Standard Model in Differential Geometry on Discrete Space $M_4 \times Z_3$

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Abstract

Standard model is reconstructed using the generalized differential calculus extended on the discrete space $M_4 \times Z_3$. $Z_3$ is necessary for the inclusion of strong interaction. Our starting point is the generalized gauge field expressed as $A(x, y) = \sum_i a_i^\dagger(x, y) d a_i(x, y), (y = 0, \pm)$, where $a_i(x, y)$ is the square matrix valued function defined on $M_4 \times Z_3$ and $d = d + \chi$ is generalized exterior derivative. We can construct the consistent algebra of $d_\chi$ with the introduction of the symmetry breaking function $M(y)$ and the spontaneous breakdown of gauge symmetry is coded in $d_\chi$. The gauge field $A_\mu(x, y)$ and Higgs field $\Phi(x, y)$ are written in terms of $a_i(x, y)$ and $M(y)$, which might suggest $a_i(x, y)$ to be more fundamental object. The unified picture of the gauge field and Higgs field as the generalized connection in non-commutative geometry is realized. Not only Yang-Mills-Higgs lagrangian but also Dirac lagrangian, invariant against the gauge transformation, are reproduced through the inner product between the differential forms. Two model constructions are presented, which are distinguished in the particle assignment of Higgs field $\Phi(x, y)$. 
§1. Introduction

Higgs-Kibble mechanism is very important for the gauge theory with the spontaneous breakdown of symmetry. Weinberg-Salam theory 1) based on this mechanism has elucidated many experimental mysteries of the electro-weak interactions since proposed in 1967. However, it seems to be rather artificial and Higgs particle as the essential ingredient of this mechanism has not been detected from now, which have persuaded one to investigate other alternatives 3) such as the technicolor model, Kaluza-Klein theory and etc.. Such alternatives , however, are by no means successful and satisfactory because of the extra physical modes like the technicolor particles.

Recently, Connes 3) has proposed the new approach which enables us to understand Higgs mechanism in non-commutative geometry on the discrete space $M_4 \times Z_2$. The unified picture of the gauge field and Higgs field as the generalized connection on the discrete space is realized. However, the mathematical settings of his theory are so difficult that one is hard to understand it. Several versions have been extended in line with Connes’s original idea to make his approach more understandable. Chamseddine, Felder and Frolich 4) provide more successful formalism to be applicable to grand unified theories with the complicated breaking pattern of symmetry. They introduce the generalized Dirac operator and get the gauge invariant lagrangian by use of Clifford algebra.

Sitarz 5) investigated this problem in more familiar form with the ordinary differential geometry. His approach is seemingly different to that of Connes. He introduced the fifth one form basis $\chi$ and the corresponding exterior derivative $d\chi$ in addition to the ordinary one form basis $dx^\mu$ and the exterior derivative $d$. His approach, however, seems to be difficult to be applied to the gauge theory with rather complicated breaking pattern like the grand unified theory.

K. Morita and the present author 6) developed this approach adopting the algebraic rules different from that of Sitarz also on $M_4 \times Z_2$. We introduced the matrix function $M(y)$ ($y$ is a variable in $Z_2$ and $y = \pm$) which is a factor to determine the scale and pattern of the spontaneous breakdown of gauge symmetry. The introduction of $M(y)$ makes the formalism more flexible. However, Model I in Ref.6) permits the appearance of gauge non-invariant term in Higgs potential and we have to discard it by hand. This defect in our first paper 6) has been overcome by taking account of the auxiliary fields according to Chamseddine et
al.4) and we could reproduce standard model in the completely gauge invariant way 7). However, the inclusion of the strong interaction into standard model was not successful and the treatment of fermion sector was unsatisfactory. The purpose of this paper is to present the complete formalism of standard model. The inclusion of strong interaction requires the discrete space $M_4 \times Z_3$, not $M_4 \times Z_2$. We can extend the formalism by use of only one form basis $\chi$ like in Ref.7) since the strong interaction does not break spontaneously. The generalization of the present formalism to $M_4 \times Z_N$ has been done by the present author 8) and $SU(5)$ grand unified theory is reconstructed therein. This paper is divided into five sections and one Appendix. The next section presents the general framework based on the generalized differential calculus on $M_4 \times Z_3$. The generalized gauge field is defined there and a geometrical picture for the unification of the gauge and Higgs fields is realized. The third section provides the gauge invariant lagrangian for both boson and fermion sectors on the same footing. The fourth section is devoted to the reconstruction of the standard model. Two model constructions will be made. The last section is devoted for conclusions and discussions. $SU(2)$ Higgs-Kibble model is treated in the non-commutative geometry on $M_4 \times Z_2$ in the Appendix.

§2. The generalized connection on $M_4 \times Z_3$

The inclusion of strong interaction into standard model requires the discrete space $M_4 \times Z_3$ because the corresponding gauge symmetry is $SU(3)_c \times SU(2) \times U(1)$. Let $A(x, y)$ be the generalized connection (gauge field) defined on discrete space $M_4 \times Z_3$, where $M_4$ is the ordinary four dimensional Minkowski space and $Z_3$ is the space with the discrete variable $y (= 0, \pm)$. According to Chamseddine et al.4), we express $A(x, y)$ to be

$$A(x, y) = \sum_i a_i^k(x, y) d a_i(x, y),$$

(1)

where $a_i(x, y)$ the square-matrix-valued function $^*$) and $d$ is the generalized exterior derivative defined as follows.

$$d a_i(x, y) = (d + d_\chi) a_i(x, y)$$

$$da_i(x, y) = \partial_\mu a_i(x, y) dx^\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu},$$

$$d_\chi a_i(x, y) = [-a_i(x, y) M(y) + M(y) a_i(x, -y)] \chi,$$

(2)

$^*$) $a_i(x, 0)$ is $3 \times 3$, $a_i(x, +)$ is $2 \times 2$ and $a_i(x, -)$ is $1 \times 1$ matrix.
Here $dx^\mu$ is ordinary one form basis, taken to be dimensionless, in $M_4$, and $\chi$ is the one form basis in the discrete space $Z_3$ assumed to be also dimensionless, and we assume $\chi^\dagger = -\chi$. We have introduced $x$-independent matrix $M(y)$ whose hermitian conjugation is given by $M(y)^\dagger = M(-y)$. The matrix $M(y)$ turns out to determine the scale and pattern of the spontaneous breakdown of the gauge symmetry. We take $M(0) = 0$ because $SU(3)_c$ as the symmetry of strong interaction is not spontaneously broken. For this reason, one-form basis in $Z_3$ is only $\chi$.

