Yang-Lee-Fisher Zeros and Julia Sets*

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Abstract

The multifractal properties of the Yang-Lee-Fisher zeros of the Potts model on a hierarchical lattice are studied through their connection to the Julia set of the renormalization-group transformation.

Prologue

This year we are celebrating Professor Yang's seventieth birthday. However, the subject of my talk is more like a celebration of Professor Yang's thirtieth birthday. In 1952 Professor Yang published three seminal papers that became classics in the field of statistical mechanics: C. N. Yang, "The Spontaneous Magnetization of a Two-Dimensional Ising Model"\(^1\); C. N. Yang and T. D. Lee, "Statistical Theory of Equations of State and Phase Transitions. I. Theory of Condensation"\(^2\); T. D. Lee and C. N. Yang, "Statistical Theory of Equations of State and Phase Transitions. II. Lattice Gas and Ising Model"\(^2\). The first paper gave the celebrated calculation of the spontaneous magnetization of the Ising model. The second paper proposed a general theory of phase transitions by studying the distribution of the roots of the grand partition function, thenceforth known as "Yang-Lee zeros." The third paper applied the general theory to the lattice gas and the Ising model and obtained the famous "Lee-Yang circle theorem." Any one of these works could be the dream of a physicist's life-time accomplishment.

The theme of this conference is the interface between mathematics and physics. In the last decade there is one branch of mathematics that has contributed significantly to the progress of physics: dynamical systems theory. This beautiful and important branch of mathematics, despite the contribution of many physicists (notably Poincaré) to its upbringing, somehow escaped the general attention of physicists until quite recently. However, the whole subject of chaos, nonlinear dynamics, complexity, etc. is all related to, if not a direct descendent of, dynamical systems theory. On the other hand, phase transitions have been one of the hottest topics of research in physics in the last twenty years. It is also one area of physics in which the most important breakthrough has been made due to the renormalization-group theory. The concept of
phase transitions now permeates through not only all branches of physics but also other branches of science as well.

The work reported here was inspired by Professor Yang's work forty years ago and it lies in the interface between dynamical systems in mathematics and phase transitions in physics.

I. INTRODUCTION

In recent years there has been much interest in the study of phase transitions on hierarchical lattices. These lattices are iteratively constructed to be exactly self-similar (see Fig.1). On the theoretical side, since the Migdal-Kadanoff renormalization group is exact, they provide a class of models whose critical behavior can be precisely studied. On the practical side, since these models are highly inhomogeneous, they serve to give insights into the understanding of such physical systems as random magnets, polymers, percolation clusters, and superconducting networks.

On a hierarchical lattice, the exact renormalization-group recursion relation defines a rational mapping of the coupling constant. Associated with such a rational mapping in the complex plane is the Julia set. Recently a remarkable observation has been made on the Julia set of the renormalization-group mapping. It has been shown that this Julia set is simply the limiting set of the Yang-Lee-Fisher (YLF) zeros of the partition function. Since the discovery of the famous Lee-Yang circle theorem, very little is known about the distribution and structure of these zeros except for a few exactly solvable cases. In the case of hierarchical lattices, fairly complete
information about the YLF zeros has thus become available due to their connection to the Julia set.

In another line of development, it has been recognized that most of the fractals in nature are actually composed of an infinite set of interwoven subfractals, and hence the name “multifractal.” To provide a more complete characterization of the global scaling properties of multifractals, one has to use a spectrum of critical exponents and their singularities. Julia sets are multifractals. In this paper we will study the multifractal properties of the Julia set of the s-state Potts model on the diamond hierarchical lattice. In Section II the relation between the YLF zeros of the partition function and the Julia set of the renormalization-group recursion relation is reviewed. In Section III the thermodynamic formalism for the multifractal properties is recapitulated. In Section IV the singularity spectra and generalized dimensions of these Julia sets are calculated. In Section V a summary is given.

II. YANG-LEE ZEROS AND JULIA SETS

We will first review the beautiful work of Derrida et al., who have discovered the connection between the the YLF zeros of the partition function and the Julia set of the renormalization transformation.

