RACAH SUM RULE AND
BIEDENHARN-ELLIOTT IDENTITY
FOR THE SUPER-ROTATION 6 – j SYMBOLS.

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Abstract

It is shown that the well known Racah sum rule and Biedenharn-Elliott identity satisfied by the recoupling coefficients or by the 6 – j symbols of the usual rotation $SO(3)$ algebra can be extended to the corresponding features of the super-rotation $osp(1|2)$ superalgebra. The structure of the sum rules is completely similar in both cases, the only difference concerns the signs which are more involved in the super-rotation case.

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I Introduction

The super-rotation algebra also called graded $su(2)$ has been investigated first by Pais and Rittenberg [1]. They have built explicitly the finite dimensional representations of this algebra and have shown that the irreducible representations are characterized by a superspin $j$, which takes integer or half integer values, and by the parity $\lambda$ which takes the values 0 or 1. Later on, Scheunert, Nahm and Rittenberg [2] have introduced the concept of grade star representation and have computed the Clebsch-Gordan coefficients that reduce the product of two super-rotation representations. It turns out that these super-rotation Clebsch-Gordan coefficients are products of the usual rotation Clebsch-Gordan coefficients, which are independent of the parities, and a scalar factor that depends both on the superspins and the parities of the representations. Berezin and Tolstoy [3] have shown that these scalar factors form a pseudo-orthogonal matrix and have derived the explicit matrix elements of the $j$ representation in standard basis.

Recently, Minnaert and Mozzrymas [4] have shown that in order to reduce the tensor product of two arbitrary representations one must introduce the most general Hermitian form in the representation space. The Racah-Wigner calculus for the super-rotation algebra was then developed with a definition of the super-rotation $S3 - (j, \lambda)$ and $S6 - (j, \lambda)$ symbols and an analysis of their properties. In subsequent papers [5,6], Daumens, Minnaert, Mozzrymas and Toshev have defined $\lambda$-parity independent super $S3 - j$ and $S6 - j$ symbols and studied their symmetries.

It is well known from the usual rotation case [7] that the study of the $6 - j$ symbols is interesting for several reasons: (i) since the $6 - j$ symbols are independent of the projection quantum numbers they are basis independent objects; (ii) the $3 - j$ symbols (or, equivalently, the Clebsch-Gordan coefficients) can be obtained as a limit of the $6 - j$ symbols; (iii) the $6 - j$ symbols are proportional to the recoupling coefficients for the addition of three angular momenta and constitute a “basis” for all recoupling coefficients in the addition of an arbitrary number of angular momenta. In some sense, it is the $6 - j$ symbols that are the basic objects of the angular momentum theory, since all other entities can be derived from them. In turn, the rotation $6 - j$ symbols are determined up to a phase convention by the orthogonality relations, the Racah sum rule, the Biedenharn-Elliott identity and their symmetry properties.

Therefore, carrying on the development of the Racah-Wigner calculus for the super-rotation algebra, we derive in the present paper the fundamental relations satisfied by the super-rotation $S6 - j$ symbols, namely the Racah sum rule and the Biedenharn-Elliott identity. It will be shown that, similarly to the (pseudo-) orthogonality relations already discussed in Refs. [4,5], the Racah sum rule and the Biedenharn-Elliott identity for the $S6 - j$ symbols possess the same structure as the corresponding relations for the rotation algebra apart from the signs which are more involved in the super-rotation case, see e.g. eqs. (3.6)-(3.7) and (4.3)-
II The Super-Rotation Algebra

The super-rotation algebra defined by Pais and Rittenberg [1] has the following (anti)commutation relations

\[ [J_i, J_j] = i \epsilon_{ijk} J_k, \]
\[ [J_i, J_\alpha] = \frac{1}{2} J_\beta (\sigma_i)_\alpha^\beta, \]
\[ \{J_\alpha, J_\beta\} = \frac{1}{2} J_i (\Gamma \sigma_i)_{\alpha\beta}, \]

where \( i, j, k = 1, 2, 3 \), \( \alpha, \beta = \pm \frac{1}{2} \), \( \sigma_i \) are the Pauli matrices and \( \Gamma = i \sigma_2 \). This superalgebra with complex structure constants is related to the orthosymplectic superalgebra \( osp(1|2) \) [2-5].