We further define the calculation rules as follows.

$$
a_i(x, y) dx^\mu = dx^\mu a_i(x, y),
\quad
f(x, y) \chi = \chi f(x, -y),
\quad
\begin{align*}
\frac{d}{\chi} \chi &= 0, \\
\frac{d}{\chi} M(y) &= M(y) M(-y) \chi,
\end{align*}
$$

(3)

where $f(x, y)$ denotes $a_i(x, y)$ or $M(y)$. The second equation in Eq.(3) expresses the non-commutativity of the geometry under consideration. We have one more assumption that $^*$

$$
\frac{d}{\chi} (p(x, y) q(x, y)) = (\frac{d}{\chi} p(x, y)) q(x, y) + (-1)^r p(x, y) (\frac{d}{\chi} q(x, y)),
$$

where $p(x, y)$ and $q(x, y)$ consist of the products of $a_i(x, y)$ and $M(y)$ like in the footnote and $r$ is the number of $M(y)$ in $p(x, y)$. According to these calculation rules we can prove Leibniz rule for the product of square matrix functions as

$$
\frac{d}{\chi} (a_i(x, y) b_i(x, y)) = (\frac{d}{\chi} a_i(x, y)) b(x, y) + a_i(x, y) (\frac{d}{\chi} b_i(x, y)).
$$

With these considerations we easily prove the nilpotency of $\chi$

$$
\frac{d}{\chi}^2 a_i(x, y) = 0, \quad \frac{d}{\chi}^2 M(y) = 0,
$$

(4)

$^*$ For example, we take the following manipulation.

$$
\frac{d}{\chi} (a_i(x, y) M(y) b_i(x, -y)) = (\frac{d}{\chi} a_i(x, y)) M(y) b_i(x, -y) \\
+ a_i(x, y) (\frac{d}{\chi} M(y)) b_i(x, -y) - a_i(x, y) M(y) (\frac{d}{\chi} b_i(x, -y)).
$$
which leads to the nilpotency of $d$ in the follows.

$$d^2 a_i(x, y) = 0, \quad d^2 M(y) = 0, \quad (5)$$

where $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$ and $dx^\mu \wedge \chi_m = -\chi_m \wedge dx^\mu$ are assumed reasonable in the ordinary differential calculus. The proof of $d^2 p(x, y) = 0$ is also easy, but we quote only Eq.(5) in this paper. Nilpotency $d^2 = 0$ is very important for the gauge invariant formulation.

Let us rewrite $A(x, y)$ in Eq.(1) by use of Eq.(2).

$$A(x, y) = \sum_i a^i(x, y) [\partial_\mu a_i(x, y) dx^\mu + (-a_i(x, y) M(y) + M(y) a_i(x, -y)) \chi]$$

$$= \sum_i a^i(x, y) \partial_\mu a_i(x, y) dx^\mu + \sum_i a^i(x, y) (-a_i(x, y) M(y) + M(y) a_i(x, -y)) \chi. \quad (6)$$

From this equation we can define the ordinary gauge field $A_\mu(x, y)$ and Higgs field $\Phi(x, y)$ as follows.

$$A_\mu(x, y) = \sum_i a^i(x, y) \partial_\mu a_i(x, y),$$

$$\Phi(x, y) = \sum_i a^i(x, y) (-a_i(x, y) M(y) + M(y) a_i(x, -y)). \quad (7)$$

Here we require according to Chamseddine et al.4)

$$\sum_i a^i(x, y) a_i(x, y) = 1, \quad (8)$$

without loss of generality, which leads to

$$A^\dagger_\mu(x, y) = -A_\mu(x, y), \quad \Phi^\dagger(x, y) = \Phi(x, -y). \quad (9)$$

Eqs.(7) and (8) make us perceive the necessity of the sum about $i$ in Eq.(1) in order for $A_\mu(x, y)$ not to become the pure gauge field. The generalized connection
\( A(x, y) \) is written in terms of \( A_\mu(x, y) \) and \( \Phi(x, y) \) as

\[
A(x, y) = A_\mu(x, y)dx^\mu + \Phi(x, y)\chi. \tag{10}
\]

Eq.(10) realizes the unified picture of the gauge field and Higgs field as the generalized connection in the geometry on the discrete space \( M_1 \times Z_3 \). Since the fifth dimension is discrete, there arises no additional physical degrees of freedom as in the Kaluza-Klein theory.

The generalized field strength \( F(x, y) \) is defined in terms of \( A(x, y) \) as

\[
F(x, y) = dA(x, y) + A(x, y) \wedge A(x, y). \tag{11}
\]

In order to explicit expression of \( F(x, y) \), we have to calculate \( dA(x, y) \) from Eq.(1). Owing to the nilpotency \( d^2 = 0 \) in Eq.(5) we find

\[
dA(x, y) = d\left( \sum_i a_i(x, y)da_i(x, y) \right) = \sum_i da_i(x, y) \wedge da_i(x, y). \tag{12}
\]

Inserting Eq.(2) into Eq.(12) and using Eqs.(7) and (8), we obtain

\[
dA(x, y) = \partial_\mu A_\nu(x, y)dx^\mu \wedge dx^\nu + [\partial_\mu \Phi(x, y) + A_\mu(x, y)M(y) - M(y)A_\mu(x, -y)]dx^\mu \wedge \chi
+ [\Phi(x, y)M(-y) + M(y)\Phi(x, -y) + M(y)M(-y) - \sum_i a_i^i(x, y)M(y)M(-y)a_i(x, y)]\chi \wedge \chi. \tag{13}
\]

The auxiliary field \( Y(x, y) \) is denoted as

\[
Y(x, y) = \sum_i a_i^i(x, y)M(y)M(-y)a_i(x, y), \tag{14}
\]

which plays an important role to insure the gauge invariance of Yang-Mills-Higgs lagrangian as shown later. \( A(x, y) \wedge A(x, y) \) in Eq.(11) is calculated by use of Eq.(9)
\[ A(x, y) \wedge A(x, y) = A_\mu(x, y) A_{\nu}(x, y) dx^\mu \wedge dx^\nu + (A_\mu(x, y) \Phi(x, y) - \Phi(x, y) A_\mu(x, y) - y)) dx^\mu \wedge \chi + \Phi(x, y) \Phi(x, -y) \chi \wedge \chi. \]  

(15)

From Eqs. (13) and (15) the generalized field strength \( F(x, y) \) is written as

\[
F(x, y) = \frac{1}{2} F_{\mu\nu}(x, y) dx^\mu \wedge dx^\nu + D_\mu \Phi(x, y) dx^\mu \wedge \chi + V(x, y) \chi \wedge \chi, \quad (16)
\]

where

\[
F_{\mu\nu}(x, y) = \partial_\mu A_\nu(x, y) - \partial_\nu A_\mu(x, y) + [A_\mu(x, y), A_\mu(x, y)],
\]

\[
D_\mu \Phi(x, y) = \partial_\mu \Phi(x, y) - (M(y) + \Phi(x, y)) A_\mu(x, -y) + A_\mu(x, y) (M(y) + \Phi(x, y))
\]

\[
V(x, y) = (\Phi(x, y) + M(y))(\Phi(x, -y) + M(-y)) - Y(x, y).
\]

(17)

If we put \( H(x, y) = \Phi(x, y) + M(y) \), the second and third equations in Eq.(17) is rewritten as

\[
D_\mu \Phi(x, y) = \partial_\mu H(x, y) + A_\mu(x, y) H(x, y) - H(x, y) A_\mu(x, -y)
\]

\[
V(x, y) = H(x, y) H(x, -y) - Y(x, y),
\]

(18)

which indicates that \( H(x, y) \) is the un-shifted Higgs field with the vacuum expectation value \( M(y) \) whereas \( \Phi(x, y) \) represents the physical field with the vanishing vacuum expectation value. This is the meaning that \( M(y) \) determines the scale and pattern of the spontaneous breakdown of gauge symmetry.