The Hamiltonian of the s-state Potts model is

\[ H = -J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \delta(\sigma_i, \sigma_j) \]

(1)

where \( \delta \) is the Kronecker delta, and the sum is over nearest neighbors. The partition function is then given by
\[ Z = \sum \exp \left[ K \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) \right], \]  
\[ \text{(2)} \]

where \( K = \beta J \). For convenience, we use the variable \( z = e^K \).

To derive the recursion relation for the partition function we look at the first two levels of the construction of the diamond hierarchical lattice. Summing over the trace, we easily obtain the following recursion relation

\[ A(z)Z_1(z') = Z_2(z), \]  
\[ \text{(3)} \]

where

\[ A(z) = (2z + s - 2)^2, \]  
\[ \text{(4)} \]

\[ z' = f(z) = \left( \frac{z^2 + s - 1}{2z + s - 2} \right)^2. \]  
\[ \text{(5)} \]

This transformation is simply the recursion relation obtained by using the Migdal-Kadanoff renormalization group, which is exact here. The general recursion relation between the partition function at the \((n-1)\)th level and the \(n\)th level is

\[ Z_n(z) = Z_{n-1}(z')[A(z)]^{4n-2}. \]  
\[ \text{(6)} \]

There are \(4^{n-1}\) bonds at the \(n\)th level; therefore, the partition function is a polynomial of degree \(4^{n-1}\) in \(z\). Let \(z_i\) be the zeros of \(Z_n(z)\):
\[ Z_n(z) = s \prod_{i=1}^{4n-1} (z - z_i) \quad \text{(7)} \]

Now one may rewrite Eq. (6) in the form of Eq. (7)

\[ s \prod_{i=1}^{4n-1} (z - z_i) = s \prod_{i=1}^{4n-2} (z' - z'_i) (2z + s - 2)^{24n-2} \quad \text{(8)} \]

where \( z'_i \) denote the zeros of \( Z_{n-1}(z) \). By using Eq. (5), Eq. (8) can be rewritten as follows

\[ \prod_{i=1}^{4n-1} (z - z_i) = \prod_{i=1}^{4n-2} \left[ (z^2 + s - 1)^2 - z'_i (2z + s - 2)^2 \right] \quad \text{(9)} \]

Each factor on the right-hand side is a fourth degree polynomial

\[ P_i(z) = (z^2 + s - 1)^2 - z'_i (2z + s - 2)^2 \quad \text{(10)} \]

We notice that in Eq. (10) there is an equivalence between the zeros of \( P_i(z) \) and the preimages of \( z'_i \) under the mapping \( f \)

\[ P_i(z) = 0 \iff z'_i = f(z) \quad \text{(11)} \]

Therefore the four zeros of \( P_i(z) \) are just the four preimages of \( z_i \) under the mapping \( f \).

This property allows one to obtain the \( 4^{n-1} \) zeros \( z_i \) of \( Z_n(z) \) by finding the preimages of the \( 4^{n-2} \) zeros \( z'_i \) of \( Z_{n-1}(z) \). Therefore, step by step, one can get the zeros of \( Z_n(z) \) from the unique zero of \( Z_1(z) = s(z + s - 1), \) i.e. \( z = 1 - s \). As \( n \to \infty \), these preimages form exactly the Julia set of the transformation \( f \) in the complex \( z \)-plane.
The physical region of the set of zeros of the partition function \( Z_n(z) \) is the positive real \( z \)-axis. There will be no point in the set which lies in the positive real \( z \)-axis if \( n \) is finite. As \( n \) increases some points will move toward the positive real \( z \)-axis. Only when \( n \) is infinite will there be points reaching the physical region.

### III. THERMODYNAMIC FORMALISM

In the thermodynamic formalism the partition function is defined by

\[
G(q, \eta) = \sum_{i=1}^{N} \frac{p_i^q}{l_i^1},
\]

(12)

where \( l_i \) is the linear dimension of the \( i \)th piece and \( p_i \) its probability. The function \( \tau(q) \) can be determined from the implicit equation

\[
\Gamma(q, \tau(q)) = 1.
\]

(13)

Once \( \tau(q) \) is obtained, the generalized dimension \( D_q \) and the exponent \( \alpha(q) \) can be calculated

\[
D_q = \frac{\tau(q)}{q - 1},
\]