An irreducible representation of this algebra [1-4] is characterized by the superspin \( j(j = 0, 1/2, 1, \ldots) \) and the parity \( \lambda (\lambda = 0, 1) \) which is the Grassmann degree of the highest weight vector in the graded representation space. The superspin \( j \) multiplet contains two rotation multiplets with spins \( \ell = j \) and \( \ell = j - 1/2 \). Its dimension is 1 if \( j = 0 \) and 4\( j + 1 \) if \( j \neq 0 \). The standard basis in the space \( V(j, \lambda) \) is denoted by \( | j, \ell, m \rangle \).

In order to reduce the tensor product of two arbitrary representations, it is necessary to consider the most general pseudo-Hermitian form in \( V(j, \lambda) \) [4]

\[ \Phi_{\varphi \psi}^{i\lambda}(j, \lambda; \ell, m | j, \lambda; \ell', m') = (-1)^{k \varphi + \psi} \delta_{\ell \ell'} \delta_{mm'}, \]

where \( \varphi \) and \( \psi \) are two binary variables that can take the values 0 or 1 (mod 2) such that \( \lambda + \varphi + \epsilon = 1 \) (mod 2). The representation space \( V(j, \lambda) \) in which acts an irreducible representation of class \( \epsilon \) and equipped with the bilinear Hermitian form \( \Phi_{\varphi \psi}^{i\lambda} \) will be denoted by \( \mathcal{H}_{\varphi \psi}^{i\lambda} \). If the bilinear Hermitian form \( \Phi \) in the tensor product space \( \mathcal{H}_{\psi_1 \psi_1}^{i\lambda_1} \otimes \mathcal{H}_{\psi_2 \psi_2}^{i\lambda_2} \) is defined by

\[ \Phi(x_1 \otimes x_2, y_1 \otimes y_2) = (-1)^{\alpha(x_2) \alpha(y_1)} \Phi_{\varphi_1 \psi_1}^{i\lambda_1}(x_1, y_1) \Phi_{\varphi_2 \psi_2}^{i\lambda_2}(x_2, y_2), \]

then [2] the tensor product of grade star representations of class \( \epsilon \) is also a grade star representation of same class and it is fully reducible ( \( \alpha(x_2) \) and \( \alpha(y_1) \) are the Grassmann degrees of the vectors \( x_2 \) and \( y_1 \), respectively ). The reduction formula for the tensor product of the spaces reads

\[ \mathcal{H}_{\psi_1 \psi_1}^{i\lambda_1} \otimes \mathcal{H}_{\psi_2 \psi_2}^{i\lambda_2} = \bigoplus_{j_{12}} \mathcal{H}_{\varphi_1 \psi_1}^{i\lambda_1} \otimes \mathcal{H}_{\varphi_2 \psi_2}^{i\lambda_2}. \]

The main difference with the usual rotation case resides in the fact that the possible values of \( j_{12} \) in the direct sum are

\[ j_{12} = | j_1 - j_2 |, | j_1 - j_2 | + \frac{1}{2}, \ldots, j_1 + j_2 - \frac{1}{2}, j_1 + j_2. \]
i.e., they vary with a step of $1/2$ and not 1. As a consequence, the sum of the three superspins, $j_1, j_2, j_{12}$ is not necessarily an integer, it can be half-integer as well.

