V} = \mathbf{id} = \Psi_{V,\mathbf{1}} \) for all \( V \). If \( \Psi^2 = \mathbf{id} \) then one of the hexagons is

superfluous and we have an ordinary symmetric monoidal category or tensor category. Braided

monoidal categories were formally introduced in [15], while being known also to specialists in

the representation theory of quantum groups[16, Sec. 7].

Crucial for us is the following diagrammatic notation for working with algebraic objects in

braided categories. Firstly, we write all morphisms pointing downwards (say) and in the case of

the braiding morphism, we use the shorthand

\[
\Psi_{V,W} = \begin{array}{cc}
V & W \\
W & V
\end{array}, \quad \Psi_{V,W}^{-1} = \begin{array}{cc}
V & W \\
W & V
\end{array}
\]  

(2)
This distinguishes between $\Psi$ and $\Psi^{-1}$, while the hexagons (1) appear as

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
V W Z \\
\end{array}
\begin{array}{c}
\begin{array}{c}
Z V W \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
V W Z \\
\end{array}
\begin{array}{c}
\begin{array}{c}
W Z V \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

(3)

The doubled lines refer to the composite objects $V \otimes W$ and $W \otimes Z$ in a convenient extension of the notation. The coherence theorem for braided categories says then that if two series of morphisms built from $\Psi, \Phi$ correspond to the same braid then they compose to the same morphism. The proof is just the same as Mac Lane’s proof in the symmetric case with the action of the symmetric group replaced by that of the Artin braid group.

Finally, we take this notation further by writing any other morphisms as nodes on a string connecting the inputs down to the outputs. Functoriality of the braiding then says that morphisms $\phi : V \to Z, W \to Z,$ etc. can be pulled through braid crossings,

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
V W \\
\end{array}
\begin{array}{c}
\begin{array}{c}
W Z \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
V W \\
\end{array}
\begin{array}{c}
\begin{array}{c}
Z V \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

(4)

Similarly for $\Psi^{-1}$ with inverse braid crossings. The simplest example is with $C = \text{SuperVec}$, the category of $\mathbb{Z}_2$-graded vector spaces and braiding

\[
\Psi(v \otimes w) = (-1)^{|v||w|} w \otimes v
\]

(5)

where $||$ denotes the degree of a homogeneous element. Of course, this example is not truly braided since $\Psi^2 = \text{id}$.

We recall also the celebrated Yang-Baxter equations or Artin braid relations. Thus, a Yang-Baxter operator is a morphism $\hat{R} : V \otimes V \to V \otimes V$ such that

\[
\hat{R}_{23} \circ \hat{R}_{12} \circ \hat{R}_{23} = \hat{R}_{12} \circ \hat{R}_{23} \circ \hat{R}_{12}
\]

(6)

where the suffices refer to the copy of $V$ in $V \otimes V \otimes V$. If $V$ is an ordinary vector space and everything is linear then we can write $\hat{R} = PR$ where $P : V \otimes V \to V \otimes V$ is the permutation operator. Then the corresponding equation for $R$ is

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}
\]

(7)
which is the so-called *quantum Yang-Baxter equation* (QYBE). The matrices of such operators are called in physics ‘R-matrices’.

There is a close relation between R-matrices and braided categories for which the objects are built on vector spaces. Obviously, if $\Psi$ is a braiding then $\Psi_{V,V}$ is an invertible Yang-Baxter operator and hence when $V$ is a finite-dimensional vector space we have an associated invertible R-matrix. Conversely, any invertible R-matrix defines a braiding on the monoidal category generated by $V$.

Note that the general theory of Sections 2,3 works in any braided monoidal category. In this case we use the word ‘operator’ etc here a bit loosely. On the other hand, our examples in Section 4 are in a $k$-linear setting where $k$ is a field, and then our operators are indeed linear maps.

2 Canonical braiding of a braided-Lie algebra

We have introduced in [4] the notion of a braided-Lie algebra or Lie-algebra-like object in a braided monoidal category $(\mathcal{C}, \Psi)$ as $(\mathcal{C}, \Delta, \epsilon, [\cdot, \cdot])$ where $(\mathcal{C}, \Delta, \epsilon)$ is a coalgebra in the category and $[\cdot, \cdot] : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is the braided Lie bracket and is required to obey

\[
\begin{align*}
(L1) & \quad \Delta \otimes \Delta = \Delta \otimes \Delta \\
(L2) & \quad [\cdot, \cdot] \otimes \Delta = \Delta \otimes [\cdot, \cdot] \\
(L3) & \quad [\cdot, \cdot] \otimes \Delta = \Delta \otimes [\cdot, \cdot]
\end{align*}
\]

We use here the diagrammatic notation described in the preliminaries. A coalgebra in the category is defined in just the same way as an algebra, but with arrows reversed. Thus, $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, the comultiplication, is coassociative in an obvious sense and $\epsilon : \mathcal{C} \to \mathbf{1}$ is a counit for it in the obvious sense. Explicitly,

\[
\begin{align*}
& \quad \Delta \otimes 1 = \Delta \\
& \quad 1 \otimes \Delta = \Delta \\
& \quad \Delta \otimes \epsilon = \epsilon \otimes \Delta
\end{align*}
\]
The condition (I.1) is called the braided-Jacobi identity axiom, (I.2) the braided-cocommutativity axiom and (I.3) the coalgebra-compatibility axiom. We refer to [4] for the justification and full explanation of these axioms. Suffice it to say that in a truly braided category the naive notions of $\Psi$-anticommutativity and $\Psi$-Jacobi identity are not appropriate and one needs a genuinely new idea. The new idea in [4] is to allow ourselves a more general coalgebra $\Delta$ instead of the primitive coalgebra structure $\Delta \xi = \xi \otimes 1 + 1 \otimes \xi$ on $k \oplus g$ which is implicitly assumed in the theory of Lie algebras.

The basic theory of braided-Lie algebras has also been developed in [4]. This includes such things as (in the Abelian category case) a braided-enveloping bialgebra $U(\mathcal{L})$, braided-Killing forms and braided-Casimirs etc. To this theory we want to add now the following theorem announced in [17].

**Theorem 2.1** Let $(\mathcal{L}, \Delta, \epsilon, [ , ])$ be a braided-Lie algebra. Then

$$\hat{R} = \Delta \left( \frac{\mathcal{L} \otimes \mathcal{L}}{1} \right) \left( \frac{\mathcal{L} \otimes \mathcal{L}}{1} \right) = ([ , ] \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\Delta \otimes \text{id}) : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$$

is a Yang-Baxter operator.
Proof We do this diagrammatically in Figure 1, using the notation explained in the preliminaries. The vertices are $\Delta = \bigwedge$ and $[ , ] = \bigvee$ throughout. The first expression is the right-hand side of (6) for $\mathbf{R}$ as stated. The first equality is (I.3). The second equality is coassociativity (9) and functoriality to put the diagram in a form suitable for (I.2), which is the third equality. The fourth equality is coassociativity (9) again. The fifth then uses our braided-Jacobi identity axiom (I.1). The sixth is coassociativity once more and finally we use functoriality to slide the diagram into the final form, which is the left hand side of (6) for $\mathbf{R}$. □

Moreover, it is evident from its diagrammatic definition in [4] that the braided enveloping algebra $U(\mathcal{L})$ is generated by 1 and $\mathcal{L}$ with the relations

$$\cdot \circ \hat{R} = \cdot$$

of braided commutativity.