In order to construct gauge invariant lagrangian we have to investigate the gauge transformation property. The gauge transformation is defined by

\[
A^g(x, y) = g^{-1}(x, y) A(x, y) g(x, y) + g^{-1}(x, y) d g(x, y),
\]

(19)

where \( g(x, y) \) is the gauge function with respect to the corresponding unitary group.
We take the operation of $d = d + \chi$ on $g(x, y)$ as

$$dg(x, y) = (d + d\chi)g(x, y)$$

$$= \partial_\mu g(x, y) dx^\mu + [-a_i(x, y) M(y) + M(y) a_i(x, -y)] \chi,$$

which together with Eq.(19) defines the gauge transformation law for $A_\mu(x, y)$ and $\Phi(x, y)$,

$$A_\mu^g(x, y) = g^{-1}(x, y) \partial_\mu g(x, y) + g^{-1}(x, y) A_\mu(x, y) g(x, y)$$

$$\Phi^g(x, y) = g^{-1}(x, y) \Phi(x, y) g(x, -y) + g^{-1}(x, y) (M(y) g(x, -y) - g(x, y) M(y)) \tag{20}$$

For the unphysical Higgs field $H(x, y)$, we have

$$H^g(x, y) = g^{-1}(x, y) H(x, y) g(x, -y). \tag{21}$$

This is the usual gauge transformation law of Higgs field when the spontaneous breakdown of symmetry is not still switched on. The gauge transformation Eqs.(19) and (20) require the gauge transformation for the fundamental field $a_i(x, y)$ to be

$$a_i^g(x, y) = a_i(x, y) g(x, y). \tag{22}$$

Then, the gauge transformation of field strength is then subject to

$$F^g(x, y) = g^{-1}(x, y) F(x, y) g(x, y), \tag{23}$$

which shows $F(x, y)$ gauge covariant. In components, Eq.(23) reads

$$F^g_{\mu\nu}(x, y) = g^{-1}(x, y) F_{\mu\nu}(x, y) g(x, y),$$

$$D^g_\mu \Phi(x, y) = g^{-1}(x, y) D_\mu \Phi(x, y) g(x, -y), \tag{24}$$

$$V^g(x, y) = g^{-1}(x, y) V(x, y) g(x, y).$$

In order for the third equation in Eq.(24) to be consistent, Eq.(22) is inevitable. With these gauge transformations, the gauge invariant Yang-Mills-Higgs lagrangian is obtainable.
§3. Gauge invariant lagrangian

In order to get the Yang-Mills-Higgs lagrangian from the field strength $F(x, y)$ which is the differential two form, we have to decide the metric structure of the discrete space $M_4 \times Z_3$. The ordinary Minkowski metric in $M_4$ is to be supplemented by additional assumption that the fifth direction is orthogonal to $M_4$ and the metric along the fifth direction is left arbitrary:

$$<dx^\mu, dx^\nu> = g^{\mu\nu}, \quad g^{\mu\nu} = \text{diag}(1, -1, -1, -1),$$

$$<\chi, dx^\mu> = 0,$$

$$<\chi, \chi> = -\alpha^2.$$  \hspace{1cm} (25)

The inner products of two-forms are taken to be

$$<dx^\mu \wedge dx^\nu, dx^\rho \wedge dx^\sigma> = g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho},$$

$$<dx^\mu \wedge \chi, dx^\nu \wedge \chi> = -\alpha^2 g^{\mu\nu},$$

$$<\chi \wedge \chi, \chi \wedge \chi> = \beta^4,$$  \hspace{1cm} (26)

while other inner products among the basis two-forms vanish. It should be noticed that $\beta$ is not necessarily equal to $\alpha$ because of the non-commutative property of geometry under consideration.

**Yang-Mills-Higgs sector**

According to these metric structures and Eq.(16) we can give the formula for gauge-invariant Yang-Mills-Higgs lagrangian

$$\mathcal{L}_{YM_H}(x) = - \text{tr} \sum_{y=0, \pm} \frac{1}{g_y^2} <F(x, y), F(x, y)>$$

$$= - \text{tr} \sum_{y=0, \pm} \frac{1}{2g_y^2} F^\dagger_{\mu\nu}(x, y) F^{\mu\nu}(x, y)$$

$$+ \text{tr} \sum_{y=0, \pm} \frac{\alpha^2}{g_y^2} (D_\mu \Phi(x, y))^{\dagger} D^\mu \Phi(x, y) - \text{tr} \sum_{y=0, \pm} \frac{\beta^4}{g_y^2} V^\dagger(x, y) V(x, y),$$  \hspace{1cm} (27)

where $g_y$ is the gauge coupling constant and tr denotes the trace over internal symmetry matrices. In Ref.6) we took $g_y$ to be common value for $y = \pm$. However, in this paper we take $g_y$ to be different values for $y = 0, \pm$ to discuss the inclusion of strong interaction. It should be remarked that Eq.(27) is Yang-Mills-Higgs
lagrangian with the same number of parameters as in the ordinary Weinberg-Salam theory. We can not predict anything about Weinberg angle or Higgs mass in contrast to the situation in the non-commutative geometry approach. This is because $\beta$ is not necessarily equal to $\alpha$. It may be interesting to investigate under what conditions $\beta$ is equal to $\alpha$. It probably determines the non-commutativity of geometry under consideration.

**fermionic sector**

We propose the general framework to construct the gauge invariant Dirac lagrangian of the fermionic sector. Chamseddine et al. and others usually give the Dirac lagrangian by use of Clifford algebra. We want to get it in the same way as Yang-Mills-Higgs lagrangian in Eq.(27) is introduced by taking the inner product of the differential forms.

Remarking the equation $\mathbf{d} + A(x, y)$, we define

$$\mathcal{D}\psi(x, y) = (\mathbf{d} + A^f(x, y))\psi(x, y),$$

which we call the covariant spinor one-form. Here, $A^f(x, y)$ corresponds to the differential representation for $\psi(x, y)$ such that $A^f(x, y) = A^f_\mu(x, y)dx^\mu + \Phi(x, y)\chi$. Since the role of $d_\chi$ makes the shift $\Phi(x, y) \rightarrow \Phi(x, y) + M(y)$ as shown previously, we define

$$d_\chi\psi(x, y) = M(y)\psi(x, -y)\chi.$$  \hspace{1cm} (29)

$d + A^f(x, y)$ is Lorentz invariant, and so $\mathcal{D}\psi(x, y)$ is transformed as a spinor just like $\psi(x, y)$ against Lorentz transformation. Let $\rho$ be the representation for $\psi(x, y)$ so that the gauge transformation is expressed as

$$\psi^\rho(x, y) = \rho(g^{-1}(x, y))\psi(x, y).$$

As $A^f_\mu(x, y)$ is Lee algebra for the fermion representation of gauge group under consideration, it is transformed as

$$(A^f_\mu(x, y))^\rho = \rho(g^{-1}(x, y))A^f_\mu(x, y)\rho(g(x, y)) + \rho(g^{-1}(x, y))\partial_\mu\rho(g(x, y)).$$

In addition, we have to assign $\psi(x, y)$ in the model building in order that the
following relation is satisfied.

\[ H^g(x,y)\psi^g(x,-y) = \rho(g^{-1}(x,y))H(x,y)\psi(x,-y), \]

which is sufficiently taken into account in the following section.

From these, \( \mathcal{D}\psi(x,y) \) is shown to be the gauge covariant;

\[ \mathcal{D}^g \psi^g(x,y) = \rho(g^{-1}(x,y))\mathcal{D}\psi(x,y). \]  

\( \mathcal{D}\psi(x,y) \) just introduced above is the gauge covariant spinor one form. In order to give the gauge invariant Dirac lagrangian by taking the inner product of differential form, we have to introduce the corresponding spinor one form. For this purpose we consider the following spinor one form.

\[ \tilde{\mathcal{D}}\psi(x,y) = \gamma_\mu \psi(x,y)dx^\mu + i\psi(x,y)\chi, \]  

where \( c \) is a constant related with the Yukawa coupling between Higgs particle and fermion. \( \tilde{\mathcal{D}}\psi(x,y) \) is evidently transformed against Lorentz transformation just as \( \psi(x,y) \) and also gauge covariant.

