The well-defined phase of simplicial quantum gravity in four dimensions

Wolfgang Beirl, Erwin Gerstenmayer, Harald Markum, and Jürgen Riedler

Institut für Kernphysik, Technische Universität Wien, A-1040 Vienna, Austria

We analyze simplicial quantum gravity in four dimensions using the Regge approach. The existence of an entropy dominated phase with small negative curvature is investigated in detail. It turns out that observables of the system possess finite expectation values although the Einstein-Hilbert action is unbounded. This well-defined phase is found to be stable for a one-parameter family of measures. A preliminary study indicates that the influence of the lattice size on the average curvature is small. We compare our results with those obtained by dynamical triangulation and find qualitative correspondence.

PACS number(s): 04.60.+n, 12.25.+e
I. INTRODUCTION

General relativity relates classical gravitation to the curvature of space-time and therefore quantum gravity is essentially a quantum theory of space-time geometry [1]. In the search for such a theory the sum-over-histories approach has proved to be an elegant and powerful tool [2-4]. It leads directly from the classical action to the quantization of space-time using the Feynman path integral. The problems of this approach are well known. Its perturbation theory is not renormalizable and a unique prescription for the summation over all 4-geometries does not exist.

Moreover, in the case of pure gravity the action is unbounded from below due to rapid conformal fluctuations. At first sight this means that the corresponding Euclidean path-integral is ill-defined and the quantum theory has no ground state. However, there are exceptions such as the Hydrogen atom whose quantum mechanics is well-defined although the Euclidean action is unbounded from below. In the case of quantum gravity numerical simulations within the Regge calculus indicate that the entropy of the system can suppress geometries with large curvature leading to a phase with finite expectation values [5,6,8]. It is the aim of this article to further investigate the conditions for the existence of the well-defined phase. We do not give a complete picture of the phase structure and the details of the phase transition but refer the reader to appropriate literature [5-7].

We use the path integral

\[ Z = \int Dg e^{-I_E(g)} \]  \hspace{1cm} (1)

as starting point for the non-perturbative investigation of quantum gravity. The functional integral extends over a class of closed 4-geometries \( g \) in the Euclidean sector with pure gravitational action

\[ -I_E(g) = L_P^{-2} \int d^4x \sqrt{g} (R - 2\Lambda), \]  \hspace{1cm} (2)

where \( L_P \) is the Planck length, \( R \) the curvature scalar, and \( g \) the determinant of the metric. In general the action contains the cosmological constant \( \Lambda \).
Additional prescriptions are necessary to define the path integral as a limit of systematic approximations. A direct route to such approximations is the Regge calculus that uses simplicial lattices to discretize general relativity in a coordinate independent and thus very elegant way [9,10]. One considers a simplicial net being the triangulation of a given 4-topology and introduces a metric by assigning a length to each link in the net. Increasing systematically the number of simplices and decreasing the link lengths one reaches the classical continuum limit if certain conditions are fulfilled [11–13].

The Regge approach is a valuable tool to approximate the path integral on a simplicial net with fixed incidence matrix and varying link lengths [6,8,13,14]. A complementary method using fixed link lengths and varying incidence matrices is known as dynamical triangulation [15–17]. However, the severe difficulties of the functional integral (1) are also present on the simplicial lattice,

(i) In general the integration should include a summation over all possible 4-topologies, but topologies are not classifiable in 4 dimensions [18–20].

(ii) In the case of pure gravity the action is unbounded due to rapid conformal fluctuations [3,21,22]. This divergence is also present in the Regge action if one does not restrict or even fix all link lengths. The introduction of a cosmological constant term guarantees a finite volume of the simplicial lattice, but this is only a necessary and not a sufficient condition for finite link lengths and a bounded action. In Sect. 2 we give an explicit example of a configuration with finite volume but infinite link lengths and action. One has to explain why such configurations do not contribute to the path integral in the well-defined phase and this is the aim of the entropy argument of Berg: The integral \( Z = \int dI n(I) e^{-I} \) is well defined if the density of states \( n(I) \) vanishes rapidly enough for \( I \to -\infty \). Previous computations indicate that the entropy of the system can indeed compensate the unbounded action leading to a phase with small and negative average curvature [5,8] and one has assumed that a finite volume will always lead to finite link lengths [6].

To give, at least numerically, further evidence for this assumption we investigate in Sect. 3 the well-defined phase of the Regge approach with three different methods. Furthermore,
we compare our results with those obtained by dynamical triangulation and find qualitative agreement.

(iii) A unique definition of the measure $Dg$ does not exist, since different physical arguments prefer different classes of measures to be outlined below [2,23–26]. It is an open question in what manner various choices of the measure in the path integral affect the well-defined phase although one expects universality for a certain class [7,14]. Hence, we use a one-parameter family of measures and investigate in Sect. 4 the stability of the entropy dominated region against variations of this parameter. Again, the comparison with dynamical triangulation shows a remarkable coincidence of both approaches.