3 Canonical braiding of a braided-Hopf algebra

In this section, we further generalise the result of the last section to associate to any braided-Hopf algebra at all a canonical Yang-Baxter operator. Braided-Hopf algebras were introduced by the author in [6][18][19][20] as a generalisation to braided categories of the usual notion of Hopf algebra or super-Hopf algebra. Briefly, a braided-Hopf algebra means $(B, \Delta, \epsilon, S)$ where firstly $B$ is a unital algebra in a braided monoidal category. This means it comes equipped with product and unit morphisms $B \otimes B \to B$ and $1 \to B$ respectively, obeying the obvious axioms of associativity and unity. Secondly, $\Delta : B \to B \otimes B$ and $\epsilon : B \to 1$ form a coalgebra as already encountered in Section 2. We require further that $\Delta$ is an algebra homomorphism where $B \otimes B$ is the braided tensor product algebra (as also introduced by the author). This forms a braided-bialgebra or bialgebra in a braided category. Finally, we require an antipode $S : B \to B$ obeying axioms similar to those for quantum groups or Hopf algebras, but as a morphism in our
category. In the diagrammatic notation with $\Delta = \land$ and $\cdot = \lor$, our axioms read

\[
\begin{align*}
\Delta_B \land \Delta_B & = \Delta_B \\
\Delta_B \land \cdot & = \cdot \\
\Delta_B \land \Delta_B & = \Delta_B
\end{align*}
\]

\[\tag{11}\]

The general theory of braided-Hopf algebras has also been developed by now in a diagrammatic form\cite{7,10,9}. One has left and right dual Hopf algebras (when $B$ has a dual object), regular actions and coactions, braided-adjoint actions and coactions, cross products etc. We use the braided-adjoint action now.

**Theorem 3.1** Let $(B, \Delta, \epsilon, S)$ be a braided-Hopf algebra and $\text{Ad}$ the braided-adjoint action. Then

\[
\tilde{R} = \begin{array}{c}
\Delta_B \\
\text{Ad} \\
\end{array}
\begin{array}{c}
B \\
B \\
B \\
B \\
B \\
B \\
\end{array} = \begin{array}{c}
\Delta_B \\
\Delta_B \\
\Delta_B \\
\end{array}
\begin{array}{c}
B \\
B \\
B \\
B \\
B \\
\end{array} = (\text{Ad} \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\Delta \otimes \text{id}) : B \otimes B \rightarrow B \otimes B
\]

is a Yang-Baxter operator.

**Proof** This is given in diagrammatic form in Figure 2. Some of the $\lor$ vertices are the braided-adjoint action $\text{Ad}$\cite{18,4} and the rest are the product in $B$. The $\land$ are the coproduct throughout. The first expression is the right hand side of (6) for $\tilde{R}$ as stated. The first equality uses that $\text{Ad}$ is indeed an action of $B$ on $B$. The second equality substitutes the form of $\text{Ad}$ in terms of the braided-Hopf algebra structure, as shown in the definition of $\tilde{R}$. The third equality uses the bialgebra axiom that $\Delta$ is an algebra homomorphism to the braided tensor product as on the left in (11). We also adopt the convention that repeated applications of $\Delta$ can be represented by multiple branches. Likewise for multiple products. This convention expresses coassociativity and associativity respectively. The fourth equality is the lemma proven in [7] that $S$ is a braided-anti-algebra homomorphism in the sense $S \circ \cdot = \cdot \circ \Psi \circ (S \otimes S)$. The fifth equality recognises
a loop involving the antipode and cancels it according to the left-hand antipode axiom shown in (11). We also recognise the remaining antipode $S$ as part of an application of $\text{Ad}$. The sixth equality is coassociativity and functoriality to push this $\text{Ad}$ down to the bottom of the expression. Finally, we use again that $\text{Ad}$ is an action to obtain the left hand side of (6) for $\dot{\mathbf{R}}$. □

Moreover, it is obvious from coassociativity, associativity and the axioms for the antipode that $B$ itself is braided-commutative in the sense

Let us note also that the axioms of a braided-Hopf algebra in (11) are symmetric under the operations of left-right reflection and braid crossing reversal, up-down reflection and braid crossing reversal, and rotation by 180 degrees. As explained in [1], it means that the diagrammatic method always gives three theorems for the price of one. Applying these symmetries to Figure 2
and its associated lemmas gives

\[
\tilde{R} = \begin{array}{ccc}
B & B \\
B & B
\end{array}
\]

\[
\tilde{R} = \begin{array}{ccc}
B & B \\
B & B
\end{array}
\]

\[
\tilde{R} = \begin{array}{ccc}
B & B \\
B & B
\end{array}
\]

as three other Yang-Baxter operators. \( B \) is also braided-commutative with respect to the first and braided-cocommutative with respect to the second and third.

4 Examples

In this section we describe various examples and special cases of the general constructions above. Throughout this section we work over a field \( k \) of characteristic zero. The same results apply more generally with appropriate care. As far as I know, only the case in subsection 4.4 was known before from the theory of racks and its generalisation in subsection 4.5 from Drinfeld’s quantum double construction for Hopf algebras and later from [21]. The rest appear to be a product of our construction, as announced recently in the conference proceedings[17].

4.1 Ordinary Lie Algebras

Note that an ordinary Lie algebra obeys these axioms if one puts \([1, \xi] = \xi, [\xi, 1] = 0, [1, 1] = 1\) and

\[
\mathcal{L} = k \oplus g, \quad \Delta l = 1 \otimes 1, \quad \epsilon l = 1, \quad \Delta \xi = \xi \otimes 1 + 1 \otimes \xi, \quad \epsilon \xi = 0, \quad \forall \xi \in g. \quad (12)
\]

So this structure \( \Delta, \epsilon \) is implicit for an ordinary Lie algebra but we never think about it because it has this standard form. This was our motivation in [4].