Dirac lagrangian is computed by the inner product

\[ \mathcal{L}_D(x,y) = \langle \tilde{\mathcal{D}}\psi(x,y), \mathcal{D}\psi(x,y) \rangle = \langle \tilde{\psi}(x,y)\gamma_\mu (\partial_\mu + A_\mu^f(x,y))\psi(x,y) \]

\[ + i\tilde{\psi}(x,y)c\alpha^2(M(y) + \Phi(x,y))\psi(x,-y) \rangle, \]  

where we have defined the inner product for spinor one-form, by noting Eq.\((25)\),

\[ \langle A(x,y)dx^\mu, B(x,y')dx'^\nu \rangle = \tilde{A}(x,y)B(x,y')g^{\mu\nu}, \]

\[ \langle A(x,y)\chi, B(x,y')\chi \rangle = -\tilde{A}(x,y)B(x,y')\alpha^2, \]  

with vanishing other inner products and \( \tilde{A}(x,y) = A^\dagger(x,y)\gamma^0 \). Yukawa interaction of Higgs particle to fermion is given by the last term in Eq.\((32)\).

\[ \mathcal{L}_{Yukawa}(x,y) = -g\tilde{\psi}(x,y)H(x,y)\psi(x,-y), \]  

which is evidently gauge invariant.
The total Dirac lagrangian is the sum over \( y = 0, \pm \):

\[
\mathcal{L}_D(x) = \sum_{y=0, \pm} \mathcal{L}_D(x, y),
\]  

which is gauge and Lorentz invariant and hermitian. Eqs. (27) and (35) play an important role to reconstruct standard model.

\section*{§4 Reconstruction of standard model}

In this section we present two model constructions using the formalism developed in the previous section. They are designed so as to reproduce standard model with constraints from the non-commutative geometry.

**Model I**

Let us begin with Yang-Mills-Higgs sector according to Eq.(27).

**Yang-Mills-Higgs sector**

The first model we shall consider is based on the following identification for \( A(x, y), \Phi(x, y), \) and \( M(y) \) \((y = 0, +, -)\) with obvious notations:

\[
A_\mu(x, 0) = -\frac{i}{2} \sum_{a=1}^{8} \lambda^a G^a_\mu(x),
\]

\[
\Phi(x, 0) = 0, \quad M(0) = 0
\]

\[
g(x, 0) = g_s(x), \quad g_s(x) \in SU(3)_c,
\]

where \( G^a_\mu(x) \) represents gluon field, and the second equation means \( SU(3)_c \) gauge symmetry not to break and \( \lambda^a(a = 1, 2 \cdots 8) \) are Gell-Mann zweig matrices.

\[
A_\mu(x, +) = -\frac{i}{2} \sum_{i=1}^{3} \tau^i A^i_\mu(x) - \frac{i}{2} a \tau^0 B_\mu(x),
\]

\[
\Phi(x, +) = \Phi(x) = \begin{pmatrix} \phi_+^{\prime}(x) \\ \phi_0^{\prime}(x) \end{pmatrix}, \quad M(+) = \begin{pmatrix} 0 \\ \mu \end{pmatrix},
\]

\[
g(x, +) = e^{-i\alpha(x)} g(x), \quad e^{-i\alpha(x)} \in U(1), \quad g(x) \in SU(2),
\]

\(^*)\ We can also define the Dirac lagrangian by

\[
\mathcal{L}_D(x, y) = -i \langle \mathcal{D}(x, y) \psi(x, y), \bar{\mathcal{D}}(x, y) \psi(x, y) \rangle
\]

which is the same as Eq.(32) except for the total divergence and we assign \( \psi(x, 0) = 0 \) as shown later.
where $A^i_\mu(x)$ and $B_\mu(x)$ denotes $SU(2)$ and $U(1)$ gauge fields, respectively. $\tau^i (i = 1, 2, 3)$ are Pauli matrices and $\tau^0$ is $2 \times 2$ unit matrix.

\[
A_\mu(x, -) = -\frac{i}{2}bB_\mu(x),
\]
\[
\Phi(x, -) = \Phi^+(x),
\]
\[
M(0, \mu) = M^+(\mu),
\]
\[
g(x, -) = e^{-ib\alpha(x)} \in U(1).
\]

$M(y)$ must be chosen to give the correct symmetry breakdown. It should be noticed that there are free parameters $a, b$ in Eqs.(36a) and (36b). This implies that $U(2)$ on the upper sheet $y = +$ is not orthogonal to $U(1)$ on the lower sheet $y = -$, but the gauge group is only $SU(2) \times U(1)$.

With above identifications the generalized gauge field $F(x, y)$ is expressed as

\[
F(x, 0) = -\frac{i}{4} \sum_a X^a G^a_{\mu\nu}(x) dx^\mu \wedge dx^\nu,
\]
\[
F(x, +) = \frac{i}{4} \left[ - \sum_i \tau^i F^i_{\mu\nu}(x) - a\tau^0 B_{\mu\nu}(x) \right] dx^\mu \wedge dx^\nu
\]
\[
+ D_\mu H(x) dx^\mu \wedge \chi + \left[ H(x) H^4(x) - Y(x, +) \right] \chi \wedge \chi,
\]
\[
F(x, -) = -b \frac{i}{4} B_{\mu\nu}(x) dx^\mu \wedge dx^\nu
\]
\[
+ \left( D_\mu H(x) \right)^4 dx^\mu \wedge \chi + \left[ H^4(x) H(x) - Y(x, -) \right] \chi \wedge \chi,
\]

where

\[
G^a_{\mu\nu}(x) = \partial_\mu G^a_\nu(x) - \partial_\nu G^a_\mu(x) + f^{abc} G^b_{\mu}(x) G^c_\nu(x),
\]
\[
F^i_{\mu\nu}(x) = \partial_\mu A^i_\nu(x) - \partial_\nu A^i_\mu(x) + \epsilon^{ijk} A^j_\mu(x) A^k_\nu(x),
\]
\[
B_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x),
\]
\[
H(x) = \Phi(x) + M(+),
\]
\[
D^\mu H(x) = \left[ \partial_\mu - \frac{i}{2} \left( \sum_i \tau^i A^i_\mu(x) + (a - b) \tau^0 B_\mu(x) \right) \right] H(x).
\]
By inserting Eq.(37) into Eq.(27), we immediately get

\[ \mathcal{L}_{YM} = - \sum_{y=0,\pm} \frac{1}{g_y^2} \left< F(x, y), F(x, y) \right> \]

\[ = - \frac{1}{4g_y^2} \sum_a G^a_{\mu\nu}(x) G^{a\mu\nu}(x) \]

\[ - \frac{1}{4g_+^2} \sum_i F^i_{\mu\nu}(x) \cdot F^{i\mu\nu}(x) - \frac{1}{4} \left( \frac{a^2}{g_+^2} + \frac{b^2}{2g_-^2} \right) B_{\mu\nu}(x) B^{\mu\nu}(x) \]

\[ + \left( \frac{1}{g_+^2} + \frac{1}{g_-^2} \right) \alpha^2 (D_\mu H(x))^\dagger D^\mu H(x) \]

\[ - \frac{\beta^4}{g_+^2} \text{tr}(H(x) H^\dagger(x) - Y(x, +))^2 - \frac{\beta^4}{g_-^2} (H^\dagger(x) H(x) - Y(x, -))^2. \]

Let us investigate the auxiliary fields \( Y(x, +) \) and \( Y(x, -) \) in Eq.(39) from Eq.(14).

\[ Y(x, +) = \sum_i a_i^y(x, +) M(+) M(-) a_i(x, +) \]

\[ = \sum_i a_i^y(x, +) \begin{pmatrix} 0 & 0 \\ 0 & \mu^2 \end{pmatrix} a_i(x, +). \]

This field can not be expressed in terms of gauge or Higgs fields and also not a constant so that \( Y(x, +) \) is independent. Thus, the potential term containing this auxiliary field in Eq.(39) is eliminated owing to the equation of motion. In Ref.6), the term corresponding to this gauge non-invariant term is discarded by hand, which is justified here.