(14)

\[
a(q) = \frac{d\tau(q)}{dq}.
\]

(15)

The singularity spectrum \( f(\alpha) \) is then given by the Legendre transformation
\[
f(x) = q\alpha(q) - \tau(q) .
\] (16)

In the case where the Julia set is one-dimensional or quasi-one-dimensional, \( l_i \) is well defined. But in our case the Julia sets are not quasi-one-dimensional. Fig. 2 shows a typical Julia set of the Ising model \((s = 2)\) on the diamond hierarchical lattice. For such a Julia set the points are no longer well-ordered. The usual way of defining \( l_i \) seems difficult to apply here. To overcome this difficulty we will use instead the method of derivatives.\(^7\)

Consider a map \( f \) defined in the complex plane. We divide the set into \( N \) subsets, each containing only one point. We denote by \( l(z_i) \) the size of the \( i \)th subset, i.e. \( l(z_i) \) is the size of the piece containing the point \( z_i \). Similarly, \( l(z_{i-1}) \) is the length for \( z_{i-1} \), where \( z_{i-1} \) is the backward iterate of \( z_i \), \( z_{i-1} = f(z_i) \). The scaling function \( \sigma(z_i) \)

\[
s(z_i) = \frac{l(z_i)}{l(z_{i-1})} .
\] (17)

can be approximated by

\[
\sigma(z_i) = \frac{1}{|f'(z_i)|} .
\] (18)

Since

\[
l(z_i) = |f'(z_i)|^{-1} l(z_{i-1}) \]

continuing the backward iteration gives
\[ I(z_i) = |f'(z_i)|^{-1} |f'(z_{i-1})|^{-1} ... |f'(z_0)|^{-1} I_0 \]

\[ = |f'(z_i)|^{-1} |f'(f(z_i))|^{-1} |f'(f^2(z_i))|^{-1} ... |f'(f^n(z_i))|^{-1} I_0. \] \hfill (20)

Backward iteration is a method that renders unstable fixed points stable. Here \( z_0 \) is the fixed point of \( z_i \). \( z_0 = f^n(z_i) \). \( I_0 \) is of order one, and therefore

\[ I_i = |f'(z_i)|f'(f(z_i)) ... f'(z_0)|^{-1} \] \hfill (21)

To test the validity of this method, we have applied it to calculate \( f(\alpha) \) and \( D_q \) for the maps \( f(z) = z^n + c \) (\( n > 2 \)). The results are the same as those obtained by the direct method.

**IV. MULTIFRACTAL PROPERTIES**

Since the complex rational map Eq. (5) is of power four, there are four fixed points given by the equation

\[ z^* = \left( \frac{z^{*2} + s - 1}{2z^* + s - 2} \right)^2. \] \hfill (22)

It is easy to see that there is always a trivial fixed point for any \( s \) value:

\[ z^* = 1. \] Excluding this trivial fixed point from Eq. (22), we get a cubic equation

\[ z^3 - 3z^2 + (3 - 2s)z - (s - 1)^2 = 0. \] \hfill (23)
As observed by Itzykson and Luck, Eq. (23) has three real roots corresponding to two unstable fixed points and one stable fixed point for

\[
\frac{s^4 - 8s^3}{4 \cdot 27} < 0,
\]

i.e. \( 0 < s < s_0 \), where \( s_0 = 32/27 \). If

\[
\frac{s^4 - 8s^3}{4 \cdot 27} > 0,
\]

i.e. \( s > s_0 \) or \( s < 0 \), there will be only one fixed point on the real axis, the other two fixed points being complex conjugates. For \( s > s_0 \), there are three unstable fixed points, and only one is real; whereas for \( s < s_0 \), there are two unstable fixed points, both of which are real. As a consequence \( f(\alpha) \) and \( D_Q \) behave quite differently.