(iv) The discretization of general relativity destroys the diffeomorphism group and it is not known if it can be recovered in the continuum limit [10,27]. Until now the existence of a reasonable continuum limit has not been demonstrated for simplicial quantum gravity. Therefore, the meaning of our numerical results for the continuum remains unclear. However, it is possible to study the influence of finite-volume effects on the computed expectation values and it seems that the entropy dominated phase survives increasing lattice volumes as demonstrated in Sect. 4. This might be an important result for computations within the 'fundamental length scale' scenario proposed recently by Berg et al. [28].

The article is organized in the following way: The concept of the Regge calculus and the simplicial path integral are discussed in Sect. 2. In Sect. 3 we investigate the entropy dominated phase by checking various observables for well-defined expectation values. The stability of this phase against a variation of the measure and finite-volume effects are studied in Sect. 4. The conclusion follows in Sect. 5.

II. SIMPLICIAL QUANTUM GRAVITY

The triangulation of a manifold $\mathcal{M}$ is determined if a simplicial manifold $K$ homeomorphic to $\mathcal{M}$ is given [10,29]. A regular triangulation of the 4-torus $T^4$ used in the following is provided by an algorithm that divides $T^4$ into $n^4$ hypercubes and then each hypercube into 24 4-simplices [27]. For the total numbers $N_d$ of d-simplices one has $N_0 = n^4$, $n \geq 3$, and
$N_1 = 15N_0$, $N_2 = 50N_0$, $N_3 = 60N_0$ and $N_4 = 24N_0$. A triangulation of $T^4$ with a smaller number of simplices is described in [30]. In case of the hypercubic triangulation of $T^4$ the coordination numbers $N_n/N_m$ are close to those of a random triangulation as proposed by Christ et al. [31].

A simplicial net obtains a metric structure if one assigns a length to each link of the lattice. In the following we denote the squared length of link $l$ by $q_l \in R^4$. Since we consider the Euclidean sector, we require $q_l > 0$ and have to check the Euclidean triangle inequalities for all simplices [6,9,10]. In other words, for every 4-simplex $s$ in the lattice we demand that it can be constructed in $R^4$. After fulfilling the Euclidean triangle inequalities for every 4-simplex one has a consistent configuration \{q_l\} that determines the simplicial lattice.

One can calculate several quantities of a simplicial lattice such as the area $A_t$ of the triangle $t = 1, ..., N_2$ and the 4-volume $V_s$ of the 4-simplex $s = 1, ..., N_4$. Another important quantity is the deficit angle $\delta_t$ associated with each triangle $t$. It is defined as

$$\delta_t = 2\pi - \sum_{s \supset t} \theta_{s,t},$$

where the sum runs over all 4-simplices $s$ sharing the triangle $t$. The interior angle $\theta_{s,t}$ is the dihedral angle of the 4-simplex $s$ with the basis triangle $t$ [6,9,10].

From the viewpoint of differential geometry a simplicial lattice is a singular 4-geometry with the curvature concentrated on the triangles and we consider the simplicial lattice as an approximation to a smooth Riemannian manifold [5,6,9–13]. The connection of the deficit angle with the curvature tensor is exhibited by parallel transporting a vector along a closed loop around a triangle $t$ [6,9,32].

In consideration of the proportionality between the curvature tensor and the deficit angle and guided by the dimension of the action Regge proposed [9]

$$\int d^4x \sqrt{g} R \rightarrow 2 \sum_t A_t \delta_t,$$

An alternative derivation and an action with the same continuum limit can be found in [32].

One can show that the Einstein-Hilbert action of a smooth 4-geometry converging towards a simplicial lattice approaches the Regge action [11], but for our purposes it is more
important to study the opposite limit, approximating a given smooth 4-geometry by increasing the number of simplices of the lattice and decreasing the link lengths systematically. A necessary prerequisite in this discussion is the definition of the fatness $\phi_s$ of a 4-simplex $s$. It is a relation between the maximal link length and the 4-volume given by

$$\phi_s = C^2 \frac{V_s^2}{\max_{t \in s}(q_t^4)},$$

where the constant is set to $C = 24$ in the following. (Cheeger et al. use a slightly different but equivalent definition [13].) If the lattice skeleton approaches a smooth geometry such that $\phi_s \geq f = \text{const} > 0$ one recovers general relativity in the continuum limit [12,13].

Notice that the Regge calculus is the only known approximation scheme of general relativity incorporating the Bianchi identities [2,6,9]. Since these relations connect the dynamical and constraint equations of general relativity they are essential for any attempt to quantize gravitation. In the following we try to assign a concrete meaning to the path integral (1) in the language of the Regge calculus.

(i) The integration symbol $\int$ stands for a summation over all 4-geometries $(\mathcal{M}, g)$ without boundary. This can be replaced by a sum over distinct simplicial lattices $(\mathcal{K}, \{q_t\})$. One usually restricts the class of incidence matrices to include only simplicial manifolds $\mathcal{K}$ with a fixed topology. Further simplifications lead to two somehow complementary methods.