**Proposition 4.1** Let \( V = k \oplus g \) and define the linear map

\[
\tilde{R}(1 \otimes 1) = 1 \otimes 1, \quad \tilde{R}(1 \otimes \xi) = \xi \otimes 1, \quad \tilde{R}(\xi \otimes 1) = 1 \otimes \xi
\]

\[
\tilde{R}(\xi \otimes \eta) = \eta \otimes \xi + [\xi, \eta] \otimes 1, \quad \forall \xi, \eta \in g.
\]

Then \( \tilde{R} \) is a braiding iff \([ \ , ] : g \odot g \to g \) obeys the Jacobi identity. It has minimal polynomial

\[
(\tilde{R}^2 - \text{id})(\tilde{R} + \text{id}) = 0 \quad (13)
\]

iff \([ \ , ] \) is non-zero and antisymmetric.
Proof The forward direction is a special case of Theorem 2.1 where we view \( \mathcal{L} = k \oplus g \) as a braided-Lie algebra with trivial braiding \( \Psi \) as explained above. On the other hand, at least in this setting, one can compute it explicitly and see that it is reversible. Thus

\[
\hat{R}_{23} \circ \hat{R}_{12} \circ \hat{R}_{23}(\xi \otimes \eta \otimes \zeta) = \zeta \otimes \eta \otimes \xi + \zeta \otimes [\xi, \eta] \otimes 1 + [\xi, \zeta] \otimes \eta \otimes 1
\]

\[
+ [\eta, \zeta] \otimes 1 \otimes \xi + [\xi, [\eta, \zeta]] \otimes 1 \otimes 1
\]

\[
\hat{R}_{12} \circ \hat{R}_{23} \circ \hat{R}_{12}(\xi \otimes \eta \otimes \zeta) = \zeta \otimes \eta \otimes \xi + [\xi, \zeta] \otimes \eta \otimes 1 + [\eta, \zeta] \otimes 1 \otimes \xi
\]

\[
+ [\eta, [\xi, \zeta]] \otimes 1 \otimes 1 + \zeta \otimes [\xi, \eta] \otimes 1 + [[\xi, \eta], \zeta] \otimes 1 \otimes 1
\]

so that the only condition is the Jacobi identity in the form that \( [\xi, ] \) acts like a Lie derivation. The braid relations involving the basis element 1 are all empty. Secondly, we compute

\[
(\hat{R}^2 - \text{id}) \circ (\hat{R} + \text{id})(\xi \otimes \eta) = ([\xi, \eta] + [\eta, \xi]) \otimes 1 + 1 \otimes ([\xi, \eta] + [\eta, \xi])
\]

so this vanishes iff the bracket is antisymmetric. Finally, we compute

\[
(\hat{R} \pm \text{id})(\xi \otimes \eta) = \eta \otimes \xi + [\xi, \eta] \otimes 1 \pm \xi \otimes \eta
\]

\[
(\hat{R}^2 - \text{id})(\xi \otimes \eta) = [\eta, \xi] \otimes 1 + 1 \otimes [\xi, \eta]
\]

\[
(\hat{R} + \text{id})^2(\xi \otimes \eta) = 2\xi \otimes \eta + [\eta, \xi] \otimes 1 + 1 \otimes [\xi, \eta] + 2\eta \otimes \xi + 2[\xi, \eta] \otimes 1
\]

which are all non-zero for some \( \xi, \eta \) if \( [\, , ] \) is non-zero. Hence in this case (13) is the minimum polynomial. Conversely, if this is the minimum-polynomial then in each case there exist \( \xi, \eta \) such that the expression is non-zero. In particular, \( (\hat{R}^2 - \text{id}) \neq 0 \) implies that \( [\xi, \eta] \neq 0 \) for some \( \xi, \eta \). \( \square \)

This says that the definition of a Lie algebra is mathematically completely equivalent to looking for a braiding of a certain form. We say accordingly that a Yang-Baxter operator obeying (13) is of Lie type.

We next discuss the braided-enveloping algebra. To avoid confusion here we denote the basis element of \( k \) in \( k \oplus g \) by \( \lambda \) rather than 1 as above. So our starting point is that that \( \Psi \) trivial (usual transposition) and

\[
\Delta \lambda = \lambda \otimes \lambda, \quad \Delta \xi = \xi \otimes \lambda + \lambda \otimes \xi, \quad \epsilon \lambda = 1, \quad \epsilon \xi = 0
\]

\[
[\lambda, \lambda] = \lambda, \quad [\lambda, \xi] = \xi, \quad [\xi, \lambda] = 0, \quad \forall \xi \in g
\]
extends a usual Lie algebra $g$ to a braided Lie algebra $L = k \oplus g$. The braided enveloping algebra in this case is $U(L) = \widetilde{U(g)}$, a homogenised form of the enveloping algebra $U(g)$. It is an ordinary bialgebra since the braiding $\Psi$ is trivial. On the other hand we know from the definition of $U(L)$ that it is braided-commutative in the sense (10) relative to $\widehat{R}$. Moreover, everything descends to the quotient $\lambda = 1$ so we recover the usual enveloping algebra too as braided-commutative in this sense, a fact which is in any case evident from the form of $\Psi$.

Note also that if we allow $\lambda^{-1}$ then this form of $\widetilde{U(g)}$ has $\xi \lambda^{-1}$ primitive and $\lambda$ group-like, and forms a Hopf algebra. It is only for this part of its structure that antisymmetry of the Lie bracket of $g$ is needed. For a general braided-Lie algebra I do not know any very natural notion of antisymmetry for the bracket.

Finally, we give the concrete matrix version in which we choose a basis $g = \{x_i\}$ for $i = 1, 2, \cdots, n - 1$ and $k = \{x_0\}$ for $V = k \oplus g$. Then

$$
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & I & 0 & c \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{pmatrix}
$$

(14)

where $I$ are identity matrices and $c_{ijk}$ are the structure constants of the bracket $[\ ,\ ]$ on $g$. The basis for $V \otimes V$ used here is $\{x_0 \otimes x_0, x_0 \otimes x_j, x_i \otimes x_0, x_i \otimes x_j\}$. Explicitly,

$$
R_{i,j}^{0,k} = c_{ij}^k, \ R_{j,i}^{k,l} = \delta^i_j \delta^k_l, \ R_{0,i}^{0,j} = \delta^i_j = R_{i,0}^{0,j}, \ R_{0,0}^{0,0} = 1
$$

and zero for the rest. This obeys the QYBE and has minimal polynomial of Lie type iff $c$ defines a non-zero Lie algebra. The quantum $R$-plane or Zamolodchikov algebra

$$
x_i x_j = x_j x_i R_{i,j}^{a,b}
$$

for this $R$-matrix recovers the homogenised enveloping algebra above in our basis. The indices here range $0, \cdots, n - 1$ and summation of the repeated indices is understood.

Finally, we note that homogenised Lie algebras have recently been studied in [22][23] as examples of a kind of non-commutative geometry based on ‘projective line modules’. It would be interesting to try connect this with the braided-geometrical picture developed above.
4.2 Super-Lie Algebras

A super-Lie algebra is a $\mathbb{Z}_2$-graded or ‘super’ vector space $g$ with a degree-preserving map $[\ ,\ ] : g \otimes g \rightarrow g$ obeying the axiom of graded-antisymmetry and the graded-Jacobi identity:

$$[[\xi, \eta], \zeta] + [\eta, [[\xi, \zeta], \eta]] + [[\xi, \eta], \zeta] = 0$$

on homogeneous elements $\xi, \eta, \zeta \in g$. One can view any super-Lie algebra as a braided-Lie algebra $\mathcal{L} = k \oplus g$ in the category of super vector spaces with braiding given by super-transposition (5) and the remaining structure as in subsection 4.1 in (12). The $k$ part of $\mathcal{L}$ is given degree zero.