The basis vectors in the space $\mathcal{H}_{j_1 j_2 j_{12}}^{j_1 j_2}$ are denoted by $| (j_1 \lambda_1; j_2 \lambda_2) j_{12} \lambda_{12}; \ell_{12} m_{12} \rangle$. They are related to the tensor product basis by the super-rotation Clebsch-Gordan coefficients (SRCG)

$$| (j_1 \lambda_1; j_2 \lambda_2) j_{12} \lambda_{12}; \ell_{12} m_{12} \rangle = \sum_{\ell_1, \ell_2, m_{12}} | j_1 \lambda_1; \ell_1 m_1 \rangle \otimes | j_2 \lambda_2; \ell_2 m_2 \rangle | j_1 \lambda_1; j_2 \lambda_2; \ell_1 m_1; j_2 \lambda_2; \ell_2 m_2; j_{12} \lambda_{12}; \ell_{12} m_{12} \rangle (2.8)$$

The SRCG have been computed in [2,3] where it was shown that they can be factorized into two factors

$$(j_1 \lambda_1; j_2 \lambda_2; \ell_1 m_1; j_2 \lambda_2; \ell_2 m_2; j_{12} \lambda_{12}; \ell_{12} m_{12}) = \left( \begin{array}{cc} j_1 \lambda_1 & j_2 \lambda_2 \\ \ell_1 & \ell_2 \end{array} \right) (\ell_1 \ell_2) (\ell_1 m_1 \ell_2 m_2; \ell_{12} m_{12}) (2.9)$$

The second factor is the usual rotation Clebsch-Gordan coefficient that is independent of the parities whilst the first factor, called the scalar factor because it is independent of the magnetic quantum numbers, depends on the parities of the representations. The main point, stressed in [4], is that the scalar factors satisfy pseudo-orthogonality relations which are easily understandable in terms of pseudo-Hermitian forms that satisfy relations (2.5). Of course, since the usual rotation Clebsch-Gordan coefficients are orthonormalized, the SRCG satisfy pseudo-orthogonality relations similar to those of the scalar factors [4,5].

From eq.(2.8), using (2.5) and the symmetry properties of the super-rotation Clebsch-Gordan coefficients derived in [4]:

$$(j_1 \lambda_1; j_2 \lambda_2; \ell_1 m_1; j_2 \lambda_2; \ell_2 m_2; j_{12} \lambda_{12}; \ell_{12} m_{12}) = (-1)^{\ell_1 + \ell_2 + (\lambda_1 + \lambda_2 + \ell_1 + \ell_2)} (j_2 \lambda_2; \ell_2 m_2; j_1 \lambda_1; \ell_1 m_1; j_{12} \lambda_{12}; \ell_{12} m_{12})$$

we obtain the following useful formula concerning the symmetry properties of the basis vectors in $\mathcal{H}_{j_1 j_2 j_{12}}^{j_1 j_2}$:

$$| (j_1 \lambda_1; j_2 \lambda_2) j_{12} \lambda_{12}; \ell_{12} m_{12} \rangle = (-1)^{(\lambda_1 + \ell_1 + j_{12}) + [j_1 + j_2 - j_{12}]} | (j_2 \lambda_2; j_1 \lambda_1) j_{12} \lambda_{12}; \ell_{12} m_{12} \rangle. (2.11)$$

where $[j_1 + j_2 - j_{12}]$ is the integer part of $j_1 + j_2 - j_{12}$.

Let us consider the reduction of the tensor product of three representations $(j_i, \lambda_i), i = 1, 2, 3$. This reduction can be done in two different ways. Either, one couples first the representations $(j_1, \lambda_1)$ and $(j_2, \lambda_2)$ and then the result $(j_{12}, \lambda_{12})$ is coupled to the representation $(j_3, \lambda_3)$ in order to give as a final result the representation $(j, \lambda)$. Or, one couples $(j_1, \lambda_1)$ with the result $(j_{23}, \lambda_{23})$
of the coupling of the representations \((j_2, \lambda_2)\) and \((j_3, \lambda_3)\) in order to yield \((j, \lambda)\). The corresponding basis vectors denoted by \(|(j_1 \lambda_1; j_2 \lambda_2)_{j_12} j_3 \lambda_3 j_3 \lambda \ell m\rangle\) and \(|(j_1 \lambda_1; j_2 \lambda_2; j_3 \lambda_3)_{j_123} j_3 \lambda \ell m\rangle\) are related by the so called super-rotation re-coupling coefficients \([4]\)