Dynamical triangulation: One assigns the same length to each link, $q_l = 1 \ \forall l$, and performs a summation over all possible triangulations $\mathcal{K}$ of a given topology

$$Z = \sum_{\mathcal{K}} e^{-I_G(\mathcal{K})}. \quad (6)$$

The main motivation for this approach is its success in two dimensions [15–17,34]. In four dimensions the action simplifies for equal link lengths to $-I_E = c_2 N_2 - c_4 N_4$ with coupling parameters $c_2, c_4$. It is bounded from below (and above) for a finite number of simplices and, moreover, since the inequality $N_2 < 10N_4$ holds for every simplicial lattice one has $I_E > 0$ for $c_4 > 10c_2$. This can be fulfilled by making the cosmological constant large enough, in contrary to the continuum theory and the Regge approach. Since we try to investigate the possible existence of an entropy dominated phase for unbounded gravitational action, we use the
Regge approach: The triangulation is kept fixed and the path integral \( Z \) reduces to a summation over different configurations \( \{ q_l \} \). Only the link lengths are allowed to vary but not the incidence matrices \([5,8,10,14]\),

\[
Z = \int Dq e^{-I_E(\{q_l\})}.
\]

(7)

Important motivations for this approach are studies of the weak field limit which show the correspondence with linearized quantum gravity [27]. Comparisons of Regge calculus and mini-superspace approximations have also been performed with rather promising results [10,33].

(ii) The gravitational action \( I_E(g) \) is unbounded in the continuum due to conformal fluctuations and this unpleasant feature is also present in the Regge approach yet for a finite number of simplices [4,10]. If some of the 4-simplices collapse the corresponding areas can grow, \( A_l \to \infty \), although their 4-volumes stay finite. To give an example consider a configuration \( \{ q_l \} \) where first all link lengths are equal to \( a \), i.e. \( q_l = a^2 \forall l \). Then choose one vertex \( v_0 \) of the lattice arbitrarily and assign to each link of this vertex a length \( L > a \). It is easy to check that the Euclidean triangle inequalities are all satisfied in this case. Now consider \( a \to 0 \) and \( L \to \infty \) so that \( a^2 L = \text{const} \). In this case the 4-volume of the 4-simplices with \( v_0 \) as a vertex vanishes like \( a^4 \) and since the 4-volume of the other simplices disappears as \( a^4 \) the total 4-volume of the lattice approaches zero, \( \sum V_s \to 0 \). In contrast, the area of triangles with \( v_0 \) as a vertex diverges as \( aL \) and therefore the action grows without limit, \( \sum A_l \delta_l \to \infty \).

As mentioned in the introduction, it seems that such configurations do not contribute to the path integral [5,6,8] in the well-defined phase and we investigate this possibility in the following section. Of course, the described divergence cannot be present for dynamical triangulation with all link lengths equal.

(iii) A unique definition of the gravitational measure \( Dg \) does not exist since it is not clear which quantities have to be identified with the 'true' physical degrees of freedom [2,14,24–26]. Within the Regge approach it seems natural to translate the gravitational measure into a product over the links of the simplicial lattice [5]
where the function $\mathcal{F}$ ensures that the computations are performed in the Euclidean sector being equal 1 if the generalized triangle inequalities are fulfilled and 0 otherwise. The weight function $\mu$ is a (perhaps complicated and non-local) function of $\{q_l\}$ associated with each link $l$ [35,36]. The simplest choice is the uniform measure $\mu(l) = 1 \quad \forall l$ and one has argued that it corresponds to the DeWitt measure [7,10]. However, considering the results of Misner and Faddeev-Popov it seems natural to choose $\mu(l)$ so that a scale-invariant measure results [5,8,13].

Relying on this background the presented numerical studies have been performed with

$$\mu(l) = q_l^{-\sigma - 1},$$

where the parameter $\sigma \geq 0$ determines the behavior of the measure under rescaling. The case $\sigma = 1$ corresponds to the uniform measure while for $\sigma = 0$ a scale-invariant measure in its simplest form results.

Metropolis simulations within the Regge approach proceed in the following way [5,6,8,14]. Starting with a Euclidean configuration $\{q_l\}_0$ a Markov chain of configurations $\{q_l\}_\tau$ is generated to approximate the canonical ensemble. Proposing a new configuration one has to check first the Euclidean triangle inequalities. If the new configuration is not in the Euclidean sector it is rejected immediately. Otherwise one has to check the Metropolis condition to fulfill the requirement of detailed balance. Given a Euclidean configuration one changes usually only one link length at once which simplifies the test of the triangle inequalities. Scanning one time through the whole lattice is called a sweep. After thermalization of the system the averages over the generated configurations are taken to approximate the expectation values.