**Proposition 4.2** Let $g$ be a $\mathbb{Z}_2$-graded vector space and $V = k \oplus g$ with $k$ given degree zero, and $[\ ,\ ] : g \otimes g \rightarrow g$ a degree-preserving linear map. Then

$$\hat{R}(1 \otimes 1) = 1 \otimes 1, \quad \hat{R}(1 \otimes \xi) = \xi \otimes 1, \quad \hat{R}(\xi \otimes 1) = 1 \otimes \xi$$

$$\hat{R}(\xi \otimes \eta) = (-1)^{[\xi][\eta]} \eta \otimes \xi + [\xi, \eta] \otimes 1, \quad \forall \xi, \eta \in g.$$  

obeys the braid relations iff $[\ ,\ ] : g \otimes g \rightarrow g$ obeys the graded-Jacobi identity. Moreover, it has minimal polynomial (13) iff $[\ ,\ ]$ is graded-antisymmetric and non-zero.

**Proof** That $\hat{R}$ obeys the braid relations follows from Theorem 2.1 in the category of $\mathbb{Z}_2$-graded vector spaces where $V$ is viewed as a braided-Lie algebra in this category as explained. Conversely, an explicit computation along the same lines as the proof of Proposition 4.1 gives that the braid relations force $[\ ,\ ]$ to obey the graded Jacobi identity. Similarly for the minimal polynomial by explicit computation. $\Box$

The braided-enveloping algebra $U(\mathcal{L})$ in this case is a homogenised super-bialgebra version of the enveloping super-Hopf algebra $U(g)$. It is the Zamolodchikov or quantum plane algebra for the matrix $R$ corresponding to $\hat{R}$ in this case.

This generalisation of the preceding subsection is immediate because the category is not truly braided, i.e. one has $\Psi^2 = \text{id}$ and hence all the properties familiar in the category of vector spaces. The same applies if we work in any symmetric monoidal category of vector spaces with $\Psi^2 = \text{id}$ and the same form of coproduct $\Delta$ and $[\ ,\ ]$ on an object $\mathcal{L} = k \otimes g$. The analogue of Proposition 4.2 recovers the obvious axioms of a $\Psi$-Lie algebra as studied, for example, in [24]. We still find (13) for $\hat{R}$ even for this more general case.
4.3 Matrix braided-Lie algebras

The data we need is a matrix solution $R \in M_n \otimes M_n$ of the QYBE which is bi-invertible. The ‘second inverse’ $\tilde{R}$ which we suppose here is characterised by

$$\tilde{R}^i_{\ a \ j} R^a_{\ b \ \mathfrak{k}} = \delta_i^j \delta^b_a = R^i_{\ a \ j} \tilde{R}^a_{\ b \ \mathfrak{k}}.$$

We assume summation of repeated indices throughout this section. These $R, \tilde{R}$ generate a braided monoidal category $\mathcal{C}$ and this has an associated braided group $\text{Aut}(\mathcal{C})[20][6]$ which has in turn, a braided-Lie algebra $\mathcal{L}$. Explicitly[11][4],

$$\mathcal{L} = k^{n^2} = \{u^i_j\}, \quad \Delta u^i_j = u^i_k \otimes u^k_j, \quad \epsilon u^i_j = \delta^i_j. \quad (15)$$

$$\Psi(u^i_j \otimes u^l_k) = u^i_k \otimes u^l_j \Psi^j^l_K, \quad \Psi^j^l_K = R^{i_\mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} R^{-1 \mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} R^{i_\mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} \tilde{R}^{i_\mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} \quad (16)$$

$$[u^i_j, u^l_k] = u^i_k \mathcal{E}_{i_\mathfrak{a} \mathfrak{a}} K^j^l_J, \quad \mathcal{E}_{i_\mathfrak{a} \mathfrak{a}} J^j^l = \tilde{R}^{i_\mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} R^{-1 \mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} R^{i_\mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} \tilde{R}^{i_\mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} \quad (17)$$

where we write $J = (i_0, i_1)$ etc as multi-indices. We changed conventions here from [4] to lower indices for the $\{u^i_j\}$. There is also a nice compact notation used in physics where subscripts refer to the positions in a matrix tensor product (as in the QYBE above). In this notation,

$$\Delta u = u \otimes u, \quad \epsilon u = \text{id}$$

$$\Psi(R_{12}^{-1} u_1 \otimes R_{12} u_2) = u_2 R_{12} u_1 R_{12}, \quad R_{21}[u_1, R_{12} u_2] = u_2 R_{21} R_{12}.$$ 

**Corollary 4.3** Let $R \in M_n \otimes M_n$ be a bi-invertible solution of the QYBE and $\tilde{\mathcal{L}}(R)$ its associated matrix braided-Lie algebra. Then the associated canonical braiding from Theorem 2.1 is

$$\tilde{\mathcal{R}}(u^i_j \otimes u^l_k) = u^i_k \otimes u^l_j \tilde{\mathcal{R}}^j^l_K, \quad \tilde{\mathcal{R}}^j^l_K = R^{-1 \mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} R^{i_\mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} \tilde{\mathcal{R}}^{i_\mathfrak{a} \mathfrak{a}}_{i_\mathfrak{a} \mathfrak{a}} \quad \text{and} \quad R \in M_n \otimes M_n$$ necessarily obeys the QYBE.

**Proof** We compute the canonical braiding for the matrix braided-Lie algebra above. In fact, the necessary computation was done already in the proof of [4, Prop. 5.2] in the course of computing the relations of $U(\mathcal{L})$. We need only the matrix form of $\Delta$ and the formulæ for $\Psi = \mathcal{Y}, \mathcal{Y} = \mathcal{Y}$ in (15)-(17). Hence from Theorem 2.1 we conclude that $\tilde{\mathcal{R}}$ obeys the QYBE too. □

The braided enveloping algebra $U(\mathcal{L})$ for this class was computed and identified in [4, Prop. 5.2] as the braided-bialgebra of $B(R)$ of ‘braided matrices’ as introduced in [18]. This is the
associative algebra generated by 1 and \( u = \{ u^i_j \} \) with the braided-commutativity relations (10) which are now

\[
u_j u_L = u_K u_I R^I_J R^J_K, \quad \text{i.e.} \quad R_{21} u_1 R_{12} u_2 = u_2 R_{21} u_1 R_{12}
\]

where the second puts two of the \( R \)'s to the left and uses the matrix notation. The coproduct \( \Delta u = u \otimes u \) extends as an algebra homomorphism \( B(R) \to B(R) \otimes B(R) \) to the braided tensor product algebra determined by \( \Psi \), i.e. according to the axiom on the left in (11). Note that the motivation in [18] for \( B(R) \) was as a braided-version of quantum or super matrices, with braid statistics \( \Psi \), i.e. the generators are to be regarded as, by definition, the braided-commutative ring of co-ordinate functions on a braided space. Hence it is remarkable that this \( B(R) \) is also the enveloping algebra of a braided-Lie algebra. We obtain in the corollary a new and conceptual proof that the matrix \( \mathbf{R} \) that describes its relations indeed obeys the \text{QYBE}.