\[
| (j_1 \lambda_1; (j_2 \lambda_2; j_3 \lambda_3)_{j_23} \lambda_23) j \lambda \ell m \rangle = \sum_{j_12} | (j_1 \lambda_1; (j_2 \lambda_2; j_3 \lambda_3)_{j_23} \lambda_23) j \lambda | ((j_1 \lambda_1; j_2 \lambda_2)_{j_12} j_3 \lambda_3 \lambda) j \lambda \rangle \times | ((j_1 \lambda_1; j_2 \lambda_2)_{j_12} j_3 \lambda \lambda \lambda) \ell m \rangle,
\]

which are equal, up to a sign, to the parity dependent super-rotation \(S6 - (j, \lambda)\) symbols

\[
| (j_1 \lambda_1; (j_2 \lambda_2; j_3 \lambda_3)_{j_23} \lambda_23) j \lambda | ((j_1 \lambda_1; j_2 \lambda_2)_{j_12} j_3 \lambda_3 \lambda) j \lambda \rangle = (-1)^{(I_{23}+1)I_{23} + I_{12} \lambda_3 + I_{23} \lambda_3 + [j_1 + j_2 + j_3 + j_3]} \sum_{j_12} \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_3 \\ j \end{array} \right\}^S \left\{ \begin{array}{c} j_1 \lambda_1 \\ j_2 \lambda_2 \\ j_3 \lambda_3 \\ j \lambda \end{array} \right\} \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_3 \\ j \end{array} \right\}^S.
\]

We use the notation \(I_{i, j, k} = 2(j_i + j_j + \ldots + j_k + j_{i, j, k}) \mod 2\). Note that sometimes we use \(j\) instead of \(j_{123}\) or \(j_{1234}\) but the meaning is always obvious from the context.

Finally, the parity independent super-rotation \(S6 - j\) symbols \([5]\) are defined by

\[
\left\{ \begin{array}{c} j_1 \\ j_2 \\ j_12 \\ j \end{array} \right\}^S = (-1)^{\Psi(\lambda_1, \lambda_2, \lambda_3)} \left\{ \begin{array}{c} j_1 \lambda_1 \\ j_2 \lambda_2 \\ j_3 \lambda_3 \\ j \lambda \end{array} \right\}
\]

(2.14)

where the phase \(\Psi\) depends on three independent parities \(\lambda_1, \lambda_2, \lambda_3\) in the following way

\[
\Psi(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1 + 2j_1)I_{23} + (\lambda_2 + 2j_2)(I_{23} + I_{12} + I_{23}) + (\lambda_3 + 2j_3)I_{12},
\]

(2.15)

The super-rotation \(S6 - j\) symbols satisfy the pseudo-orthogonality relations

\[
\sum_{j_{12}} (-1)^{I_{12}+1} \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_12 \\ j \end{array} \right\}^S \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_12 \\ j \end{array} \right\}^S = (-1)^{I_{123}+1} \delta_{j_23} j_23.
\]

(2.16)

Using the analytical formulae \([5]\) for the \(S6 - j\) symbols where one of the superspins is equal to \(\frac{1}{2}\) we obtain a simple corollary of the pseudo-orthogonality relations (2.16) which almost coincide with the corresponding equation in the rotation case

\[
\sum_{j_{12}} (-1)^{I_{12}+1} \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_{12} \\ j \end{array} \right\}^S \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_{12} \\ j \end{array} \right\}^S = \delta_{j_{12} j_{12}}.
\]

(2.17)
III Racah Sum Rule

The simplest way of deriving the Racah sum rule and the Biedenharn-Elliott identity is to use the recoupling theory of, respectively, three and four superspins, the calculations consisting in repetitive applications of eqs. (2.11) and (2.12).