A priori one expects that simulations of simplicial quantum gravity with unbounded action might not find an equilibrium. Indeed, one has to face this problem for computations on non-regular triangulations [14,37]. To investigate systematically if the entropy of the system can stabilize the unbounded action we use the following method: Guided by the analysis
of Cheeger et al. [13] we set a lower limit for the fatness of each 4-simplex, $\phi_s \geq f > 0$, restricting the configuration space. In this way we introduce a scale-invariant cutoff allowing the system to reach an equilibrium after a finite number of sweeps. As long as $f > 0$ the gravitational action is bounded for a finite number of 4-simplices. Decreasing stepwise the value of $f$ we investigate the limit $f \rightarrow 0$ and conclude from the convergence of this process if a well-defined phase does exist or not. Since decreasing $f$ enlarges the significant configuration space one has to increase the number of sweeps to ensure equilibrium. In the actual computations the number of sweeps varies from $5k$ ($f = 10^{-3}$) up to $20k$ ($f = 10^{-6}$).

III. ENTROPY VERSUS ACTION

In the following we employ the action in the form

$$-I_E = \beta \sum_i A_i \delta_t - \lambda \sum_s V_s,$$

with a bare cosmological constant $\lambda$ fixing the scale and we restrict our attention to the simplicial measure (9) with the parameter $\sigma$. Considering the behavior of the path integral $Z(\beta, \lambda)$ under rescaling $q_i \rightarrow r q_i$ one obtains

$$Z(\beta, \lambda) = \left(\frac{\beta}{\lambda}\right)^{\sigma N_1} Z\left(\frac{\beta^2}{\lambda}, \frac{\beta^2}{\lambda}\right)$$

(11)

after setting $r = \beta/\lambda$. Differentiating $\ln Z$ with respect to $\beta$ and $\lambda$ gives

$$-\beta \langle \sum_i A_i \delta_t \rangle + 2 \lambda \langle \sum_s V_s \rangle = \sigma N_1$$

(12)

in the well-defined phase [6]. We therefore set the parameter $\lambda$ equal to $\sigma$ so that $\langle V_s \rangle$ takes the value $N_1/2N_4$ for $\sigma > 0$ at $\beta = 0$. For $\beta > 0$, $\langle V_s \rangle$ decreases slightly as well as $\langle q_i \rangle$. To compensate the changes of the lattice size as a function of $\beta$ and $\sigma$ we use the dimensionless quantity

$$\ell^2 = \frac{\beta}{2} \langle q_i \rangle$$

(13)

to compare the results. The factor $\frac{1}{2}$ respects the original form of the Regge action (4) and since $\beta$ corresponds to $2L_F^2$ the dimensionless observable $\ell^2$ expresses the average squared
lattice spacing in units of the bare Planck length. Different to conventional lattice field-
theory the simplicial lattice itself is the quantum object, i.e. the lattice spacing is an
observable rather than a parameter and different definitions are possible [8].

Another important observable of simplicial quantum gravity is the scale invariant average
curvature of a lattice configuration [5,6]

\[
\hat{R} = \frac{\sum_s A_s \delta t}{\sum_s V_s} \left( \frac{1}{N_t} \sum \ell q^\ell \right).
\]

(14)
The expectation value \( \langle \hat{R} \rangle \) can be understood as effective cosmological constant measured
in lattice units. This can be seen from the classical field equations deriving \( \Lambda = \frac{1}{4} \hat{R} \) and obtaining

\[
\Lambda = \frac{1}{4} \frac{\int d^4 x \sqrt{g} R}{\int d^4 x \sqrt{g}} \Rightarrow \frac{1}{2} \hat{R},
\]

(15)
with \( \Lambda \) the classical cosmological constant, \( \hat{R} \) the curvature scalar and \( g \) the determinant of
the metric tensor.

In Fig. 1 the behavior of the expectation value \( \langle \hat{R} \rangle \) as a function of \( \ell \) is depicted. The
most interesting region \( \langle \hat{R} \rangle < 0 \) is displayed in more detail in the lower plot. The value of
the lower limit \( f \) for the fatness decreases as follows: \( f = 10^{-n}, n = 3, 4, 5, 6, \infty \). For every
\( f \) we increase \( \beta \) stepwise leading to increasing \( \ell \). Since the significant configuration space
becomes larger with smaller values of \( f \) we performed 20k sweeps for each data point for
\( f \leq 10^{-5} \) and 5k sweeps otherwise. Averages are taken over the last 10k and 2.0k sweeps,
respectively, so that the statistical error is smaller than the symbol size. This computation
was performed with the uniform measure (\( \sigma = 1 \)) on the regular hypercubic triangulation
of the 4-torus with \( 4^4 \) vertices. We will see later that the well-defined phase is stable against
changes of the parameter \( \sigma \) in a certain range.