This class of examples generalises those of Section 4.1 for ordinary Lie algebras and Section 4.2 for super-Lie algebras, as well as including the case of Lie algebras defined in an obvious way relative to any background braiding where \( \Psi^2 = \text{id} \). The way to obtain these from the notion of braided-Lie algebras is explained in [4]. One uses \( \bar{\chi} = \frac{u - \text{id}}{h} \) along with 1 as generators of \( U(\mathcal{L}) \) in place of 1 and \( u \), where \( R \) is parametrised in such a way that \( R_{21} R = O(h) \). The standard \( \mathbf{R} \)-matrices associated to semisimple Lie algebras \( g \) in [25] then give deformations as braided-bialgebras of their homogenised enveloping algebras \( \widetilde{U}(g) \) from this point of view.

Unfortunately, the simplest non-trivial example of a matrix braided-Lie algebra has to be four-dimensional. We mention the standard one from [4], namely the braided-Lie algebra \( \mathcal{L} = gl_{2, \mathbb{Q}} \) with basis

\[
gl_{2, \mathbb{Q}} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \}, \quad t = q^{-1} a + q d, \quad x = \frac{b + c}{2}, \quad y = \frac{b - c}{2i}, \quad z = d - a
\]

if we work over \( \mathbb{C} \). The braided-Lie bracket is obtained from (17) with the standard \( sl_2 \) \( \mathbf{R} \)-matrix, and given explicitly in [4, Example 5.5]. The braided-enveloping algebra here in terms of the \( \chi \) variables is a deformation of \( U(gl_2) \) or from another point of view, of \( U(\mathfrak{sl}_2) \).

The canonical braiding \( \hat{\mathbf{R}} \) from Theorem 2.1 for this example is the braided-commutativity relations for the algebra of 2 \( \times \) 2 braided matrices[18] and with the braided-determinant

\[
\text{BDET} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \frac{q^2}{(q^2 + 1)^2} t^2 - q^2 x^2 - q^2 y^2 - \frac{(q^2 + 1) q^2}{2(q^2 + 1)^2} z^2 + \left( \frac{q^2 - 1}{q^2 + 1} \right)^2 q \frac{q^2 - 1}{2} z^2
\]
it can be viewed as a braided $q$-deformation of the algebra of functions on Minkowski space with its Lorentzian metric\cite{26}\cite{27}. These $t, x, y, z$ are the non-commutative spacetime co-ordinates. The algebra here also agrees with the proposal for $q$-Minkowski space based on spinors in the approach \cite{28}\cite{29}. We note that the FRT bialgebra $A(\mathbb{R})$ associated to this canonical braiding has a Hopf algebra quotient $SO_q(1, 3)$, the $q$-Lorentz group in the interpretation above. On the other hand, this is also closely related to the dual of the quantum double of $U_q(su_2)$. These points are described in detail elsewhere.

The braided-enveloping algebra here of $2 \times 2$ braided matrices is also isomorphic to a degenerate form of the 4-dimensional Sklyanin algebra as shown in \cite{11}, so the latter has the R-matrix form (18). More recently, some remarkable homological properties of braided-matrix algebras have been found in \cite{30}.

4.4 Finite Groups and Racks

A rack is a set $X$ and a map $X \times X \to X$ denoted $x \times y \mapsto \tau y$ obeying the ‘rack-identity’

$$(\tau y)(\tau z) = \tau(y z), \quad \forall x, y, z \in X.$$ 

One usually adds to this that the map $\tau(\ )$ is bijective for each $x$, but we do insist on this here. One may also have conventions in which the notation is $y^x$ rather than $\tau y$. Such objects have a long history and some applications in algebraic topology\cite{5}. It is easy to see that every rack provides an example of a braided-Lie algebra if we take as our category $\mathcal{C}$ as the category of sets, with tensor product provided by the direct product of sets, and with the usual permutation map as $\Psi$. This is a symmetric monoidal category rather than a truly braided one. We just take

$$[x, y] = \tau y, \quad \Delta x = x \times x$$

and note that the axiom (L1) in (8) then becomes the rack identity above, while the others are empty. At the level of sets the braiding from Theorem 2.1 is

$$\hat{\mathcal{R}}(x \times y) = [x, y] \times x$$

and recovers the braiding associated to a rack in [5].

For a $k$-linear setting over a field we let $\mathcal{L} = kX$, the vector space with basis $X$, and the above definitions extended $k$-linearly. So $\mathcal{L}$ is the coalgebra with basis $X$ and all basis elements
grouplike. Then we have a (trivially braided) braided-Lie algebra in the category of vector spaces and the canonical braiding from Theorem 2.1 is just

\[ \tilde{R}(x \otimes y) = [x, y] \otimes x. \]

The braided-enveloping algebra \( U(L) \) consists of the algebra generated by elements of \( X \) modulo the relations \( xy = [x, y]x \) for all \( x, y \). This is the bialgebra generated by the rack monoid. Here the rack monoid is the free monoid generated by symbols from \( x, y \) modulo such relations, and is also a classical construction for racks.

Our examples of matrix braided-Lie algebras in subsection 4.3 are a deformation of a mixture of some Lie-algebra like elements as in subsection 4.1 and some rack-like elements. In the \( gl_{2, q} \) example mentioned there, the rack-like element is proportional to the ‘time’ direction \( t \) and the Lie-algebra like elements are the ‘space’ directions \( x, y, z \).

We see that even when the braiding is trivial, the notion of a braided-Lie algebra in (8) is still useful. Moreover, it is more general than a rack because we are free to specify a more general coproduct \( X \to X \times X \) than the diagonal map, as long as we obey (L1)–(L3) in the category of sets. The simplest way to obey (1.2) is for \( \Delta \) to be cocommutative.

The classic example of a rack is a group with the rack operation \( [x, y] = xyx^{-1} \). The braiding \( \tilde{R} \) in the \( k \)-linear setting is then the braiding associated to the quantum double Hopf algebra \( D(X)[2] \). The latter is defined for \( X \) finite but the rack point of view is more general and works for any group. The associated 3-manifold invariants in this case are well-known, see for example [31]. On the other hand, we have super-racks, etc. just as easily as examples of braided-Lie algebras, and their associated braidings may prove more interesting.

### 4.5 Ordinary Hopf Algebras

If \( H \) is any Hopf algebra then it acts on itself by the Hopf-algebra adjoint action \( Ad_h(g) = \sum h_{(1)} g S h_{(2)} \), where we use the notation \( \Delta h = \sum h_{(1)} \otimes h_{(2)} \) of [32] for the coproduct. Theorem 3.1 reduces for ordinary Hopf algebras (with trivial braiding) to

\[ \tilde{R}(h \otimes g) = \sum Ad_{h_{(1)}}(g) \otimes h_{(2)}, \quad \forall h, g \in H. \quad (19) \]

One can easily verify in a couple of lines that \( \tilde{R} \) obeys the braid relations. This was perhaps first explicitly remarked in [21]. The case of \( H \) a group algebra clearly reduces us to the rack
braiding in subsection 4.4.