In the case of recoupling of three superspins $j_1$, $j_2$ and $j_3$ one can present the transformation (2.12) (using eq. (2.11) and summing over the intermediate states containing $j_{13}$) as a product of two successive recouplings

$$| (j_1 \lambda_1; (j_2 \lambda_2; j_3 \lambda_3) j_{23} \lambda_{23}) j \lambda; \ell m \rangle$$

$$= (-1)((\lambda_2 + I_{23})(\lambda_3 + I_{23}) + [j_2 + j_3 - j_{23}]) \sum_{j_{13}} (-1)^{\Theta_{RC}^R} \left| (j_{11} \lambda_1; (j_3 \lambda_3; j_2 \lambda_2) j_{23} \lambda_{23}) j \lambda; \ell m \right\rangle$$

$$= (-1)^{(\lambda_2 + I_{23})(\lambda_3 + I_{23}) + [j_2 + j_3 - j_{23}] \sum_{j_{13}} (-1)^{(\lambda_1 + I_{12})(\lambda_2 + I_{23}) + [j_1 + j_2 - j_{12}]}$$

$$\times \left| (j_1 \lambda_1; (j_3 \lambda_3; j_2 \lambda_2) j_{23} \lambda_{23}) j \lambda; \ell m \right\rangle$$

$$\times \sum_{j_{12}} (-1)^{(\lambda_1 + I_{12})(\lambda_2 + I_{23}) + [j_1 + j_2 - j_{12}]}$$

$$\times \left| (j_2 \lambda_2; (j_1 \lambda_1; j_3 \lambda_3) j_{12} \lambda_{12}) j \lambda; \ell m \right\rangle$$

$$\times \left| (j_1 \lambda_1; (j_2 \lambda_2; j_3 \lambda_3) j_{12} \lambda_{12}) j \lambda; \ell m \right\rangle$$

which gives us the Racah sum rule for the super-rotation recoupling coefficients in the form

$$[ (j_1 \lambda_1; (j_2 \lambda_2; j_3 \lambda_3) j_{23} \lambda_{23}) j \lambda; ((j_1 \lambda_1; j_2 \lambda_2) j_{12} \lambda_{12}; j_3 \lambda_3) j \lambda]$$

$$= \sum_{j_{13}} (-1)^{\Theta_{RC}^R} \left[ (j_{11} \lambda_1; (j_3 \lambda_3; j_2 \lambda_2) j_{23} \lambda_{23}) j \lambda; ((j_1 \lambda_1; j_3 \lambda_3) j_{13} \lambda_{13}; j_2 \lambda_2) j \lambda \right\rangle$$

$$\times \left[ (j_2 \lambda_2; (j_1 \lambda_1; j_3 \lambda_3) j_{13} \lambda_{13}) j \lambda; ((j_2 \lambda_2; j_1 \lambda_1) j_{12} \lambda_{12}; j_3 \lambda_3) j \lambda \right\rangle.$$  \hspace{1cm} (3.2)

The phase $\Theta_{RC}^R$

$$\Theta_{RC}^R = (\lambda_2 + I_{23})(\lambda_3 + I_{23}) + [j_2 + j_3 - j_{23}]$$

$$+ (\lambda_2 + I_{23})(\lambda_3 + I_{23}) + [j_2 + j_3 - j_{23}] + (\lambda_1 + I_{12})(\lambda_2 + I_{12}) + [j_1 + j_2 - j_{12}]$$

$$= \lambda_1(I_{12} + I_{13} + I_{23}) + \lambda_2(I_{12} + I_{23} + I_{12}) + \lambda_3(I_{13} + I_{23} + I_{12})$$

$$+ [j_1 + j_2 - j_{12}] + [j_2 + j_3 - j_{23}] + [j_2 + j_3 - j_{23}] + I_{12} + I_{23} + I_{12} + I_{13} I_{13}.$$  \hspace{1cm} (3.3)

takes the values 0 or 1 (mod 2). In terms of the parity-dependent $S6 - (j, \lambda)$ symbols eq. (3.2) reads