For all values of the cutoff \( f \) we find a well-defined phase with small and negative curva-
ture. Increasing \( \ell \) one suddenly enters a region of large positive curvature. For \( f \leq 10^{-5} \) it
is difficult to reach a stable equilibrium across the transition point due to a trend towards
non-reproducible states. This coupling regime is therefore called the ill-defined phase.
In every case it seems that \( \langle \hat{R} \rangle < 0 \) as long as \( \ell < \ell_c \), with \( \ell_c = 0.45(1) \) for sufficiently small \( f \). Furthermore, we observe for \( f \to 0 \) a convergence of the function \( \langle \hat{R}(\ell) \rangle \) at every \( \ell < \ell_c \). As long as \( \ell < 0.3 \) one sees practically no difference in the average curvature \( \langle \hat{R} \rangle \) for all values of the cutoff \( f \). This is a strong indication that configurations with small fatness \( \phi_\ast \) give almost no contribution in this region. Increasing \( \ell \) further one begins to realize differences in \( \langle \hat{R} \rangle \) for larger \( f \). However, for \( f \leq 10^{-5} \) there are (nearly) no differences to see over the whole well-defined phase \( 0 \leq \ell < \ell_c \). This convergence for \( f \to 0 \) indicates that the entropy of the system is indeed able to compensate the unbounded gravitational action. Configurations with large average curvature occur with small probability and give (practically) no contribution to the path integral in the well-defined phase. Therefore, it might be possible to perform computations even without any restriction \( (f = 0) \) and obtain reasonable expectation values. One should be aware that the scale of the curvature radius should be larger than the average lattice spacing and smaller than the size of the system, \( \langle q_i \rangle \langle \langle \hat{R} \rangle \rangle^{-1} < N_0^{1/2} \langle q_i \rangle \) [7].

To further analyze the probability of 'crumpled' configurations with collapsing simplices we display the behavior of the average fatness \( \langle \phi_\ast \rangle \). Fig. 2 shows \( \langle \phi_\ast \rangle \) as a function of \( \ell \) for decreasing cutoff \( f \). The parameters of the system and the configurations sampled for the statistics are the same as before. It turns out that \( \langle \phi_\ast \rangle \) is much larger than the lower bound \( f \) in the well-defined region. The convergence of the function \( \langle \phi_\ast(\ell) \rangle \) for decreasing \( f \to 0 \) is seen clearly in the well-defined phase. With the transition to large values of \( \langle \hat{R} \rangle \) the average fatness decreases suddenly as expected. This indicates the collapse of 4-simplices and the transition to a 'crumpled' lattice.

Returning to the well-defined phase let us investigate the behavior of the expectation value \( \langle \delta_i^2 \rangle \). The following argument will supply us with a further indication for the stability of the well-defined phase. Imagine that one incorporates an additional term \( -\alpha \sum_i \delta_i^2 \) in the action. Differentiation of \( \ln Z \) then gives

\[
\frac{\partial \ln Z}{\partial \beta} = \langle \sum_i \lambda \delta_i \rangle , \quad \frac{\partial \ln Z}{\partial \alpha} = -\langle \sum_i \delta_i^2 \rangle
\]
\[
\Rightarrow \frac{\partial \langle \delta_t^2 \rangle}{\partial \beta} = - \frac{\partial \langle A_t \delta_t \rangle}{\partial \alpha}.
\] (16)

All computations are performed at \( \alpha = 0 \) but we assume that pushing \( \alpha \) slightly to a positive value would shift \( \langle A_t \delta_t \rangle \) towards zero since the additional term prefers flat configurations and supresses the curved. (Recall the definition of \( \hat{R} \) to see that \( \hat{R} < 0 \) corresponds to \( \sum_t A_t \delta_t < 0 \)). This means that \( \langle A_t \delta_t \rangle \) will increase for negative values and decrease for positive. We thus have \( \frac{\partial}{\partial \alpha} \langle A_t \delta_t \rangle > 0 \) and therefore \( \frac{\partial}{\partial \beta} \langle \delta_t^2 \rangle < 0 \) in the well-defined phase with negative curvature. Since \( \ell \) is proportional to \( \beta^{1/2} \) the fact that \( \langle A_t \delta_t \rangle \) is negative induces that \( \langle \delta_t^2 \rangle \) decreases in this phase with \( \ell \). As seen in Fig. 3 \( \langle \delta_t^2 \rangle \) indeed decreases as expected with the minimum just around the transition point. This is an indication that the average curvature is really negative in this phase and at least shows the self consistency of the numerical results.

It is practically impossible to determine a systematic error for Metropolis simulations. One cannot exclude for example the possibility that the well-defined phase is only a numerical artefact due to metastable states preventing the system from reaching configurations with large positive curvature. To check this we performed computations with inhomogeneous start configurations from the ill-defined phase having large positive average curvature.

Without a restriction of the fatness (\( f = 0 \)) and a non-zero coupling \( \beta \) the well-defined phase \( \ell < \ell_c \) was reached after a large number of sweeps. The average curvature returned to small and negative values of \( \hat{R} \) although the unbounded gravitational action prefers a large positive curvature. The history of such a run is depicted in the upper plot of Fig. 4 and compared with the history for a start with a homogeneous configuration. The independence of \( \langle \hat{R} \rangle \) from the start configuration is the strongest indication that entropy is indeed able to compensate the unbounded action in the well-defined phase.