Once again, this braiding can be viewed as originating in Drinfeld’s quantum double construction $D(H)[2]$, this time applied to a general finite-dimensional Hopf algebra $H$. Drinfeld introduced $D(H)$ as a quasitriangular Hopf algebra defined by generators and relations in a basis. Here the quasitriangular structure $R \in D(H) \otimes D(H)$ obeys Drinfeld’s axioms which are such as to ensure that its image in any representation obeys the QYBE. We introduced a form of this in [33] built explicitly on the vector space $H^* \otimes H$ with product

$$(a \otimes h)(b \otimes g) = \sum \langle S(h_{(1)}, b_{(2)}) b_{(3)} a \otimes h_{(3)} g \langle h_{(4)}, b_{(5)} \rangle), \quad \forall a, b \in H^*, \; h, g \in H$$

and tensor product unit and coalgebra. In writing this we switch also to conventions with $H$ and $H^{op}$ (the opposite algebra) as sub-hopf algebras, rather than Drinfeld’s original conventions with $H, H^{op}$ (the opposite coalgebra). Let $\{e_a\}$ be a basis of $H$ and $\{f^a\}$ a dual basis then

$$R = \sum_a (f^a \otimes 1) \otimes (1 \otimes e_a)$$

is Drinfeld’s quasitriangular structure in these conventions.

**Proposition 4.4** $D(H)$ acts on $H$ by

$$(1 \otimes h) \triangleright g = \sum h_{(1)} g S h_{(2)}, \quad (a \otimes 1) \triangleright g = \sum \langle a, h_{(2)} \rangle h_{(3)}$$

and the associated braiding is (19).

**Proof** It is easy to see that this defines an action of $D(H)$ (this is modelled on quantum mechanics and could be called the ‘Schroedinger representation’ of the quantum double). Then the action of $R$ is $R \triangleright (h \otimes g) = \sum_a (f^a \otimes 1) \triangleright h \otimes (1 \otimes e_a) \triangleright g = \sum h_{(1)} \otimes (1 \otimes h_{(2)}) \triangleright g = \sum h_{(1)} \otimes \Delta h_{(2)} (g)$ giving exactly (19) for the corresponding $\hat{R}$. □

Thus the braiding from Theorem 3.1 does not give anything genuinely new for an ordinary Hopf algebra. It is slightly more general than the braiding coming from the quantum double in that it does not require $H$ to be finite dimensional, but this issue too can be dealt with in other ways[2]. On the other hand, it is still a useful observation, as is the fact which is obvious from (19) that $\cdot \circ \hat{R} = \cdot$ holds in $H$. See for example [34] where such an observation recently proved very useful for some Hopf algebraic constructions.
4.6 Super Hopf Algebras

To obtain something new from Theorem 3.1 we can consider Hopf algebras in categories other than the usual one of vector spaces. The simplest setting is that of $\mathbb{Z}_2$-graded or super Hopf algebras. These are defined in the obvious way with all maps degree-preserving, where the degree of a term in a tensor product is the sum of the degrees in a homogeneous decomposition. Plenty of super-Hopf algebras are known, not least in algebraic topology[35].

The braiding in this case is

$$\tilde{R}(h \otimes g) = \sum \text{Ad}_{h_{(1)}}(g) \otimes (-1)^{|h_{(1)}||h_{(2)}|} h_{(2)}; \quad \text{Ad}_k(g) = \sum h_{(1)} g S h_{(2)} (-1)^{|h_{(1)}||h_{(2)}|}$$

where it is assumed that all tensor product elements are decomposed homogeneously.

4.7 The Braided Line $k[x]$

The previous subsection is still not a truly braided example of Theorem 3.1. Truly braided-Hopf algebras were first introduced and studied by the author through a number of papers. In this subsection we compute Theorem 3.1 for the simplest of these[36], where the braiding is still a factor but not necessarily ±1 as it was in subsection 4.6.

The braided-line $k[x]$ as an algebra is nothing other than the polynomials in one variable. However, we regard it as an algebra in the category of $\mathbb{Z}$-graded vector spaces. As such, it has a braiding

$$\Psi(x^m \otimes x^n) = q^{mn} x^n \otimes x^m$$

where $q \neq 0, 1$ is a fixed but otherwise arbitrary element of $k$. The ideas here are from [36]. We define $\Delta x = x \otimes 1 + 1 \otimes x$, $\epsilon x = 0$ and extend to products as a braided-Hopf algebra according to (11). It is easy to see that

$$\Delta x^m = \sum_{r=0}^{m} \binom{m}{r}_q x^r \otimes x^{m-r}, \quad \binom{m}{r}_q = \frac{[m; q]!}{[r; q]! [m - r; q]!}, \quad [m; q] = \frac{q^m - 1}{q - 1}.$$

(21)

The $q$-integers and $q$-binomial coefficients here are well-known[37] but we use them in a novel way as defining a braided-Hopf algebra structure[13]. Using the lemma that $S$ is a braided-antialgebra map we have also that

$$S x^m = q^{\frac{m(m-1)}{2}} (-x)^m$$

(22)
to complete the braided-Hopf algebra structure. Finally, we note that this braided-Hopf algebra approach to $q$-analysis derives the standard $q$-derivative

$$(\partial_q f)(x) = \frac{f(qx) - f(x)}{x(q - 1)}, \quad \partial_q x^m = [m; q] x^{m-1}$$

as infinitesimal translations[13], a point of view which generalises at once to $n$-dimensional quantum plane algebras.

**Lemma 4.5** The braided adjoint action of $k[x]$ on itself as in Theorem 3.1 is

$$\text{Ad}_f(g)(x) = f(x^2(1-q)\partial_q)g(x), \quad \forall f, g \in k[x]$$

**Proof** We know from the general theory of braided-Hopf algebras[9] that the braided-adjoint action is an action and also that it acts on itself as a braided module-algebra. The latter condition means in the present case that it acts as a braided-derivation

$$\text{Ad}_x(fg) = \text{Ad}_x(f)g + \Psi(\text{Ad}_x \otimes f)g = \text{Ad}_x(f)g + L_q(f)\text{Ad}_x(g); \quad L_q(f)(x) = f(qx).$$

Since $\text{Ad}_x(x) = xx - qx = (1-q)x^2$ we deduce from this $q$-derivation property of the adjoint action that $\text{Ad}_x(x^n) = x^{n+1}(1-q^n) = x^2(1-q)\partial_q x^n$. Since $\text{Ad}_x$ is an action, we deduce the result stated. Explicitly, this has on monomials the form

$$\text{Ad}_x^n(x^n) = (1-q^n) \cdots (1-q^{n+m-1}) x^{n+m} = (1-q)^m \frac{[n+m-1; q]!}{[n-1; q]!} x^{n+m}.$$

$\square$

We note in passing that our derivation here depends strongly on the properties of $\text{Ad}$ proven in [12][18] using the same novel diagrammatic techniques as in Figure 2. If we try to compute it directly from (21)–(22) on monomials then we derive the novel $q$-identity

$$\sum_{r=0}^{m} \binom{m}{r} \frac{q^{(r-1)}(-1)^{r} q^{r} x^{r}}{[n-1; q]!} = (1-q)^m \frac{[n+m-1; q]!}{[n-1; q]!}.$$ 

This is the content of the lemma from the point of view of $q$-analysis.