\[ Y(x, -) = \sum_i a_i^y(x, -) M(-) M(+) a_i(x, -) \]

\[ = \sum_i a_i^y(x, -) \mu^2 a_i(x, -) = \mu^2, \]

which is a constant and yields the meaningful Higgs potential term in the lagrangian Eq.(39).
After rescaling

\[ G_\mu(x) \rightarrow g_s G_\mu(x), \]
\[ A_\mu^i(x) \rightarrow g A_\mu^i(x), \]
\[ B_\mu(x) \rightarrow \frac{g_+ g_-}{\sqrt{a^2 g_-^2 + b^2 g_-^2}} B_\mu(x), \]
\[ H(x) \rightarrow \frac{g_+ g_-}{\sqrt{g_+^2 + g_-^2}} H(x) = g_H H(x), \]

we find the standard Yang-Mills-Higgs lagrangian in standard model.

\[ \mathcal{L}_{YM-H} = -\frac{1}{4} \sum_a G^{a\nu}_\mu(x) \cdot G^{a\mu\nu}(x) \]
\[ -\frac{1}{4} \sum_i F^{i\mu}_\nu(x) \cdot F^{i\nu\mu}(x) + B_\mu(x) \cdot B^{\mu\nu}(x) \]
\[ + (D_\mu H(x))^\dagger (D^\mu H(x)) - \lambda (H^\dagger H(x) - \mu^2)^2, \]  

where

\[ G^{a\nu}_\mu(x) = \partial_\mu G^{a\nu}_\mu(x) - \partial_\nu G^{a\mu}_\mu(x) + g_s f^{abc} G^{b}_\mu G^{c}_\nu(x) \nu^{\mu}(x), \]
\[ F^{i\mu}_\nu(x) = \partial_\mu A^i_\nu(x) - \partial_\nu A^i_\mu(x) + g e^{ijk} A^j_\mu A^k_\nu(x) \nu^j(x), \]
\[ D_\mu H(x) = [\partial_\mu - \frac{i}{2}(g \sum_i \tau^i \cdot A^i_\mu(x) + g' \tau_0 B_\mu(x))] H(x), \]

with \( g_+ = g, \quad g' = \frac{(a - b)g_+ g_-}{\sqrt{a^2 g_-^2 + b^2 g_-^2}}, \quad \lambda = \frac{\beta^4 g_+^4 g_-^2}{\alpha^4 (g_+^2 + g_-^2)^2}, \quad \text{and} \quad \mu' = \frac{\sqrt{g_+^2 + g_-^2} \alpha^2}{g_+ g_-}. \)

Eq.(43) expresses Yang-Mills-Higgs lagrangian of the gauge theory with the symmetry \( SU(3)_c \times SU(2)_L \times U(1) \) spontaneously broken to \( SU(3)_c \times U(1)_{em} \). Denoting \( W, Z \) gauge bosons and photon by

\[ W_\mu = \frac{1}{\sqrt{2}} (A_\mu^1 - i A_\mu^2), \quad W_\mu^\dagger = \frac{1}{\sqrt{2}} (A_\mu^1 + i A_\mu^2), \]
\[ Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (-g A_\mu^3 + g' B_\mu), \quad A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g A_\mu^3 + g' B_\mu), \]

we get the gauge boson mass term to be

\[ \mathcal{L}_{\text{gauge boson mass}} = m_w^2 W_\mu^\dagger W^\mu + \frac{1}{2} m_z^2 Z_\mu Z^\mu. \]  

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The gauge boson masses are explicitly given as

\begin{align}
  m_W &= \sqrt{1 + \delta^2} \frac{\alpha \mu}{2}, \\
  m_Z &= \sqrt{1 + \delta^2} \left( 1 + \frac{2(a-b)^2}{2a^2 + \delta^2 b^2} \right) \alpha \mu.
\end{align}

where \( \delta = \frac{g_+}{g_-} \). Weinberg angle is determined by two parameters \( r = \frac{b}{a} \) and \( \delta \):

\[
  \sin^2 \theta_W = 1 - \frac{m_W^2}{m_Z^2} = \frac{2(1 - r)^2}{4 - 4r + (2 + \delta^2)r^2}.
\]

One usually impose the condition \([3]\) \( \text{tr} A_\mu(x,+) = \text{tr} A_\mu(x,-) \), which yields \( r = 1 \) and so \( \sin^2 \theta_W = \frac{1}{2 + 2\delta^2} \). Furthermore, putting \( \delta = 1 \) as usual, \( \sin^2 \theta_W = \frac{1}{4} \), which value is predicted by the original Connes’s work. If \( \beta = \sqrt{\eta} \alpha \), the Higgs mass is given from Eq.(45) by

\[
  m_H = \frac{2\eta \delta}{\sqrt{1 + \delta^2}} \alpha \mu,
\]

which is related to the W gauge boson mass via

\[
  m_H = \frac{2\sqrt{2}\eta \delta}{1 + \delta^2} m_w.
\]

If we put \( \delta = 1, \ \eta = 1 \) as usual, \( m_H = \sqrt{2} m_w \), which is typical 3) in non-commutative geometry. \( \alpha = \beta (\eta = 1) \) is natural in the commutative geometry. It is interesting to investigate whether there is a symmetry corresponding to \( \alpha = \beta \) in the non-commutative geometry on the discrete space \( Z_3 \).

**leptonic sector**

Let us reconstruct the fermionic sector of W-S theory using the formalism presented in section III. We first consider the leptonic sector in the first generation. Since leptons do not interact with gluons through the strong interaction, we assign \( \psi(x,0) = 0 \). In conformity with the chiral nature of leptons we assign \( SU(2) \) doublets with left handed chirality on the second sheet \( y = + \), and \( SU(2) \) singlets with right handed chirality on the third sheet \( y = - \). Consequently, in this context
$A_{\mu}(x, +)$ is given by $A_{\mu}(x, +)$ in Eq.(36b) with $a \to a_L$ and $A_{\mu}(x, -)$ by $A_{\mu}(x, -)$ in Eqs.(36c) with $b \to a_R$, where $a_{L,R}$ are related to hypercharges of the leptons. In order that the $U(1)$ invariance is consistently realized in Eqs.(34), $b - a = a_R - a_L$ is necessary. In view of Eqs.(44) in which gauge fields and Higgs particle are scaled out, we may set $a_{L,R} = Y$ where $Y$ is the hypercharge of lepton. With this in mind we make the following assignments for the leptons in first generation.

$$\psi(x, +) = l_L(x) = \begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}, \quad a_L = -1,$$

$$\psi(x, -) = l_R(x) = e_R(x), \quad a_R = -2,$$

where the left and right-handed spinors are defined by $\psi_L = \frac{1 - \gamma_5}{2} \psi$ and $\psi_R = \frac{1 + \gamma_5}{2} \psi$, respectively with $\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$. To be more precise the left-handed leptons form a doublet with hypercharge $Y = a_L = -1$ and the right-handed electron a singlet with hypercharge $Y = a_R = -2$ as usual. After the scale transformations of gauge fields and Higgs field as in Eq.(42), $a_{L,R}$ is nothing but the hypercharge $Y$. Gauge invariance of Yukawa coupling (44) is satisfied by $a_L - a_R = a - b = 1$. With these assignments, Dirac lagrangian for leptonic sector can be written as

$$\mathcal{L}_\mu(x) = \frac{1}{2} \bar{l}_L(x) \gamma^\mu (\partial_\mu - \frac{i}{2} g \sum_i \tau^i A_{\mu}^i(x) + \frac{i}{2} g' \gamma^0 B_\mu(x)) l_L(x)$$
$$+ \bar{l}_R(x) \gamma^\mu (\partial_\mu + i g' B_\mu(x)) l_R(x)$$
$$- g_v \left[ \bar{l}_L(x) H(x) l_R(x) + \bar{l}_R(x) H^\dagger(x) l_L(x) \right].$$