Near the transition point very long runs are necessary to reach the equilibrium for very inhomogeneous start configurations. Limited resources therefore do not allow to perform the above check for \( \hat{R}_{\text{start}} \to \infty \) and \( \ell \to \ell_c \). Thus, we cannot exclude the possibility that near \( \ell_c \) metastable states occur if the phase transition is first order as reported in [7]. However, applying a (small) lower limit for the fatness (\( f = 10^{-5} \)) we found no influence of the start
configuration for $\ell \leq 0.45$ with rather large $\hat{R}_{\text{start}}$. Of course, the curvature of the start configuration is limited after applying a lower limit on the fatness. The lower plot of Fig. 4 shows such a history reaching the equilibrium after 50k sweeps.

The two complementary approaches to simplicial quantum gravity are compared in Fig. 5 by plotting where we depict our results within the Regge approach and those obtained by Brügmann within dynamical triangulation [34]. The lower curve gives $\langle \hat{R} \rangle$ as a function of $\ell$ for $f = 10^{-5}$ and uniform measure taken from Fig. 1. The upper curve shows the results of Metropolis simulations performed by Brügmann dynamically triangulating the 4-sphere with $N^e_4 = 16k$ [34]. We depict $\langle \hat{R} \rangle$ as a function of $\ell$ and since $q_l = 1 \forall l$ we have

$$\hat{R} = \frac{a(2\pi N^e_2 - 100N^e_4)}{vN^e_4},$$

(17)

where $a, v, \theta$ denote the area, 4-volume and interior angle associated with equilateral simplices of link lengths equal to 1 and $\ell = \sqrt[2]{2}$. $

For a given number of 4-simplices the number of triangles in the lattice is limited. Since $N^e_2 < 10N^e_4$ there is an upper limit for the average curvature, $\hat{R} < (a/v)(2\pi - 10\theta)$, different to the Regge approach. However, it seems that both dynamical triangulation and the Regge approach lead to similar results at least qualitatively. (One has to remember that the underlying topology is different.) The average curvature begins for small $\ell$ with small positive values and undergoes a transition to large positive curvatures. The transition point seems to be located close to that of the Regge approach, but it is unclear if this is pure coincidence.

IV. INFLUENCE OF THE MEASURE AND FINITE-VOLUME EFFECTS

In this section we address first the question of the measure in the well-defined phase of simplicial quantum gravity and investigate then the influence of finite-volume effects.

Fig. 6 shows the expectation value $\langle \hat{R} \rangle$ as a function of $\ell$ for measures with different values of $\sigma$ increasing stepwise from 0 to 1.5 including the scale-invariant and the uniform measure. For the scale-invariant measure ($\sigma = \lambda = 0$) we use directly the constraint $\sum s V_s = \text{const}$ rescaling the lattice at every sweep as proposed by Berg [8]. The computations
have been performed on the regular hypercubic triangulation of the 4-torus with 4⁴ vertices. To facilitate numerical simulations a lower bound \( f = 10^{-5} \) has been applied for the fatness \( \phi_s \) of each 4-simplex. The results of Sect. 3 suggest that such a restriction of allowed
configurations practically does not affect the behavior of the system in the well-defined phase. Averages are taken over at least 10k sweeps after thermalization so that the statistical error is smaller than the symbol size.

For \( \sigma \leq 1 \) and \( 0 \leq \ell \leq 0.3 \) the value of \( \langle \hat{R} \rangle \) seems to be practically independent of \( \sigma \). Such a stability against a variation of the measure was reported first by Hamber in the context of \( R^2 \) theory [5]. A direct comparison of scale-invariant and uniform measure for pure gravity with a cutoff for the link lengths is described in Ref. [14]. Near the transition point, \( \ell \approx 0.1 \), the influence of \( \sigma \) becomes more pronounced.

The result for \( \sigma = 1.5 \) differs over the whole range of \( \ell \) from those obtained for \( \sigma \leq 1 \). At \( \ell = 0 \) the value of \( \langle \hat{R} \rangle \) is significantly larger than for \( \sigma \leq 1 \). With increasing \( \ell \) we find a stable equilibrium even for \( \langle \hat{R} \rangle > 0 \). This suggests that the considered measures fall into two qualitatively different classes: While we see almost no dependence of the expectation values in the interval \( 0 \leq \sigma \leq 1 \) the influence of the measure becomes visible for \( \sigma > 1 \). The situation for \( \ell = 0 \) is further illustrated in the lower plot of Fig. 6 showing \( \langle \hat{R} \rangle \) as a function of \( \sigma \). Setting \( \sigma = 2 \) we found no equilibrium even at \( \beta = 0 \).

To gain more information about the lattice geometry the behavior of the areas and deficit angles is examined separately. Fig. 7 displays the scale-invariant quantity \( \langle A_i \rangle / \langle q_i \rangle \) as a function of \( \ell \) for the different values of \( \sigma \). For \( 0 \leq \ell < 0.4 \) this ratio stays almost constant somewhat below the value \( \frac{\sqrt{3}}{4} \approx 0.433 \) of equilateral triangles. The value for \( \sigma = 1.5 \) lies significantly below the other data points. Across the transition at \( \ell \approx 0.45 \) the ratio \( \langle A_i \rangle / \langle q_i \rangle \) decreases, indicating a distortion of the triangles.