**Corollary 4.6** The braiding $\check{\text{R}} : k[x] \otimes k[x] \to k[x] \otimes k[x]$ obtained from Theorem 3.1 is

$$\check{\text{R}}(x^m \otimes x^n) = \sum_{r=0}^{m} \binom{m}{r} \frac{q^{r} x^{r} (1-q)^{m-r} [n+r-1; q]!}{[n-1; q]!} x^{n+r} \otimes x^{n-r}.$$
Proof We compute from Theorem 3.1 in our case. Thus from the formula above for $\Delta$ we have
\[ \hat{R}(x^m \otimes x^n) = \sum_{r=0}^{\min(m,n)} [m; q] \text{Ad}_x(x^n) \otimes x^{m-r} y^{n-r} \]
and putting in the form of $\text{Ad}$ computed in Lemma 4.5 gives the result stated. ~

We now provide a braided-geometrical picture of this $\hat{R}$ as an operator on polynomials in two variables. Thus we distinguish the two copies of $k[x]$ in both the input and output, so that $\hat{R}: k[y] \otimes k[x] \rightarrow k[x] \otimes k[y]$ say. Next we consider these variables to be non-commuting with the quantum-plane relations, i.e.
\[ \hat{R} : k[x, y; q] \rightarrow k[x, y; q], \quad k[x, y; q] = \frac{k(x, y)}{yx - qxy}. \]
This is a purely notational device because this algebra has a basis $\{y^n x^m\}$ so as a linear space can be identified with $k[y] \otimes k[x]$, but also has a basis $\{x^m y^n\}$ and so can be identified with the linear space $k[x] \otimes k[y]$ as well.

**Proposition 4.7** $\hat{R}$ in the form $k[x, y; q] \rightarrow k[x, y; q]$ is the operator
\[ \hat{R} = e_q^{x(1-q)\partial_{x,y}} = \sum_{m=0}^{\infty} \frac{(x^2(1-q)\partial_{x,y})^m \partial_{x,y}^m}{[m; q]!} \]
where the $\mid$ denotes that the $q$-exponential is to be understood as ordered in the form shown and $\partial_{x,y}$ and $\partial_{x,y}$ are as in (23) acting on $x, y$ respectively.

**Proof** Note that the expression is a well-defined operator since on any polynomial the powerseries always terminates. The $q$-exponential is the standard one except that we have adopted the ordering convention stated. This is such that we have
\[ e_q^{x(1-q)\partial_{x,y}} f(y) = (\Delta f)(x, y) = f(x + y) \tag{24} \]
where we recall that $\Delta$ is an algebra homomorphism to the braided tensor product algebra which we identify as $k[x] \otimes k[y] = k[x, y; q]$ for the braiding (20). This (24) describes a ‘braided-Taylors theorem’ as explained in [13], where it is also generalised to $n$-dimensions. Applying the braided-adjoint representation $x \mapsto \text{Ad}_x = (1 - q)x^2 \partial_{x,y}$ to both sides of (24) allows us to recompute $\hat{R}$ from Theorem 3.1 as
\[ \tilde{R}(f(y)g(x)) = f(x^2(1-q)\partial_{x,y} + y)g(x) = e_q^{-x(1-q)\partial_{x,y}} f(y)g(x) \]

21
which is the form stated. Here $f, g$ are arbitrary polynomials and a general polynomial in $x, y$
can be written as a linear combination of such products. □

In this form, our Yang-Baxter operator $\hat{R}$ has some similarities with the quasitriangular
structure $\mathcal{R}$ of $U_q(sl_2)$. To see this we note that classically one can embed $sl_2$ inside the Witt
algebra of ‘vector fields’ on $k[x]$ by

$$L_0 = x \frac{d}{dx}, \quad L_1 = x^2 \frac{d}{dx}, \quad L_{-1} = \frac{d}{dx}$$

so $\hat{R}$ resembles the factor $q^{(1-\xi)L_1 \otimes L_{-1}}$ occurring in the formula for $\mathcal{R}$ in [38], cf[2] and elsewhere.

On the other hand, there is no Gaussian factor $q^{L_0 \otimes L_0}$ as to be found there.

If one works over $\mathbb{C}[[h]]$ rather than over a field and sets $q = e^h$ then $\hat{R}$ has a an expansion

$$\hat{R} = P \circ (\text{id} + hr + O(h^2)); \quad r = x \frac{d}{dx} \otimes x \frac{d}{dx} - \frac{d}{dx} \otimes x^2 \frac{d}{dx}$$

which $r$ is necessarily an operator realisation $k[x] \otimes k[x] \to k[x] \otimes k[x]$ of the Classical Yang-
Baxter equation (CYBE). Here $P$ is permutation. To obtain this formula one can work from
Corollary 4.6 or else from Proposition 4.7 provided one remembers the contribution (the first
term in $r$) coming from the fact that the output of $\hat{R}$ has to be viewed in $k[x] \otimes k[y]$ while its
input is viewed in $k[y] \otimes k[x]$. In this case $r$ is indeed the image in the Witt algebra of the
Drinfeld-Jimbo solution[2][3] of the CYBE on $sl_2$ when represented as vector fields on $k[x]$. The
connection with theory of quantum groups is provided by the ‘bosonisation theorem’ introduced
in [10]. In the present case this turns constructions on $k[x]$ into equivalent ones on the Hopf
algebra $U_q(b_+)$ in [2].

On the other hand, we have obtained this $\hat{R}$ starting from nothing other than $k[x]$ regarded
as a braided-Hopf algebra (the braided-line) and Theorem 3.1. Moreover, we can suppose that
$q \in k^\times$ is a root of unity and proceed with the same calculations. More precisely, we can repeat
the above calculations for the braided-Hopf algebra $U_n(k) = k[x]/x^n$ introduced in [36], where $q$
is a primitive $n$’th root of 1. This ‘anyonic line’ braided-Hopf algebra is $n$-dimensional and hence
$\hat{R}$ corresponds to a matrix solution of the QYBE in $M_n \otimes M_n$. An elementary computation gives for example,

Example 4.8 The braiding from Theorem 3.1 applied to the anyonic line for $n = 3$ has minimal polynomial

$$(\hat{R}^2 - 1)^2(\hat{R} - q) = 0$$
and corresponding matrix solution of the QYBE

\[ R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 - q & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q^2 & 0 & q - 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q
\end{pmatrix} \]

where \( q^3 = 1 \) is a primitive root and the basis is \( \{1 \otimes 1, 1 \otimes x, \ldots, x^2 \otimes x^2 \} \).