The leptonic mass term is simply

$$\mathcal{L}_{\text{lepton mass}} = - g_v \left[ \left( \bar{\nu}_L, \bar{e}_L \right) M(+) e_R + \left( \bar{e}_L, \bar{\nu}_L \right) M(-) \left( e_R, \bar{\nu}_L \right) \right]$$
$$= - m_\ell (\bar{e}_L e_R + \bar{\nu}_L e_L) = - m_\ell \bar{e} e,$$

where $m_\ell = g_v \mu$. Electron becomes massive whereas the neutrino remains massless as expected. Inclusion of three generation is very easy for leptonic sector and so we skip it.
quark sector

We consider the quark sector in three generations. In contrast to lepton, quark interacts with gluon via the strong interaction and up-quark is massive. In addition to these, quark mixing between three generations is taken into account. If we put $q'_r = (d', s', b')$ and $q_f = (d, s, b)$ quark mixing is expressed as

$$q'_r = \sum_{f=1}^{3} U_{rf} q_f,$$

where $U$ is Kobayashi-Maskawa mixing matrix. In order for up-quark and down-quark to be massive, two kind of assignment are necessary. The first assignment is to give the down-quark

$$\psi(x, 0) = 0,$$

$$\psi(x, +) = a_1 q_L = a_1 \left( \begin{array}{c} u_L(x) \\ d_L(x) \end{array} \right), \quad a_L = \frac{1}{3},$$

$$\psi(x, -) = d_R(x), \quad a_R = -\frac{2}{3},$$

where we abbreviate subscripts about triplet representation of $SU(3)_c$ and $a_1$ exists for normalization of quark fields in the final expression. We have to carefully investigate the differential representation $A^f(x, y)$ in eq.(34). $\psi(x, +)$ is triplet for $SU(3)_c$, doublet for $SU(2)$, and of course singlet for $U(1)$ and transformed according to $g(x, 0) \otimes g^f(x, +)$, where $g^f(x, +)$ is given as $e^{-ia_L \sigma(x) g(x)}$. $\psi(x, -)$ is triplet for $SU(3)_c$ and singlet for $SU(2) \times U(1)$ and transformed according to $g(x, 0) \otimes g^f(x, -)$, where $g^f(x, -)$ is $e^{-ia_R \sigma(x)}$. As a result $A^f(x, y)$ is determined to be

$$A^f(x, +) = (-\frac{i}{2} g_s \sum_{a=1}^{8} \lambda^a G^a_p(x) - \frac{i}{2} g \sum_{i=1}^{3} \tau^i A^i_{\mu}(x) - \frac{i}{6} g' \tau^0 B_{\mu}(x)) dx^\mu + g_H H(x) \chi,$$

$$A^f(x, -) = (-\frac{i}{2} g_s \sum_{a=1}^{8} \lambda^a G^a_p(x) + \frac{i}{3} g' B_{\mu}(x)) dx^\mu + g_H H(x) \chi.$$

The associated spinor one form is taken to be

$$\psi(x, 0) = 0,$$

$$\psi(x, +) = a_1 q_L = a_1 \left( \begin{array}{c} u_L(x) \\ d_L(x) \end{array} \right), \quad a_L = \frac{1}{3},$$

$$\psi(x, -) = d_R(x), \quad a_R = -\frac{2}{3},$$

where we abbreviate subscripts about triplet representation of $SU(3)_c$ and $a_1$ exists for normalization of quark fields in the final expression. We have to carefully investigate the differential representation $A^f(x, y)$ in eq.(34). $\psi(x, +)$ is triplet for $SU(3)_c$, doublet for $SU(2)$, and of course singlet for $U(1)$ and transformed according to $g(x, 0) \otimes g^f(x, +)$, where $g^f(x, +)$ is given as $e^{-ia_L \sigma(x) g(x)}$. $\psi(x, -)$ is triplet for $SU(3)_c$ and singlet for $SU(2) \times U(1)$ and transformed according to $g(x, 0) \otimes g^f(x, -)$, where $g^f(x, -)$ is $e^{-ia_R \sigma(x)}$. As a result $A^f(x, y)$ is determined to be

$$A^f(x, +) = (-\frac{i}{2} g_s \sum_{a=1}^{8} \lambda^a G^a_p(x) - \frac{i}{2} g \sum_{i=1}^{3} \tau^i A^i_{\mu}(x) - \frac{i}{6} g' \tau^0 B_{\mu}(x)) dx^\mu + g_H H(x) \chi,$$

$$A^f(x, -) = (-\frac{i}{2} g_s \sum_{a=1}^{8} \lambda^a G^a_p(x) + \frac{i}{3} g' B_{\mu}(x)) dx^\mu + g_H H(x) \chi.$$

The associated spinor one form is taken to be
Adding to this assignment we make one more assignment to give mass to up-quark.

\[
\psi^l(x, 0) = 0, \\
\psi^l(x, +) = a_2 \begin{pmatrix} d_{\beta}^c(x) \\ -u_{\alpha}^c(x) \end{pmatrix}, \quad a_\beta = \frac{-1}{3}, \\
\psi^l(x, -) = u_{\alpha}^c(x), \quad a_\alpha = -\frac{4}{3},
\]

where superscripts \(c\) on quark fields represent the charge conjugation and \(a_1^2 + a_2^2 = 1\) is satisfied for normalization of quark field. \(u^l\) denotes the up-quark mixed by \(U^\dagger\) matrix. It should be noted that \(\psi^l(x, +)\) is the right handed spinor and \(\psi^l(x, -)\), left handed. \(\psi^l(x, +)\) is \(3^*\) representation for \(SU(3)_c\), doublet for \(SU(2)\) and of course singlet for \(U(1)\), and then transforms according to \(g(x, 0)^* \otimes g^f(x, +)\), where \(g^f(x, +)\) is given as \(e^{-ia_{\beta} \alpha(x)}g(x)\). \(\psi^l(x, -)\) is \(3^*\) representation for \(SU(3)_c\) and singlet for \(SU(2) \times U(1)\), and transforms according to \(g(x, 0)^* \otimes g^f(x, -)\), where \(g^f(x, -)\) is \(e^{-ia_{\alpha} \alpha(x)}\). For this assignment, \(A^f(x, y)\) are

\[
A^f(x, +) = \left( i/gs \sum_{a=1}^{8} \lambda^{s_{a}} G_{\mu}^{a}(x) - i/2 g \sum_{i=1}^{3} \tau^i A^l_{\mu}(x) + i/6 g' \tau^0 B_{\mu}(x) \right) dx^\mu + g_n H(x) \chi,
\]
\[
A^f(x, -) = \left( i/gs \sum_{a=1}^{8} \lambda^{s_{a}} G_{\mu}^{a}(x) + 2i/3 g' B_{\mu}(x) \right) dx^\mu + g_n H^\dagger(x) \chi
\]

The associated one form spinor is taken to be

\[
\mathcal{D}\psi^l(x, y) = \gamma_{\mu} \psi^l(x, y) dx^\mu + ic_w \psi^l(x, y) \chi.
\]

With these considerations, we can obtain the final expression as follows.