The expectation value of the average deficit angle \( \langle \delta_i \rangle \) is depicted as a function of \( \ell \) in Fig. 8. Surprisingly \( \langle \delta_i \rangle \) stays negative for all values of \( \ell \) even after the transition to large positive curvature. In the well-defined phase the absolute value of \( \langle \delta_i \rangle \) is rather small compared to \( \pi \). One can understand this behavior by examining the hypercubic triangulation of the 4-torus.
that contains two different types of triangles. The number of 4-simplices sharing a triangle of
the first type is 6 while it is 4 for the second type. As noticed first by Berg the contributions
of these two types almost cancel each other leaving a small negative average deficit angle
\[8\].

The dependence of the simplicial path integral on the measure has been studied also
within the framework of dynamical triangulation. To investigate the influence of different
types of the gravitational measure Brügmann considered an additional term $n \sum_v \ln(o(v))$
in the action where $o(v)$ is the 'order' of the vertex $v$, i.e. the number of 4-simplices that
contain this vertex [31]. Although the correspondence with the continuum is not entirely
clear, it is plausible that this term reproduces a measure of the form $\prod_x g^{x/2} \prod_{i<j} dgamma$ with
$n = -5$ the scale-invariant and $n = 0$ the uniform measure of DeWitt. Results of these
computations are depicted in Fig. 9 where again the expectation value $\langle \tilde{R} \rangle$ is plotted versus
$\ell$ for different values of $n$. Since Brügmann considers negative couplings $\beta < 0$ we extend $\ell$
to negative values.

Besides a shift in $\ell$ for dynamical triangulation, a comparison of Figs. 6 and 9 shows
a remarkable qualitative coincidence of the results with the choice $n = 0$ for dynamical
triangulation corresponding to the case $\sigma = 1$ of the Regge approach. Again the influence
of the measure seems to be small between the scale-invariant and the uniform measure, but
turns out to be significant for $n = 5$.

As for every lattice field theory it is important to study finite-volume effects for simplicial
quantum gravity [7]. To investigate the stability of the well-defined phase for larger lattices
we increased the number of vertices from $3^4$ to $8^4$. The results are displayed in Fig. 10 and
show $\langle \tilde{R} \rangle$ as a function of $\ell$ computed on the hypercubic triangulation of the 4-torus with
the uniform measure ($\sigma = 1$) and a lower limit $f = 10^{-5}$ for the fatness. For the smaller
lattices with $N_0 = 3^4, 4^4, 6^4$ we performed at least 10k sweeps after thermalization. Statistics
is rather poor in the case of $N_0 = 8^4$ where the number of sweeps in equilibrium is only 2.5k.
Therefore, one has to consider the three data points for the $8^4$-triangulation as preliminary.

First results indicate that the critical value of the coupling $\beta$ and therefore $\ell_c$ slightly
decreases for increasing $N_0$. But it seems that increasing the number of vertices $N_0$ has only a weak influence on the computed expectation values of one-point functions and the well-defined phase survives on larger lattices.

V. CONCLUSION

The Regge calculus provides a direct route to systematic approximations of the quantum-gravity path-integral. It allows numerical studies of non-perturbative quantum gravity and has led to astonishing results.

The most important finding is the existence of a well-defined phase. We studied this phase by applying a lower bound on the fatness of each simplex and decreasing it systematically. The convergence of this process supports the 'entropy hypothesis'. Although the gravitational action is unbounded due to conformal fluctuations the entropy of the system seems to stabilize the expectation values. In a certain range of the gravitational coupling a phase with small negative average curvature occurs within the Regge approach. The predicted behavior of the squared deficit angle and the independence of the simulation from the start configuration further support the stability of the entropy dominated phase.

The well-defined phase turns out to be stable against variations of the measure and an increase of the lattice size. The influence of the measure has been studied using a one-parameter family of simple local functions. However, a wider class of different measures should be investigated including also non-local types and non-regular triangulations. The dependence of the results on the lattice size has been studied to some extent. The fact that the entropy dominated phase survives on larger lattices is encouraging for studies of a 'fundamental length scale' scenario [28].

ACKNOWLEDGMENT

We are grateful to Bernd Brügmann for transmitting the data of his simulations enabling the comparison of Regge approach and dynamical triangulation. This work was supported
in part by "Fonds zur Förderung der wissenschaftlichen Forschung" under contract P9522-PHY.