This braided-line which we have studied above is only the very simplest example of a braided-Hopf algebra. The next simplest is probably the quantum plane \( k[x, y; q] \) this time regarded itself as a braided-Hopf algebra with braiding provided by the standard \( sl_2 \) R-matrix corresponding to the Jones knot polynomial[39]. Explicitly,

\[ \Psi(x \otimes x) = q^2 x \otimes x, \Psi(x \otimes y) = qy \otimes x, \Psi(y \otimes y) = q^2 y \otimes y \]

\[ \Psi(y \otimes x) = qx \otimes y + (q^2 - 1)y \otimes x. \]

Using this, one obtains the braided adjoint action on generators as

\[ \text{Ad}_x(x) = x^2(1 - q^2), \text{Ad}_x(y) = (1 - q^2)xy, \text{Ad}_y(x) = yx(1 - q^2), \text{Ad}_y(y) = y^2(1 - q^2). \]

Extending this to higher products along the lines above, one obtains the braided-vector fields for the braided-adjoint action as

\[ \text{Ad}_x = (1 - q^2)(x \partial_{t,x} + y \partial_{t,y}), \quad \text{Ad}_y = (1 - q^2)(y \partial_{t,x} + y \partial_{t,y}) \]

where \( \partial_{t,x} \) and \( \partial_{t,y} \) are the partial derivatives on the quantum plane[40] in the form obtained by an infinitesimal translation in [13]. From this one can compute the braiding \( \hat{R} \) from Theorem 3.1 as represented by \( q \)-deformed vector fields on a quantum plane. We have not found an explicit exponential formula for it along the lines of Proposition 4.7.

This class of examples generalises further to any quantum plane algebra of the R-matrix type. The necessary braided-Hopf algebra structure, braided differential calculus and R-binomial theorem are in [13]. For example, the \( q \)-Minkowski space example with generators \( t, x, y, z \) fits into this setting with additive braided-Hopf algebra structure found in [27]. In these cases the
appropriate q-exponential map needed for the analogue of Proposition 4.7 is not yet known. In a
different direction, we can take the free braided-Hopf algebra $k(x_1, \ldots, x_n)$ with braiding deter-
determined by $R$ as a generalisation of the example $k[x]$ above. In this case the relevant exponential
map is provided in [13].

A Extended braid relations and braided cocommutativity

Here we give an application of the notion, introduced by the author in [20] of a braided-
cocommutative module of a braided-bialgebra. We do not need a braided-antipode in this
section. Given $B$ a bialgebra in a braided tensor category $\mathcal{C}$ as in Section 3, a $B$-module in the
category means $(V, \alpha_V)$ where $V$ is an object and $\alpha_V : B \otimes V \to V$ is a morphism obeying the
obvious notion of an action of $B$ on $V$. We say that $B$ is braided-cocommutative with respect to
$V$ (or simply that the module is braided cocommutative when the Hopf algebra is understood) if

$$
\begin{array}{c}
\Delta B & V \\
\Delta & \\
B & \alpha_V
\end{array}
\begin{array}{c}
\Delta B & V \\
\Delta & \\
B & \alpha_V
\end{array}
= (id \otimes \alpha_V) \circ \Delta : B \otimes V \to B \otimes V
$$

(25)

So for example, the braided-enveloping bialgebra $U(\mathcal{L})$ in Section 2 acts cocommutatively on
$\mathcal{L}$ by $[\ , \ ]$. The braided-Hopf algebras that arise by transmutation from quantum groups[20]
are likewise braided-cocommutative with respect to all the corresponding braided-modules that
come from modules of the quantum group. So this is a large class.

Now, for any $B$-module $(V, \alpha_V)$ we define the associated operator

$$
U_{B,V} = \begin{array}{c}
\Delta B \\
\Delta \\
B \otimes V
\end{array} = (id \otimes \alpha_V) \circ \Delta : B \otimes V \to B \otimes V
$$

Working with this is equivalent to working with our original module since we can recover the
latter as $(\epsilon \otimes id) \circ U_{B,V} = \alpha_V$. We have

**Theorem A.1** Let $(V, \alpha_V), (W, \alpha_W)$ be modules of a braided-bialgebra $B$, with $(W, \alpha_W)$ cocom-
mutative. Then the corresponding operators $U$ obeys the extended braid relations

$$
U_{B,V} \circ \Psi_{W,V} \circ U_{B,W} \circ \Psi_{V,W} = \Psi_{W,V} \circ U_{B,W} \circ \Psi_{V,W} \circ U_{B,V}
$$
Proof \ This is shown in our diagrammatic notation in Figure 3. The $\bigvee$ vertices are the module action $\alpha_V$ or $\alpha_W$ while the $\bigwedge$ vertices are the coproduct $\Delta$ throughout. We begin with the left hand side of the extended braid relations with $U$ of the form stated above and $\Psi = X$ as usual. The first equality is (25) for $W$. The second slides this group containing $\alpha_W$ over to the left using functoriality. The third likewise pushes the group containing $\alpha_V$ up to the top of the expression. The fourth is coassociativity of $\Delta$. The fifth pulls the $\alpha_W$ vertex down and to the right. The sixth now uses the cocommutativity condition (25) again for $W$, to obtain the right hand side of the extended braid relations as required. $\square$

We note that this proof used nothing more than coassociativity of the coproduct $\Delta$ and the cocommutativity axiom (25) as it was introduced in [20], being essentially equivalent to it when we demand it for all $V$. Another equivalent way to write the condition is that $\alpha_{V \otimes W} \cong \alpha_{W \otimes V}$ by the braiding $\Psi_{V,W}$, i.e. that the module $V$ is central in the category of $B$-modules, up to the trivial isomorphism provided by the background braiding. This is the key property of representations of ordinary groups (which are commutative up to ordinary transposition) and was the main motivation behind the theory of braided groups in [7]. See [9, Eqn. (64)] for the connection with physics from this point of view.
To describe a class of examples we observe that exactly the same definitions and theorem hold for a braided-commutative module of a braided-Lie algebra $\mathcal{L}$ (as defined in [4]) so it is not necessary to work with an entire braided-bialgebra here. In particular, $\mathcal{L}$ acts on itself by $\alpha_\mathcal{L} = [\ , \ ]$, the adjoint representation of the braided-Lie algebra. So we have

$$U_{12} \circ \Psi_{23} \circ U_{12} \circ \Psi_{23} = \Psi_{23} \circ U_{12} \circ \Psi_{23} \circ U_{12}$$

as operators on $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}$. Such matrix representations of the extended Artin braid group are useful in defining invariants of links in the complement of the trivial knot, as explored in [8]. The canonical extension to an action of $U(\mathcal{L})$ puts us into the setting of Theorem A.1.

Note also that one can of course turn all these diagram-proofs upside-down. Then we have another operator $U_{V,B}$ for every right $B$-comodule $V$. This time the upside-down (25) becomes the condition of $V$ a commutative right-comodule with respect to $B$, as studied in[6]. An example is provided by $B = B(R)$ as the bialgebra of braided matrices. It has a braided-coaction on the Zamolodchikov or quantum $\mathbb{R}$-plane algebra $V$ with generators $\{x_i\}$ and coaction $\beta(x_i) = x_j \otimes w_j^i$, which is braided-commutative because it comes from transmutation[12]. Hence we can apply Theorem A.1 in dual form. This gives another point of view on the braided-commutativity relations (18) of $B(R)$.

References


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