For \( s > s_0 \), the Julia sets become denser as \( s \) decreases; for \( s < s_0 \), it is just the opposite. This feature is best reflected in the Hausdorff dimension \( D_0 \). \( D_0 = 1 \) at \( s = 0 \). It increases to its maximum value at \( s = s_0 \), then decreases again to its minimum value 1 as \( s \to \infty \). Except for the limiting cases \( s = 0 \) and \( s \to \infty \), the Hausdorff dimensions of the Julia sets are all greater than one. This is expected because these Julia sets are not quasi-one-dimensional but planar. For \( s > 3 \), there is only one point, \( z_F > 1 \) on the positive real axis. This corresponds to a ferromagnetic transition. For \( s < 3 \), another point \( 0 < z_A < 1 \) appears on the real axis. This corresponds to an antiferromagnetic transition. Moreover, when \( s < 2 \) additional points appear in the interval \( 0 < z < 1 \) on the real axis. We do not know if there is any physical significance that can be attributed to these points.
The singularity spectrum $f(\alpha)$ and the generalized dimension $D_q$ are calculated for $s$ between 0.5 and 5.0. See Fig. 3 for an example. For $s > s_0$, both $f(\alpha)$ and $D_q$ change continuously with $s$. As $s$ decreases, the $f(\alpha)$-curve moves to the right and becomes higher while $D_q$ moves up. However, as $s$ passes through $s_0$, this trend ceases. $f(\alpha)$ for $s = 1.0$ and 0.5 is shorter and bigger than that for $s = 1.5$.

Correspondingly, $D_q$ becomes a higher kink. For $s < s_0$, as $s$ decreases, $f(\alpha)$ becomes shorter and stretches along the real axis. $D_q$ at the same time becomes a higher kink. Eventually when $s \to 0$, $f(\alpha)$ will extend to the whole semi-infinite real axis, i.e. $\alpha_{\text{min}} \to 0$, $\alpha_{\text{max}} \to \infty$, and $D_q$ will be an infinitely high kink.

For $s > s_0$, $f(\alpha)$ and $D_q$ also differ for $s > 3$ and $s < 3$. When $s > 3$, among the three repelling fixed points, the point of the type $(a,0)$ is located at the most rarefied point (corresponding to $\alpha_{\text{max}}$) whereas the other two conjugate points of types $(a,b)$ and $(a,-b)$ are located at the most concentrated points (corresponding to $\alpha_{\text{min}}$). Because of these special fixed points the $f(\alpha)$-curves, no matter how low the level of calculation is, have the exact $\alpha_{\text{min}}$ and $\alpha_{\text{max}}$. Therefore the $f(\alpha)$- and $D_q$-curves converge very quickly. The $f(\alpha)$ curves for $s = 4.5$ at the low levels 5 and 4 almost coincide. But as we go below $s = 3$, none of these repelling fixed points correspond to $\alpha_{\text{min}}$ or $\alpha_{\text{max}}$, and the rate of convergence for both $f(\alpha)$ and $D_q$ becomes very slow and irregular. The convergence of $f(\alpha)$ for $s = 1.5$ at the higher levels 7 and 6 is much worse than that for $s = 4.5$. The gap between $f_{\text{min}}$ and zero in this case indicates that one needs to go to higher levels to obtain better convergence.
V. SUMMARY

We have given a brief account of the global scaling properties of the Julia sets of the YLF zeros of the s-state Potts model on the diamond hierarchical lattice. Because of limitation of space, we will refer the interested reader to Ref. 8 for a more detailed treatment. The singularity spectrum $f(\alpha)$ and the generalized dimension $D_q$ are calculated for various $s$ values. We divided $s$ into three groups: $s < s_0$, $s_0 < s < s_1$, and $s > s_1$, where $s_0 = 32/27$ and $s_1 = 16$. For $s > 3$, there is only a ferromagnetic transition point; for $s < 3$, another antiferromagnetic transition point appears. Moreover, for $s < 2^s$, there are additional points that appear on the positive real axis. We are however not sure about the physical significance of these additional points. Nevertheless, one should not conclude prematurely that these points, or, for this matter, even points away from the real axis, are neither interesting nor physically relevant. The most important remaining question is of course what direct physical meaning one can attribute to the singularity spectrum of the Julia set of the YLF zeros.

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References


Figure Captions

Figure 1 Diamond hierarchical lattice
Figure 2 Julia set for the Ising model (s=2).
Figure 3 (a) Singularity spectrum f(α) and, (b) generalized dimension D_q for the s=1.5, 2 and 2.5 state Potts model.
Fig. 3(a)