Fig. 1. Expectation value $\langle \hat{R} \rangle$ within the Regge approach as a function of $\ell$ for decreasing lower limit $f$ of the fatness of each 4-simplex. Computations were performed on a regular triangulation of the 4-torus with $4^4$ vertices using the uniform measure ($\sigma = 1$). The upper plot gives an overview of all results available whereas the lower plot magnifies the well-defined phase. The existence of a well-defined phase and the convergence of $\langle \hat{R} \rangle$ for $f \to 0$ are clearly seen. The transition point to positive curvature is shifted from $\ell_c \approx 0.8$ for $f = 10^{-3}$ down to $\ell_c \approx 0.45$ for $f \leq 10^{-5}$. 
Fig. 2. Average fatness $\langle \phi_s \rangle$ versus $\ell$ for decreasing cutoff $f$ for the fatness of each 4-simplex. The value of $\langle \phi_s \rangle$ stays much larger than $f$ in the well-defined phase and convergence for $f \to 0$ is seen. With the transition to large curvatures $\langle \phi_s \rangle$ decreases suddenly indicating the collapse of 4-simplices into degenerate configurations. Configurations and statistics are the same as in Fig. 1.
Fig. 3. Average squared deficit angle $\langle \delta_t^2 \rangle$ as function of $\ell$ for decreasing cutoff $f$. A non-monotonic behavior of $\langle \delta_t^2 \rangle$ is seen for increasing $\ell$. The fact that the average squared deficit angle decreases with increasing $\ell$ is a strong indication for the stability of the well-defined phase. Near the transition point the minimum deviation from flat space is reached and a further increase of $\beta$ and thus $\ell$ leads to the ill-defined region with fast increasing squared deficit angles (see text). Configurations and statistics are the same as in Fig. 1.
Fig. 4. (In)dependence of the numerical results in the well-defined phase from the start configuration. The upper plot shows the histories of $\tilde{R}$ for a start with homogeneous and inhomogeneous configurations, respectively. Both simulations converge to the same equilibrium in the well-defined phase for non-zero $\beta$ ($\ell \approx 0.29$). The computer time is denoted by $\tau$ with one unit representing 10 sweeps. Although the unbounded action ($f = 0$) favors a large positive curvature the entropy moves $\tilde{R}$ to negative values. With a small bound for the fatness ($f = 10^{-5}$) independence from the start is seen even starting with very inhomogeneous configurations as depicted in the lower plot.
Fig. 5. Expectation value $\langle \hat{R} \rangle$ versus $\ell$ for the two complementary approaches to simplicial quantum gravity. The upper curve shows results from dynamical triangulation obtained by Brügmann approximating the 4-sphere by 16$k$ 4-simplices [34]. The lower curve was obtained within the Regge approach on a regular triangulated 4-torus with $4^4$ vertices. The behavior of both functions is rather similar although the underlying topologies differ. A phase of small $\langle \hat{R} \rangle$ turns into a region with large curvature for increasing $\ell$. The black dot indicates a value close to the transition point $\ell_c$ for the Regge approach.
Fig. 6. Expectation value $\langle \hat{R} \rangle$ as a function of $\ell$ within the Regge approach for different types of the measure parametrized by $\sigma \geq 0$. Computations have been performed on a regular triangulated 4-torus with $4^4$ vertices applying a cutoff $f = 10^{-5}$ for faster convergence. The upper plot shows that the behavior of $\langle \hat{R} \rangle$ in the region of small $\ell$ is almost independent of $\sigma$ as long as $\sigma \leq 1$. This stability ceases for $\sigma > 1$ as depicted in some detail in the lower plot where $\langle \hat{R} \rangle$ is given for increasing $\sigma$ in the case of pure entropy, $\ell = 0 \leftrightarrow \beta = 0$. 

25
Fig. 7. Ratio $\langle A_i \rangle / \langle q_i \rangle$ as a function of $\ell$ for different measures parametrized by $\sigma \geq 0$. In the well-defined phase this scale-invariant quantity stays almost constant and somewhat below the maximum value $\frac{2\sqrt{3}}{3} \approx 0.433$ for equilateral triangles. For the scale-invariant measure ($\sigma = 0$) the upper curve results and the expectation values decrease slightly with increasing $\sigma$. Configurations and statistics are the same as in Fig. 6.
Fig. 8. Expectation value $\langle \delta_t \rangle$ as a function of $\ell$ for different measure parameters $\sigma$. It is remarkable that $\langle \delta_t \rangle$ stays always negative even after the transition to positive curvature. The curves lie close together for $0 \leq \sigma \leq 1$ while the behavior differs significantly for $\sigma = 1.5$. The configurations are those of Fig. 6.
Fig. 9. Investigations of different measures with dynamical triangulation of the 4-sphere performed by Brügmann [34]. An additional term \( n \sum_v \ln(o_v) \) in the action mimics different proposed measures. Besides a shift in \( \ell \) the picture has a striking similarity to Fig. 6 if one identifies \( \sigma = 0 \) with \( n = -5 \) (scale-invariant measure) and \( \sigma = 1 \) with \( n = 0 \) (uniform measure). Notice the small influence of \( n \) in the range \( -5 \leq n \leq +1 \) and the exceptional behavior of \( n = +5 \).
Fig. 10. Influence of the lattice size on $\langle \tilde{R} \rangle$ for the Regge approach. The number $N_0$ of vertices increases from $3^4$ to $8^4$. The well-defined phase survives an extension of the triangulated manifold which has almost no influence on the behavior of the computed expectation values $\langle \tilde{R} \rangle$. 