\[
\mathcal{L}_q = \mathcal{L}^q_{\text{kin.}} + \mathcal{L}^q_{\text{Yukawa}},
\]

\[\text{--- 19 ---}\]
where

\[
\mathcal{L}_{\text{kin.}}^q = i \bar{q}_L \gamma^\mu (\partial_\mu - \frac{i}{2} g_s \sum_{a=1}^{8} \lambda^a G^a_{\mu} - \frac{i}{2} g \sum_{i=1}^{3} \tau^i A^i_{\mu} - \frac{i}{6} g' \tau^0 B_{\mu}) q_L \\
+ i \bar{u}_R' \gamma^\mu (\partial_\mu - \frac{i}{2} g_s \sum_{a=1}^{8} \lambda^a G^a_{\mu} - \frac{i}{3} g' B_{\mu}) u_R' \\
+ i \bar{d}_R' \gamma^\mu (\partial_\mu - \frac{i}{2} g_s \sum_{a=1}^{8} \lambda^a G^a_{\mu} + \frac{1}{3} g' B_{\mu}) d_R',
\]

(58b)

and

\[
\mathcal{L}_{\text{Yukawa}}^q = -g_V (\bar{q}_L' H d_R' + \bar{d}_R' H q_L') - g_V' (\bar{q}_L' H u_R' + \bar{u}_R' H q_L'),
\]

(58c)

with \(g_V\) and \(g'_V\) appropriately chosen Yukawa coupling constants. \(q_L'\) and \(\tilde{H}\) in Eq.(58c) is given as

\[
q_L' = \begin{pmatrix} u_L' \\ d_L' \end{pmatrix}, \quad \tilde{H} = i \tau^2 H^*.
\]

For the brevity, we set \(q_L' = q_L\). To be accurate, Eq.(58a) is the lagrangian for the first generation. We have add the lagrangians over the three generations to obtain the full lagrangian.

We have shown that it is possible to reconstruct standard model with the symmetry \(SU(3)_c \times SU(2) \times U(1)\) spontaneously broken to \(SU(3)_c \times U(1)_{em}\) based on Model I. \(a, b\) in Eqs.(36b,c) play an fairly important role. \(a, b\) are the hypercharge of the particle after the rescaling of fields in Eq.(42). However, \(a - b = a_L - a_R\) is important in the particle assignments to preserve \(U(1)\) symmetry. Thus, Weinberg angle in Eq.(56) is written in terms of two parameters \(r = \frac{b}{a}\) and \(\delta\). As shown hereafter, Model II can determine \(r\) due to the physical requirement. The fermion sector is easily incorporated in the formalism. Thus, we skip it.

**Model II**

Gauge and Higgs fields are assigned as follows, characteristic to satisfy so called
the custodial symmetry. The notations are same as in Model I.

\[ A_\mu(x, 0) = -\frac{i}{2} \sum_{a=1}^{8} \lambda^a G_\mu^a(x), \]
\[ \Phi(x, 0) = 0, \quad M(0) = 0 \]
\[ g(x, 0) = g_s(x), \quad g_s(x) \in SU(3)_c, \]

\[ A_\mu(x, +) = -\frac{i}{2} \sum_i \tau^i A^i_\mu(x) - \frac{i}{2} a \tau^0 B_\mu(x), \]
\[ \Phi(x, +) = \Phi(x) = \begin{pmatrix} \phi_+(x) & -\phi^*_0(x) \\ \phi_0(x) & \phi_-(x) \end{pmatrix}, \]
\[ M(+) = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}, \]
\[ g(x, +) = e^{-i\sigma(x)} g(x), \quad e^{-i\sigma(x)} \in U(1), \quad g(x) \in SU(2). \]

and

\[ A_\mu(x, -) = -\frac{i}{2} \begin{pmatrix} b B_\mu(x) & 0 \\ 0 & 0 \end{pmatrix}, \]
\[ \Phi(x, -) = \Phi^\dagger(x, +), \]
\[ M(-) = M^\dagger(+), \]
\[ g(x, -) = \begin{pmatrix} e^{-i\sigma(x)} & 0 \\ 0 & 1 \end{pmatrix}. \]

In this assignment, the auxiliary fields are

\[ Y(x, \pm) = \sum_i a_i^\dagger(x, \pm) \mu^2 \tau^0 a_i(x, \pm) = \mu^2 \tau^0, \]

because

\[ M(+) M(-) = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} = \mu^2 \tau^0 = M(-) M(+). \]

Thus, both auxiliary fields \( Y(x, \pm) \) are constant fields, which allows both Higgs potential terms to remain in Yang-Mills-Higgs lagrangian. This model is very interesting to enable us to reconstruct standard model in the gauge invariant way, also in Ref 6).
In order to determine \( r = \frac{b}{a} \), the gauge boson mass term is investigated from Eqs.\( (18) \) and \( (27) \). It is given to be the trace of

\[
A_\mu(x,+)M(+) - M(+)A_\mu(x,-) = -\frac{i}{2} \sqrt{g} \left( \begin{array}{cc} A_\mu^1 - i A_\mu^2 & -a B_\mu - A_\mu^3 \\ (a - b) B_\mu - A_\mu^3 & -(A_\mu^1 + i A_\mu^2) \end{array} \right),
\]

multiplied by the Hermite conjugate of this term. This equation yields \( a - b = -a \rightarrow b = 2a \ (r = 2) \) since photon is massless. Putting \( r = 2 \) and using the same notations in Model I, we get Yang-Mills-Higgs lagrangian as

\[
\mathcal{L}_{YM} = -\frac{1}{4g_s^2} \sum_a G_{\mu\nu}(x) G^{\mu\nu}(x)
- \frac{1}{4g^2} \left( \sum_i F_{\mu\nu}^i(x) \cdot F^{i\mu\nu}(x) + (1 + 2\delta^2) a^2 B_{\mu\nu}(x) B^{\mu\nu}(x) \right)
+ \frac{2\alpha^2}{g^2(1 + \delta^2)} \left( (D_\mu H(x))^\dagger (D^\mu H(x)) - \frac{2\alpha^4}{g^2(1 + \delta^2)}(H^\dagger H(x) - \mu^2)^2 \right). \tag{60}
\]

By changing the scale of fields as \( G_\mu(x) \rightarrow g_s G_\mu(x) \), \( A_\mu^{i}(x) \rightarrow g A_\mu^{i}(x) \), \( B_\mu(x) \rightarrow \frac{g}{\sqrt{1 + 2\delta^2}} B_\mu(x) \), and \( H(x) \rightarrow \frac{g}{\sqrt{2 + 2\delta^2} G} H(x) \), and

\[
\mathcal{L}_{YM} = -\frac{1}{4} \sum_a G_{\mu\nu}(x) \cdot G^{\mu\nu}(x)
- \frac{1}{4} \left( \sum_i F_{\mu\nu}^i(x) \cdot F^{i\mu\nu}(x) + B_{\mu\nu}(x) B^{\mu\nu}(x) \right)
+ (D_\mu H(x))^\dagger (D^\mu H(x)) - \lambda (H^\dagger H(x) - \mu^2)^2, \tag{61}
\]

with \( g' = \frac{g}{\sqrt{1 + 2\delta^2}} \), \( \lambda = \frac{\eta^2 g^2}{2(1 + \delta^2)} \), and \( \mu' = \sqrt{\frac{2(1 + \delta^2)}{g}} \mu \).

From these equations we can predict in Model II,

\[
\sin^2 \theta_w = \frac{1}{4} \cdot \frac{2}{1 + \delta^2},
\]

\[
m_H = \sqrt{2} \eta m_w \cdot \sqrt{\frac{2}{1 + \delta^2}}. \tag{62}
\]

The first equation in Eq.\( (62) \) lead to the inequality \( \sin^2 \theta_w \leq \frac{1}{2} \). Elimination of \( \delta \)
from Eq. (62) yields

\[ m_H = 2\sqrt{2}\eta m_w \sin \theta_w. \]  \hspace{1cm} (63)

If we put \( \eta = 1 \), Eq. (63) is written as \( m_H = 2\sqrt{2}m_w \sin \theta_w \) in tree level, which leads to \( m_H = 109.1 \text{GeV} \) when \( \sin^2 \theta_w = 0.233 \), and \( m_w = 79.9 \text{GeV} \) as the experimental values. It is interesting to investigate under what conditions \( \eta = 1 \) is realized in non-commutative geometry.