$$\left\{ \begin{array}{ccc} j_1 \lambda_1 & j_2 \lambda_2 & j_3 \lambda_3 \\ j_2 \lambda_2 & j_3 \lambda_3 & j_{12} \lambda_{12} \end{array} \right\} = \sum_{j_{13}} (-1)^{\Theta_{RC}^R} \left\{ \begin{array}{ccc} j_1 \lambda_1 & j_2 \lambda_2 & j_3 \lambda_3 \\ j_1 \lambda_1 & j_3 \lambda_3 & j_{13} \lambda_{13} \end{array} \right\} \left\{ \begin{array}{ccc} j_2 \lambda_2 & j_3 \lambda_3 & j_{12} \lambda_{12} \\ j_3 \lambda_3 & j_{13} \lambda_{13} \\ j_{13} \lambda_{13} \end{array} \right\}.$$  \hspace{1cm} (3.4)
where

\[ \Theta_{\lambda}^R = \Theta_{\lambda RC} + (I_{123} + 1)I_{13} + [j_1 + j_2 + j_3 + j] \]
\[ = \lambda_1(I_{12} + I_{13} + I_{123}) + \lambda_2(I_{12} + I_{23} + I_{123}) + \lambda_3(I_{13} + I_{23} + I_{123}) + I_{12} + I_{23} + I_{13} \]
\[ + I_{123} + [j_1 + j_2 - j_2] + [j_2 + j_3 - j_3] + [j_2 + j_{13} - j] + [j_1 + j_2 + j_3 + j] \] (3.5)

From (3.4) using eq. (2.14) we obtain the Racah sum rule for the super-rotation $S6 - j$ symbols

\[ \left\{ \begin{array}{ccc} j_1 & j_2 & j_12 \\ j_3 & j & j_{23} \end{array} \right\}^S = \sum_{j_{13}} (-1)^{\Theta^R} \left\{ \begin{array}{ccc} j_1 & j_3 & j_{13} \\ j_2 & j & j_{23} \end{array} \right\}^S \left\{ \begin{array}{ccc} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{array} \right\}^S. \] (3.6)

where

\[ \Theta^R = [j_1 + j_2 - j_12] + [j_2 + j_3 - j_23] + [j_2 + j_{13} - j] + [j_1 + j_2 + j_3 + j] \]
\[ + 2j_{12}I_{12} + 2j_{23}I_{23} + 2j_{13}I_{13} + 2jI_{123}. \] (3.7)

The sign factor in this sum rule is more involved than the corresponding factor in the ordinary rotation case. However, they almost coincide in several particular cases. For example, when the values of the super-spins in the left-hand side of eq. (3.6) are such that their sums in all the triangles $(j_1, j_2, j_{12})$, $(j_2, j_3, j_{23})$, $(j_1, j_{23}, j)$ and $(j_{12}, j_3, j)$ are integer, Racah sum rule takes the form

\[ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}^S = \sum_{j_{13}} (-1)^{\Theta_{j_{12}j_{23}j_{13}} + 2j_{13}I_{13}} \left\{ \begin{array}{ccc} j_1 & j_3 & j_{13} \\ j_2 & j & j_{23} \end{array} \right\}^S \left\{ \begin{array}{ccc} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{array} \right\}^S. \] (3.8)

which means that the terms in the right-hand side, for which the triangle sums with the participation of $j_{13}$ are also integer, have the sign factor $(-1)^{j_{12} + j_{23} + j_{13}}$, identical to the sign factor in the rotation case.

Another particular case of Racah sum rule reads

\[ \sum_{j_{12}} (-1)^{2(2(j_1 + j_2)(j_{12} + j_{12}) + j_{12})} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_1 & j_2 & j_{12} \end{array} \right\}^S = 1. \] (3.9)