§5. Conclusions and Discussions

We reconstruct standard model based on non-commutative geometry on discrete discrete space \( M_4 \times Z_3 \). The inclusion of strong interaction into formalism is nicely realized in both boson and fermionic sectors. For this purpose, we prepare three sheets (\( Z_3 \)). Our framework is more akin to the ordinary differential geometry than that of Connes and Chamseddine et al.. Although Higgs mechanism has been a mystery in particle physics, it is nicely explained owing to the theory of differential geometry on the discrete space \( M_4 \times Z_3 \). The unified picture of the gauge and Higgs fields as a generalized connection is realized. It should be stressed that the method to introduce the Dirac lagrangian in section §3 is original, not in any text book of differential geometry.

Our starting point in this paper is Eq.(1), where the unidentified factor \( a_i(x, n) \) appears. The matrix \( M(x, y) \) appears through the generalized differential derivative. The gauge field and Higgs field are written in terms of \( a_i(x, y) \) and \( M(y) \) in Eq.(13). We again quote it.

\[
A_\mu(x, y) = \sum_i a_i^\dagger(x, y) \partial_\mu a_i(x, y),
\]

\[
\Phi(x, y) = \sum_i a_i^\dagger(x, y) (-a_i(x, y) M(y) + M(y) a_i(x, -y)).
\]  \hspace{1cm} (64)

which makes us imagine that \( a_i(x, y) \) would be the more fundamental object and the gauge and Higgs bosons were composed of them. In addition to this, \( a_i(x, y) \) is restricted to be \( \sum_i a_i^\dagger(x, y) a_i(x, y) = 1 \) which seems to be the normalization condition for the field. It should be remarked whether \( a_i(x, y) \) becomes the physical object experimentally observed in future.
We can reconstruct Weinberg-Salam theory with the same number of parameters as in the standard model. Thus, quantization is performed as usual, contrary to the comment of Alvarez et al (1). However, it is fun to imagine the following thing. The inner product of two-form on the discrete space $Z_3$ is taken to be $\langle \chi \wedge \chi, \chi \wedge \chi \rangle = \beta^4$, whereas $\langle \chi, \chi \rangle = -\alpha^2$. $\alpha = \beta$ is natural in the commutative geometry. It is very interesting to investigate under what conditions $\beta$ is related to $\alpha$ in the non-commutative geometry on the discrete space. If such a condition corresponds to a certain symmetry in the discrete space, it is conceivable to investigate the response against the quantum effect.

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**Appendix Higgs-Kibble model**

Ref.6) could not treat Higgs-Kibble model because of the gauge non-invariant term appearing in the Higgs potential. In this appendix we formulate the $SU(2)$ Higgs-Kibble model in the gauge invariant way presented in this paper.

We prepare also in this model two sheets $(y = \pm)$ on which we prescribe $a_i(x, +)$ to be complex $2 \times 2$ matrix-valued function and $a_i(x, -)$ to be merely a real-valued function satisfying the relation $\sum_j a_j^i(x, -)a_i(x, -) = 1$. With this prescription we immediately get $A_\mu(x, -) = 0$. We start with the field strength Eq.(16) and assign $A_\mu(x, y), \Phi(x, y)$ and $M(y)$ the following obvious notations.

$y = +$

$$A_\mu(x, +) = -\frac{i}{2} \sum_{k=1}^3 \sigma^k A^k_\mu(x) = -\frac{i}{2} \begin{pmatrix} A^3_\mu(x) & A^1_\mu(x) - iA^2_\mu(x) \\ A^1_\mu(x) + iA^2_\mu(x) & -A^3_\mu(x) \end{pmatrix},$$

$$\Phi(x, +) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix},$$

$$M(+) = \begin{pmatrix} 0 \\ \mu \end{pmatrix}, \quad g(x, +) = g(x), \quad g(x) \in SU(2).$$

(A1a)
\[ y = - \]
\[ A_\mu(x, -) = 0, \]
\[ \Phi(x, -) = (\phi_-(x), \phi_0^e(x)) = \Phi^\dagger(x, +), \phi_-(x) = \phi_+^*(x), \quad (A1b) \]
\[ M(-) = (0, \mu) = M^\dagger(+), \quad g(x, -) = 1. \]

Inserting these equations into Eq. (27) we get
\[
\mathcal{L}_{\text{YM}} = - \frac{1}{4g^2} \sum_k F^{k\mu\nu}(x) \cdot F^{k\mu\nu}(x) + \frac{(1 + \delta^2)\alpha^2}{g^2}(D_\mu \varphi(x))^\dagger D^\mu \varphi(x) - \frac{\alpha^4}{g^2} \mathrm{tr}(\varphi(x) \varphi^\dagger(x) - Y(x, +))^2 - \frac{\delta^2\beta^4}{g^2}(\varphi^\dagger(x) \varphi(x) - Y(x, -))^2. \quad (A2) \]

Here we have introduced the notation
\[
F^{ij}_{\mu\nu}(x) = \partial_\mu A^i_\nu(x) - \partial_\nu A^i_\mu(x) + \epsilon_{ijk} A^j_\mu(x) A^k_\nu(x), \quad i, j, k = 1, 2, 3, \\
\varphi(x) = \begin{pmatrix} \varphi_+(x) \\ \varphi_0(x) \end{pmatrix}, \quad \varphi_0(x) = \phi_0(x) + \mu, \quad \varphi_+(x) = \varphi^*_-(x) = \phi_+(x), \\
D_\mu \varphi(x) = [\partial_\mu - \frac{i}{2} \sum_k \sigma^k \cdot A^k_\mu(x)] \varphi(x), \\
Y(x, y) = \sum_i a^i_{ij}(x, y) M(y) M(-y) a_i(x, y), \quad (A3) \]

After eliminating the independent auxiliary field \( Y(x, +) \) and rescaling of fields; \( A^i_\mu(x) \rightarrow g A^i_\mu(x) \) and \( \varphi(x) \rightarrow \frac{g}{\sqrt{1 + \delta^2}} \varphi(x) \), we find that
\[
\mathcal{L}_{\text{YM}} = - \frac{1}{4} \sum_k F^{k\mu\nu}(x) \cdot F^{k\mu\nu}(x) + (D_\mu \varphi(x))^\dagger (D^\mu \varphi(x)) - \lambda (\varphi^\dagger(x) \varphi(x) - \mu^2)^2, \quad (A4) \]

where
\[
F^{ij}_{\mu\nu}(x) = \partial_\mu A^i_\nu(x) - \partial_\nu A^i_\mu(x) + g \epsilon_{ijk} A^j_\mu(x) A^k_\nu(x), \quad i, j, k = 1, 2, 3, \\
D_\mu \varphi(x) = [\partial_\mu - \frac{i}{2} g \sum_k \sigma^k \cdot A^k_\mu(x)] \varphi(x), \quad (A5) \]

with \( \lambda = \frac{\delta^2\beta^4 g^2}{\alpha^2(1 + \delta^2)^2} \) and \( \mu^\prime = \frac{\sqrt{1 + \delta^2} \alpha^2}{g^2} \). If we put \( \alpha = \beta \), we can get the inequality \( m_\mu \leq \sqrt{2} m_\nu \).
References


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   See also, Jagiellonian Univ. preprint,TPJU–7/92 “ Non-commutative Geometry and Gauge Theory on Discrete Groups “.
   This Beijing group paper also reconstructed standard model by use of Sitarz’ formalism developed in the above preprint. The authors are thankful to Professor T. Saito for informing this reference to us.

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