**IV Biedenharn-Elliott Identity**

In order to derive the Biedenharn-Elliott identity we will consider the recoupling of four super-spins $j_1, j_2, j_3$ and $j_4$. Using the same technique as in the derivation of eq. (3.2) we can express the vector $|((j_2\lambda_2; j_3\lambda_3)j_3\lambda_3; (j_1\lambda_1; j_4\lambda_4)j_4\lambda_{14}; \ell m)\rangle$ first, as a result of two successive recouplings of three super-spins, and second, as a result of three recouplings summing over the intermediate states containing $j_{124}$. 7
The calculations being completely analogous to the ones presented in the previous section, we will omit the intermediate (rather tedious) calculations and will present only the results: the Biedenharn-Elliott identity for the super-rotation recoupling coefficients reads

\[
\sum_{j124}(-1)^{\Theta_{BC}^{\text{BE}}}
\times\left[\left(\begin{array}{c}
  j1 \\
  j2 \\
  j3 \\
  j14
\end{array}\right)_{S} \left(\begin{array}{c}
  j23 \\
  j4 \\
  j \\
  j14
\end{array}\right)_{S}
\right]^S
\times\left[\left(\begin{array}{c}
  j1 \\
  j2 \\
  j3 \\
  j14
\end{array}\right)_{S} \left(\begin{array}{c}
  j23 \\
  j4 \\
  j \\
  j14
\end{array}\right)_{S}
\right]^S
\]

It is interesting to note that the phase \(\Theta_{BC}^{\text{BE}}\) is independent of the parities

\[
\Theta_{BC}^{\text{BE}} = I_{12}(I_{1234} + I_{123} + I_{14}) + I_{124}(I_{1234} + I_{14} + 1) + I_{12}(I_{1234} + 1) + I_{124}I_{14}
\]

As a result, the Biedenharn-Elliott identities for the \(\lambda\)-dependent \(S6 - (j, \lambda)\) and \(\lambda\)-independent super-rotation \(S6 - j\) symbols take the same form

\[
\Theta_{BC}^{\text{BE}} = I_{12}(I_{1234} + I_{123} + I_{14} + 1) + I_{12}I_{23}
\]

Here, as well as in the case of the Racah sum rule, we see that the structure of the Biedenharn-Elliott identity for the super-rotation \(S6 - j\) symbols is exactly the same as the structure of the Biedenharn-Elliott identity [8,9] for the rotation \(6 - j\) symbols. Only the sign factor \(\Theta_{BC}^{\text{BE}}\) is somewhat more involved in the supersymmetric case. This factor almost coincides with the corresponding one in the rotation case when the values of the super-spins in the left-hand side of
eq. (4.3) are such that their sums in all the triangles \((j_1, j_2, j_{12}), (j_2, j_3, j_{23}), (j_1, j_3, j_{13}), (j_{12}, j_3, j_{13}), (j_1, j_4, j_{14}), (j_{23}, j_4, j)\) and \((j_{14}, j_{23}, j)\) are integer

\[
\begin{align*}
\{ j_1 & \quad j_2 & \quad j_{12} \}^S & \quad \{ j_2 & \quad j_3 & \quad j_{23} \}^S \\
\{ j_3 & \quad j_{12} & \quad j_{23} \} & \quad \{ j_{14} & \quad j_4 & \quad j_{14} \} \\
\sum_{j_{123}} & (-1)^{j_1 + j_2 + j_3 + j_4 + j_{12} + j_{23} + j_{14} + j_{23} + j_{12} + j} & \\
\times & \quad \{ j_2 & \quad j_1 & \quad j_{12} \}^S & \quad \{ j_2 & \quad j_3 & \quad j_{23} \}^S & \quad \{ j_{14} & \quad j_2 & \quad j_{12} \}^S.
\end{align*}
\] 

\( (4.5) \)

V Conclusion

In this paper we continued the development of the Racah-Wigner calculus for the super-rotation \(osp(1|2)\) superalgebra. In particular, we have derived the Racah sum rule and the Biedenharn-Elliott identity for the super-rotation \(S6-j\) symbols. It turned out that the structure of these relations is the same as the corresponding structure in the rotation case. Only the sign factors are somewhat more involved.

In a forthcoming publication we will use the Biedenharn-Elliott identity in order to obtain recursion relations for the super-rotation \(S6-j\) symbols. These relations will permit us to develop numerical algorithms for the evaluation of the \(S6-j\) symbols for large values of the superspins, our final goal being to obtain a Regge-Ponzano [10] type formula for the \(S6-j\) symbols.

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